Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/040116

Improved Class of Estimators for Variance under Single And Two Phase Sampling

*Rajesh Singh** and *Viplav Kumar Singh*

Department of Statistics, Banaras Hindu University, Varanasi -221005, India

Received: 24Jul. 2014, Revised: 13 Feb.2015, Accepted: 15 Feb. 2015.
Published online: 1 Mar. 2015.

Abstract: In this paper improved class of estimators for estimating S_y^2 is proposed adapting [6] and [7]. We have also extended our study to the case of two phase sampling. The asymptotic expressions for the mean square error and their optimum values have been derived up to the first order of approximation. The theoretical conditions have also been verified by numerical examples. After deep inspection, it has been shown that the proposed estimators are more efficient than usual unbiased estimator, chain ratio estimator and regression estimator.

Keywords: Consistent estimator, double sampling technique, Bias (B), Asymptotic Variance (V).

1 Introduction

It is well known that use of auxiliary information improves the precision of estimator. If information on an auxiliary variable is readily available then the ratio-type and regression- type estimators can be used for estimation of parameters of interest, due to increase in efficiency of these estimators. The problem of estimating the population variance of S_y^2 of study variable *y* received a considerable attention of the statisticians in survey sampling including [1], [2],[3],[5],[10],[11],[12],[13],[14],[16],[17] and [18].

Let ϕ_i = (1,2,……..N) be the population having N units such that y is highly correlated with the auxiliary variable x and z. We assume that a simple random sample without replacement (SRSWOR) of size n is drawn from the finite population of size N. Let $(s_y^2, s_x^2, \ s_z^2)$ be the sample variance and (S_y^2, S_x^2, S_z^2) be the population variances of variable *y, x* and *z*.

Where, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$, $S_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$, $S_z^2 = \frac{1}{N-1}\sum_{i=1}^{N}(z_i - \bar{Z})^2$.

Let a preliminary large sample of size n' is drawn from the given population with *SRSWOR*. Let again another sample of size $n < n'$ is drawn from the first phase sample with *SRSWOR*. Take measurements of variable *x* and *z* on the first phase sample and measurement of variables *y* and *x, z* on the second phase sample. Let $(s'^2_x, \ s'^2_z)$ be the sample variances of variable *x* and *z* respectively based on first phase sample of size *n'* and $(s_y^2, s_x^2, \ s_z^2)$ be the sample variances of variable *y, x* and *z* respectively based on the second phase sample of size *n*.

To estimate the population variance S_y^2 of *y*, consider two cases:

1. When the population variance S_x^2 and S_Z^2 of *x* and *z* are known.

*Corresponding author e-mail: rsinghstat@gmail.com

2. When S_x^2 is unknown but S_z^2 is known.

With this background, the usual unbiased, chain ratio and regression estimator using one and two auxiliary variable are, respectively defined by

$$\hat{S}_y^2 = s_y^2 \tag{1}$$

$$\hat{Y}_{CR} = s_y^2 \left(\frac{S_x^2}{s_x^2}\right)\left(\frac{S_z^2}{s_z^2}\right) \tag{2}$$

$$\hat{Y}_{lr,1} = s_y^2 + \hat{b}_1(S_x^2 - s_x^2) \tag{3}$$

$$\hat{Y}_{lr,2} = s_y^2 + \hat{b}_1(S_x^2 - s_x^2) + \hat{b}_2(S_z^2 - s_z^2) \tag{4}$$

Where, $(\hat{b}_1, \hat{b}_2)$ denotes the estimator of the regression coefficient $\left(\beta_1 = \frac{S_y^2 \delta_{220}^*}{S_x^2 \delta_{040}^*}, \beta_2 = \frac{S_y^2 \delta_{202}^*}{S_z^2 \delta_{004}^*}\right)$.

Further, the chain-ratio and regression estimators using two phase sampling are, respectively defined by

$$\hat{Y}'_{CR} = s_y^2 \left(\frac{s'^2_x}{s_x^2}\right)\left(\frac{S_z^2}{s'^2_z}\right) \tag{5}$$

$$\hat{Y}'_{lr,1} = s_y^2 + \hat{b}_1(s'^2_x - s_x^2) \tag{6}$$

$$\hat{Y}'_{lr,2} = s_y^2 + \hat{b}_1(s'^2_x - s_x^2) + \hat{b}_2(S_z^2 - s'^2_z) \tag{7}$$

The chain ratio and regression estimators are generally biased, but the biases of these types of estimator are negligible if the sample size is large enough. The approximate variances of $\hat{S}_y^2$, $\hat{Y}_{CR}$, $\hat{Y}_{lr,1}$, $\hat{Y}_{lr,2}$, $\hat{Y}'_{CR}$, $\hat{Y}'_{lr,1}$ and $\hat{Y}'_{lr,2}$ are, respectively, given by:

$$\text{var}(\hat{S}_y^2) = \gamma_1 S_y^4 \delta_{400}^* \tag{8}$$

$$V(\hat{Y}_{CR}) = \frac{S_y^4}{n}\left[\delta_{400}^* + \delta_{040}^* + \delta_{004}^* - 2\delta_{220}^* - 2\delta_{202}^* + 2\delta_{022}^*\right] \tag{9}$$

$$V(\hat{Y}_{lr,1}) = \frac{S_y^4}{n}\left[\delta_{400}^* + \frac{\beta_1^2}{R_1^2}\delta_{040}^* - 2\frac{\beta_1}{R_1}\delta_{220}^*\right] \tag{10}$$

$$V(\hat{Y}_{lr,2}) = \frac{S_y^4}{n}\left[\delta_{400}^* + \frac{\beta_1^2}{R_1^2}\delta_{040}^* + \frac{\beta_2^2}{R_2^2}\delta_{004}^* - 2\frac{\beta_1}{R_1}\delta_{220}^* - 2\frac{\beta_2}{R_2}\delta_{202}^* + 2\frac{\beta_1}{R_1}\frac{\beta_2}{R_2}\delta_{022}^*\right] \tag{11}$$

$$V(\hat{Y}'_{CR}) = S_y^4\left[\frac{\delta_{400}^*}{n} + \gamma\,\delta_{040}^* + \frac{\delta_{004}^*}{n'} - 2\gamma\delta_{220}^* - \frac{2\delta_{202}^*}{n'}\right] \tag{12}$$

$$V(\hat{Y}'_{lr,1}) = S_y^4\left[\frac{\delta_{400}^*}{n} + \gamma\frac{\beta_1^2}{R_1^2}\delta_{040}^* - 2\frac{\beta_1}{R_1}\gamma\delta_{220}^*\right] \tag{13}$$

$$V(\hat{Y}'_{lr,2}) = S_y^4\left[\frac{\delta^*_{400}}{n} + \gamma\frac{\beta_1^2}{R_1^2}\delta^*_{040} + \frac{\beta_2^2}{R_2^2 n'}\delta^*_{004} - 2\gamma\frac{\beta_1}{R_1}\delta^*_{220} - 2\frac{\beta_2}{R_2 n'}\delta^*_{202}\right] \quad (14)$$

Where, $R_1 = \frac{S_y^2}{S_x^2}$, $R_2 = \frac{S_y^2}{S_z^2}$ and $\gamma\left(\frac{1}{n} - \frac{1}{n'}\right)$.

The objective of this paper is to propose some modified chain- ratio, exponential and regression type estimators using single and double sampling schemes. Thus in the following section we have suggested some estimators for population variance S_y^2 based on s_x^2, s'^2_x and s'^2_z and their properties studied. Numerical illustrations are given in support of the present study.

2 Suggested Estimators

Adapting estimators due to [7], [14] and [8], we have proposed three different modified class of ratio and regression type estimator using two auxiliary variable x and z are, respectively, defined by

$$t_0 = s_y^2\left[\frac{m_1 S_x^2 + m_2 S_z^2}{m_1 s_x^2 + m_2 s_z^2}\right]^{\alpha} \quad (15)$$

Where, α is a constant and (m_1, m_2) are weights that satisfy the condition, $m_1 + m_2 = 1$.

$$t_1 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]^{\beta_1} \exp\left[\frac{S_z^2 - s_z^2}{S_z^2 + s_z^2}\right]^{\beta_2} \quad (16)$$

Where β_i (i=1,2) are suitably chosen constants.

$$t_2 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]^{\eta_1} \exp\left[\frac{S_z^2 - s_z^2}{S_z^2 + s_z^2}\right]^{\eta_2} + b_1(S_x^2 - s_x^2) + b_2(S_z^2 - s_z^2) \quad (17)$$

Where $\eta_i\ (i = 1,2)$ are suitably chosen constants and (b_1, b_2) are defined earlier in section 1.

Estimators t_0, t_1 and t_2 under two phase sampling scheme are, respectively, defined by

$$t_{0d} = s_y^2\left[\frac{m_{1d} s'^2_x + m_{2d} S_z^2}{m_{1d} s_x^2 + m_{2d} s'^2_z}\right]^{\alpha_d} \quad (18)$$

Where, α_d is a constant and (m_{1d}, m_{2d}) are weights that satisfy the condition, $m_{1d} + m_{2d} = 1$.

$$t_{1d} = s_y^2 \exp\left[\frac{s'^2_x - s_x^2}{s'^2_x + s_x^2}\right]^{\beta_{1d}} \exp\left[\frac{S_z^2 - s'^2_z}{S_z^2 + s'^2_z}\right]^{\beta_{2d}} \quad (19)$$

Where β_{id} (*i=1,2*) are suitably chosen constants.

$$t_{2d} = s_y^2 \exp\left[\frac{s'^2_x - s_x^2}{s'^2_x + s_x^2}\right]^{\eta_{1d}} \exp\left[\frac{S_z^2 - s'^2_z}{S_z^2 + s'^2_z}\right]^{\eta_{2d}} + b_1(s'^2_x - s_x^2) + b_2(S_z^2 - s'^2_z) \quad (20)$$

Where $\eta_{id}\ (i = 1,2)$ are suitably chosen constants and (b_1, b_2) are defined earlier in section *1*.

To obtain the biases and variances of the estimators t_0, t_1, t_2, t_{0d}, t_{1d} and t_{2d} we write

$$e_0 = \frac{(s_y^2 - S_y^2)}{S_y^2}, e_1 = \frac{(s_x^2 - S_x^2)}{S_x^2}, e_2 = \frac{(s_z^2 - S_z^2)}{S_z^2}, e_1^{'} = \frac{(s'^2_x - S_x^2)}{S_x^2} \text{ and } e_2^{'} = \frac{(s'^2_z - S_z^2)}{S_z^2}$$

Such that, $E(e'_i) = E(e_i) = 0, \ \forall\ (i = 0,1,2)$

And

$$E(e_0^2) = \delta^*_{400}/n,\ E(e_1^2) = \delta^*_{040}/n,\ E(e_2^2) = \delta^*_{004}/n,\ E(e'^2_1) = \delta^*_{040}/n',\ E(e'^2_2) = \delta^*_{004}/n'$$

$$E(e_0e_1) = \delta^*_{220}/n, E(e_0e_2) = \delta^*_{202}/n, E(e_1e_2) = \delta^*_{022}/n, E(e'_0e'_1) = \delta^*_{220}/n', E(e'_0e'_2) = \delta^*_{202}/n',$$
$$E(e'_1e'_2) = \delta^*_{022}/n'$$

Where

$$\delta^*_{pqr} = (\delta_{pqr} - 1),\ \delta_{pqr} = (\mu_{pqr} / \mu_{200}^{p/2}\mu_{020}^{q/2}\mu_{002}^{r/2}),\ \mu_{pqr} = \sum_{i=1}^{N}(y_i - \bar{Y})^p (x_i - \bar{X})^q (z_i - \bar{Z})/N$$

Now expressing t_0, t_1, t_2, t_{0d}, t_{1d} and t_{2d} in terms of e_i's we have

$$t_0 = S_y^2\left[\begin{array}{l}1 + e_0 + \lambda_1\alpha(m_1S_x^2e_1 + m_2S_z^2e_2) + \lambda_1\alpha(m_1S_x^2e_0e_1 + m_2S_z^2e_0e_2) \\ \qquad + \dfrac{\alpha(\alpha-1)}{2}\lambda_1^2(m_1S_x^2e_1 + m_2S_z^2e_2)^2\end{array}\right] \quad (21)$$

$$t_1 = S_y^2\left[1 + e_0 - \frac{\beta_1e_1}{2} - \frac{\beta_2e_2}{2} + \left\{\frac{\beta_1}{4} + \frac{\beta_1^2}{8}\right\}e_1^2 + \left\{\frac{\beta_2}{4} + \frac{\beta_2^2}{8}\right\}e_2^2 - \frac{\beta_1e_0e_1}{2} - \frac{\beta_2e_0e_2}{2} + \beta_1\beta_2e_1e_2\right] \quad (22)$$

$$t_2 = S_y^2\left[\left\{1 + e_0 - \frac{\beta_1e_1}{2} - \frac{\beta_2e_2}{2} + \left\{\frac{\beta_1}{4} + \frac{\beta_1^2}{8}\right\}e_1^2 + \left\{\frac{\beta_2}{4} + \frac{\beta_2^2}{8}\right\}e_2^2 - \frac{\beta_1e_0e_1}{2} - \frac{\beta_2e_0e_2}{2} + \beta_1\beta_2e_1e_2\right\} - \frac{b_1e_1}{R_1} - \frac{b_2e_2}{R_2}\right] \quad (23)$$

$$\begin{aligned}t_{0d} = S_y^2\Big[1 + e_0 &+ \lambda_1\alpha m_1S_x^2e'_1 - \lambda_1\alpha(m_1S_x^2e_1 + m_2S_z^2e'_2) - \lambda_1^2\alpha^2m_1S_x^2(m_1S_x^2e_1e'_1 + m_2S_z^2e'_1e'_2) \\ &+ \frac{\alpha(\alpha+1)}{2}\lambda_1^2(m_1S_x^2e_1 + m_2S_z^2e'_2)^2 + \frac{\alpha(\alpha-1)}{2}\lambda_1^2m_1^2S_x^4e'^2_1 + \lambda_1\alpha m_1S_x^2e_0e'_1 \\ &- \lambda_1\alpha(m_1S_x^2e_0e_1 + m_2S_z^2e_0e'_2)\Big]\end{aligned} \quad (24)$$

$$t_{1d} = S_y^2\left[1 + e_0 + \frac{\beta_1(e'_1 - e_1)}{2} - \frac{\beta_2e'_2}{2} + \frac{\beta_2e'^2_2}{4} + \frac{\beta_2^2e'^2_2}{8} - \frac{\beta_1\beta_2(e'_1e'_2 - e_1e'_2)}{4} - \frac{\beta_1(e'^2_1 - e_1^2)}{4}\right. \quad (25)$$

$$+\frac{\beta_1^2(e'_1-e_1)^2}{8}-\frac{\beta_2 e_0 e'_2}{2}+\frac{\beta_1(e_0e'_1-e_0e_1)}{2}\Bigg]$$

$$t_{2d}=S_y^2\Bigg[1+e_0+\frac{\beta_1(e'_1-e_1)}{2}-\frac{\beta_2 e'_2}{2}+\frac{\beta_2 e'^2_2}{4}+\frac{\beta_2^2 e'^2_2}{8}-\frac{\beta_1\beta_2(e'_1e'_2-e_1e'_2)}{4}-\frac{\beta_1(e'^2_1-e_1^2)}{4}$$

$$+\frac{\beta_1^2(e'_1-e_1)^2}{8}-\frac{\beta_2 e_0 e'_2}{2}+\frac{\beta_1(e_0e'_1-e_0e_1)}{2}+\frac{b_1(e'_1-e_1)}{R_1}-\frac{b_2 e'_2}{R_2}\Bigg] \tag{26}$$

Where, $\lambda_1=\dfrac{1}{m_1S_x^2+m_2S_z^2}$, $R_1=\dfrac{S_y^2}{S_x^2}$, $R_2=\dfrac{S_y^2}{S_z^2}$.

Subtracting S_y^2 and then taking expectation of both sides of (21), (22), (23), (24), (25) and (26) we get the biases of t_0, t_1, t_2, t_{0d}, t_{1d} and t_{2d} to the first degree of approximation, respectively as

$$B(t_0)=\frac{S_y^2}{n}\Bigg[\lambda_1\alpha(m_1S_x^2\delta^*_{220}+m_2S_z^2\delta^*_{202})+\frac{\alpha(\alpha-1)}{2}\lambda_1^2(m_1^2S_x^4\delta^*_{040}+m_2^2S_z^4\delta^*_{004}+2m_1m_2S_x^2S_z^2\delta^*_{022})\Bigg] \tag{27}$$

$$B(t_1)=S_y^2\Bigg[\left\{\frac{\beta_1}{4}+\frac{\beta_1^2}{8}\right\}\delta^*_{040}+\left\{\frac{\beta_2}{4}+\frac{\beta_2^2}{8}\right\}\delta^*_{004}-\frac{\beta_1\delta^*_{220}}{2}-\frac{\beta_2\delta^*_{202}}{2}+\beta_1\beta_2\delta^*_{022}\Bigg] \tag{28}$$

$$B(t_2)=S_y^2\Bigg[\left\{\frac{\beta_1}{4}+\frac{\beta_1^2}{8}\right\}\delta^*_{040}+\left\{\frac{\beta_2}{4}+\frac{\beta_2^2}{8}\right\}\delta^*_{004}-\frac{\beta_1\delta^*_{220}}{2}-\frac{\beta_2\delta^*_{202}}{2}+\beta_1\beta_2\delta^*_{022}\Bigg] \tag{29}$$

$$B(t_{0d})=S_y^2\Bigg[\frac{\alpha(\alpha+1)}{2}\lambda_1^2\left(\frac{m_1^2S_x^4\delta^*_{040}}{n}+\frac{m_2^2S_z^4\delta^*_{004}}{n'}+\frac{2m_1m_2S_x^2S_z^2\delta^*_{022}}{n'}\right)-\frac{\lambda_1^2\alpha^2m_1S_x^2}{n'}$$

$$(m_1S_x^2\delta^*_{040}+m_2S_z^2\delta^*_{004})+\frac{\alpha(\alpha-1)}{2}\lambda_1^2m_1^2S_x^4\frac{\delta^*_{040}}{n'}+\lambda_1\alpha m_1S_x^2\frac{\delta^*_{220}}{n'} \tag{30}$$

$$-\lambda_1\alpha(m_1S_x^2\frac{\delta^*_{220}}{n}+m_2S_z^2\frac{\delta^*_{202}}{n'})\Bigg]$$

$$B(t_{1d})=S_y^2\Bigg[\left(\frac{\beta_1}{4}+\frac{\beta_1^2}{8}\right)\gamma\,\delta^*_{040}+\left(\frac{\beta_2}{4}+\frac{\beta_2^2}{8}\right)\frac{\delta^*_{004}}{n'}-\frac{\beta_2\,\delta^*_{202}}{2n'}-\frac{\beta_1\gamma\delta^*_{220}}{2}\Bigg] \tag{31}$$

$$B(t_{2d})=S_y^2\Bigg[\left(\frac{\beta_1}{4}+\frac{\beta_1^2}{8}\right)\gamma\,\delta^*_{040}+\left(\frac{\beta_2}{4}+\frac{\beta_2^2}{8}\right)\frac{\delta^*_{004}}{n'}-\frac{\beta_2\,\delta^*_{202}}{2n'}-\frac{\beta_1\gamma\delta^*_{220}}{2}\Bigg] \tag{32}$$

Subtracting S_y^2 squaring and then taking expectation of both sides of (21), (22), (23), (24), (25) and (26), we get the asymptotic variances of t_0, t_1, t_2, t_{0d}, t_{1d} and t_{2d} to the first degree of approximation, respectively as

$$AV(t_0)=\frac{1}{n}\left[A_0+m_1^2B_0+m_2^2C_0-2m_1D_0-2m_2E_0+2m_1m_2F_0\right] \quad (33)$$

Partially differentiating equation (33) with respect to m_1 under the condition $m_1+m_2=1$, we get the optimum values of m_1 and m_2 as

$$m^*_{1(opt)}=\frac{\alpha\theta S_z^2\delta^*_{004}+S_y^2S_x^2\delta^*_{220}-\alpha\theta S_x^2S_z^2\delta^*_{022}-S_y^2S_z^2\delta^*_{202}}{S_x^4\delta^*_{040}+S_z^4\delta^*_{004}-2S_x^2S_z^2\delta^*_{022}} \text{ And } m^*_{2(opt)}=1-m_{1(opt)}$$

Where

$$A_0=S_y^4\delta^*_{400}, B_0=\theta^2\alpha^2S_x^4\delta^*_{040}, C_0=\theta^2\alpha^2S_z^4\delta^*_{004}, D_0=\theta\alpha S_y^2S_x^2\delta^*_{220}, E_0=\theta\alpha S_y^2S_z^2\delta^*_{202}$$

$$F_0=\theta^2\alpha^2S_x^2S_z^2\delta^*_{022}.$$

$$AV(t_1)=\frac{S_y^4}{n}\left[A_1+\beta_1^2B_1+\beta_2^2C_1-\beta_1D_1-\beta_2E_1+\beta_1\beta_2F_1\right] \quad (34)$$

Partially differentiating equation (34) with respect to β_1 and β_2, we get the optimum values of β_1 and β_2 as

$$\beta_{1(opt)}=\frac{E_1F_1-2C_1D_1}{F_1^2-4B_1C_1} \text{ And } \beta_{2(opt)}=\frac{D_1F_1-2B_1E_1}{F_1^2-4B_1C_1}$$

Where

$$A_1=\delta^*_{400}, B_1=\frac{\delta^*_{040}}{4}, C_1=\frac{\delta^*_{004}}{4}, D_1=\delta^*_{220}, E_1=\delta^*_{202}, F_1=\frac{\delta^*_{022}}{2}$$

$$AV(t_2)=A_2+\eta_1^2B_2+\eta_2^2C_2+\eta_1\eta_2D_2-\eta_1E_2-\eta_2F_2+G_2 \quad (35)$$

Partially differentiating equation (35) with respect to η_1 and η_2, we get the optimum values of η_1 and η_2 as

$$\eta_{1(opt)}=\frac{2C_2E_2-D_2F_2}{4B_2C_2-D_2^2} \text{ And } \eta_{2(opt)}=\frac{2B_2F_2-D_2E_2}{4B_2C_2-D_2^2}$$

Where

$$A_2=\frac{S_y^4\delta^*_{400}}{n}, B_2=\frac{S_y^4\delta^*_{040}}{4n}, C_2=\frac{S_y^4\delta^*_{004}}{4n}, D_2=\frac{S_y^4\delta^*_{022}}{2n}$$

$$E_2=\frac{S_y^2}{n}\left[S_y^2\delta^*_{220}-b_1S_x^2\delta^*_{040}-b_2S_z^2\delta^*_{022}\right], F_2=\frac{S_y^2}{n}\left[S_y^2\delta^*_{202}-b_1S_x^2\delta^*_{022}-b_2S_z^2\delta^*_{004}\right],$$

$$G_2=\frac{1}{n}\left[b_1^2S_x^4\delta^*_{040}+b_2^2S_z^4\delta^*_{004}-2b_1S_x^2S_y^2\delta^*_{220}-2b_2S_z^2S_y^2\delta^*_{202}+2b_1b_2S_x^2S_z^2\delta^*_{022}\right].$$

$$AV(t_{0d})=S_y^4\left[\frac{\delta^*_{400}}{n}+m_{1d}^2A'_0+m_{2d}^2B'_0-2m_{1d}C'_0-2m_{2d}D'_0\right] \quad (36)$$

Partially differentiating equation (36) with respect to m_{1d} under the condition $m_{1d}+m_{2d}=1$, we get the optimum values of m_{1d} and m_{2d} as

$$m_{1d\,(opt)} = \frac{B'_0 + C'_0 - D'_0}{A'_0 + B'_0} \text{ And } m_{2d\,(opt)} = 1 - m_{1(opt)}$$

Where

$$A'_0 = \alpha^2\lambda_1^2\gamma S_x^4\delta^*_{040},\ B'_0 = \alpha^2\lambda_1^2 S_z^4\delta^*_{004}/n',\quad C'_0 = \alpha\lambda_1 S_x^2\gamma\delta^*_{220},\ D'_0 = \alpha\lambda_1 S_z^2\gamma\delta^*_{202}/n'.$$

$$AV(t_{1d}) = S_y^4\left[A'_1 + \beta_{1d}^2 B'_1 + \beta_{2d}^2 C'_1 - \beta_{1d} D'_1 - \beta_{2d} E'_1\right] \tag{37}$$

Partially differentiating equation (37) with respect to β_{1d} and β_{2d}, we get the optimum values of β_{1d} and β_{2d} as

$\beta_{1d\,(opt)} = D'_1/2B'_1$ And $\beta_{2d\,(opt)} = E'_1/2C'_1$

Using these optimum values, Min $MSE(t'_{1d})$ is, respectively, given by

$$AV(t_{1d}) = S_y^4\left[A'_1 - \frac{D'^2_1}{B'_1} - \frac{E'^2_1}{C'_1}\right] \tag{38}$$

Where

$$A'_1 = \frac{\delta^*_{400}}{n}, B'_1 = \frac{\gamma\delta^*_{040}}{4}, C'_1 = \frac{\delta^*_{004}}{4n'}, D'_1 = \gamma\delta^*_{220}, E'_1 = \delta^*_{202}/n'$$

$$AV(t_{2d}) = S_y^4\left[A'_2 + \frac{\eta_{1d}^2 B'_2}{4} + \frac{\eta_{2d}^2 C'_2}{4} + \eta_{1d} D'_2 + \eta_{2d} E'_2 + F'_2\right] \tag{39}$$

Partially differentiating equation (39) with respect to η_{1d} and η_{2d}, we get the optimum values of η_{1d} and η_{2d}, as

$\eta_{1d\,(opt)} = -2D'_2/B'_2$ And $\eta_{2d\,(opt)} = -2E'_2/C'_2$

Using these optimum values, Min $MSE(t'_{2d})$ is, respectively, given by

$$AV(t_{2d}) = S_y^4\left[A'_2 - \frac{D'_2}{B'_2} - \frac{E'_2}{C'_2} + F'_2\right] \tag{40}$$

3 Theoretical Efficiency Comparisons

Using equation (9), (10),(11), (33), (34) and (35), we have:

$V(\hat{Y}_{CR}) \geq AV(t_0)$, if

$$S_y^4\left[\delta^*_{040} + \delta^*_{004} - 2\delta^*_{220} - 2\delta^*_{202} + 2\delta^*_{022}\right] - \left[m_1^2 B_0 + m_2^2 C_0 - 2m_1 D_0 - 2m_2 E_0 + 2m_1 m_2 F_0\right] \geq 0 \tag{41}$$

$V(\hat{Y}_{lr,1}) \geq AV(t_1)$, if

$$\left[\beta_1^2 B_1 + \beta_2^2 C_1 - \beta_1 D_1 - \beta_2 E_1 + \beta_1\beta_2 F_1\right] - \left[\frac{\beta_1^2}{R_1^2}\delta^*_{040} - 2\frac{\beta_1}{R_1}\delta^*_{220}\right] \leq 0 \tag{42}$$

$V(\hat{Y}_{lr,1}) \geq AV(t_2)$, if

$$\left[\eta_1^2 B_2 + \eta_2^2 C_2 + \eta_1\eta_2 D_2 - \eta_1 E_2 - \eta_2 F_2 + G_2\right] - \left[\frac{\beta_1^2}{R_1^2}\delta^*_{040} - 2\frac{\beta_1}{R_1}\delta^*_{220}\right] \le 0 \tag{43}$$

$V(\hat{Y}_{lr,2}) \ge AV(t_2)$

$$\left[\eta_1^2 B_2 + \eta_2^2 C_2 + \eta_1\eta_2 D_2 - \eta_1 E_2 - \eta_2 F_2 + G_2\right] - \frac{S_y^4}{n}\left[\frac{\beta_1^2}{R_1^2}\delta^*_{040} + \frac{\beta_2^2}{R_2^2}\delta^*_{004} - 2\frac{\beta_1}{R_1}\delta^*_{220} - 2\frac{\beta_2}{R_2}\delta^*_{202} + 2\frac{\beta_1}{R_1}\frac{\beta_2}{R_2}\delta^*_{022}\right] \le 0 \tag{44}$$

Using equation (12), (13),(14),(36),(38) and (40), we have

$V(\hat{Y}'_{CR}) \ge AV(t_{0d})$, if

$$S_y^4\left[m_{1d}^2 A'_0 + m_{2d}^2 B'_0 - 2m_{1d}C'_0 - 2m_{2d}D'_0\right] - S_y^4\left[\gamma\delta^*_{040} + \frac{\delta^*_{004}}{n'} - 2\gamma\delta^*_{220} - \frac{2\delta^*_{202}}{n'}\right] \le 0 \tag{45}$$

$V(\hat{Y}'_{lr,1}) \ge AV(t_{1d})$, if

$$S_y^4\left[\frac{\delta^*_{400}}{n} + \gamma\frac{\beta_1^2}{R_1^2}\delta^*_{040} - 2\frac{\beta_1}{R_1}\gamma\delta^*_{220}\right] - S_y^4\left[A'_1 - \frac{D'^2_1}{B'_1} - \frac{E'^2_1}{C'_1}\right] \ge 0 \tag{46}$$

$V(\hat{Y}'_{lr,1}) \ge AV(t_{2d})$, if

$$S_y^4\left[F'_2 - \frac{D'_2}{B'_2} - \frac{E'_2}{C'_2}\right] - S_y^4\left[\gamma\frac{\beta_1^2}{R_1^2}\delta^*_{040} - 2\frac{\beta_1}{R_1}\gamma\delta^*_{220}\right] \le 0 \tag{47}$$

$V(\hat{Y}'_{lr,2}) \ge AV(t_{2d})$, if

$$S_y^4\left[F'_2 - \frac{D'_2}{B'_2} - \frac{E'_2}{C'_2}\right] - S_y^4\left[\gamma\frac{\beta_1^2}{R_1^2}\delta^*_{040} + \frac{\beta_2^2}{R_2^2 n'}\delta^*_{004} - 2\gamma\frac{\beta_1}{R_1}\delta^*_{220} - 2\frac{\beta_2}{R_2 n'}\delta^*_{202}\right] \le 0 \tag{48}$$

4 Empirical studies

Population.1: To have a rough idea about the gain in efficiency of the estimator, defined under the situation when a prior information of population variance on two auxiliary variable is available .We have taken the data of Murthy (1967), page no.-399, which contain the data of *34* villages and we have to estimate the population variance of study variable y for a positively related *x* and z respectively given by,

y- Area under wheat in 1964.

x- Area under wheat in 1963.

z- Cultivated area in 1961.

There are some known population parameters, given as

$$\delta^*_{400} = 2.726,\ \delta^*_{040} = 1.912,\ \delta^*_{004} = 1.808,\ \delta^*_{220} = 2.105,\ \delta^*_{202} = 1.979,\ \delta^*_{022} = 1.738$$

$$S_y^2 = 22564.56,\ S_x^2 = 197095.3,\ S_z^2 = 2652.05,\ S_{yx} = 60304.01,\ S_{yz} = 22158.05$$

$n = 7, n' = 15$, N=34.

Population.2: The data for the empirical study are taken from Ahmed (1995). The population consists of 340 villages.

y- Number of literate persons.

x- Number of household.

z- Total population in the village.

To estimate the variance of y, we have used x and z as the prior information. From the data set, we have

$$N = 340, n' = 120, n = 50, S_y^2 = 71379.47, S_x^2 = 11838.85, S_z^2 = 691820.23$$

$$\nabla^*_{400} = 9.90334289, \nabla^*_{040} = 7.05448224, \nabla^*_{004} = 8.2552346, \nabla^*_{220} = 6.31398563\ \nabla^*_{202} = 8.12904924,$$
$$\nabla^*_{022} = 6.13646859.$$

The percentage relative efficiency (*PRE*) of estimator is defined as:

$$\text{PRE}(*) = \frac{\text{VAR}(\hat{S}_y^2)}{\text{MSE}(*)} \times 100$$

Table.1. PRE of Estimators with respect to s_y^2

Estimators	Percent Relative Efficiency	
	Population-1	Population-2
$\hat{S}_y^2$	100.00	100. 00
$\hat{Y}_{CR}$	155.42	115.156
$\hat{Y}_{lr,1}$	667.287	232.90
$\hat{Y}_{lr,2}$	112.127	140.89
t_0	697.710	522.71
t_1	699.0769	529.86
t_2	699.0769	529.86

Table.2. PRE of Estimators under two phase sampling with respect to s_y^2

Estimators	Percent Relative Efficiency	
	Population-1	Population-2
$\hat{S}_y^2$	100.00	100. 00
$\hat{Y}'_{CR}$	556.70	298.50
$\hat{Y}'_{lr,1}$	198.379	149.89
$\hat{Y}'_{lr,2}$	578.0769	302.71
t_{0d}	253.294	152.06
t_{1d}	578.0769	302.71
t_{2d}	578.077	578.077

5 Conclusion

The present studies lead to an overall conclusion that the estimator t_1, t_2, t_{1d} and t_{2d} are preferable to chain-ratio $(\hat{Y}_{CR}, \hat{Y}'_{CR})$ and regression estimators $(\hat{Y}_{lr,1}\hat{Y}'_{lr,1}\hat{Y}_{lr,2}\hat{Y}'_{lr,2})$. In view of these findings, if computational difficulty is not a matter of great concern, the variance estimators (t_1, t_2, t_{1d}, t_{2d}) may be considered as suitable estimator over others.

References

[1] L. K. Grover, A correction note on improvement in variance estimation using auxiliary information. Comm. *In. Stat. Theo. Meth.***39**, 753–764, (2010).

[2] C. T.Isaki,Variance estimation using auxiliary information. *J. Amer. Statist. Assoc.***78**, 117–123, (1983).

[3] H .S.Jhajj, M. K.Sharma, and L. K.Grover, An efficient class of chain estimators of population variance under sub-sampling scheme. *J. Japan Stat. Soc.*, **35(2)**, 273-286, (2005).

[4] C.Kadilar, H.Cingi, Improvement in variance estimation using auxiliary information. *Hacettepe J. Math. Statist.* **35(1)**, 111–115, (2006).

[5] C.Kadilar, H.Cingi, Improvement in variance estimation in simple random, (2007).

[6] C.Kadilar, H.Cingi, Sampling. *Commun. Statist. Theor. Meth.***36**, 2075–2081. A new estimator using two auxiliary variables. *Applied mathematics and computation***162** 901-908, (2005).

[7] J. Lu, Z. Yan, C. Ding, Z. Hong, The chain ratio estimator using two auxiliary information. International Conference on Computer and Communication Technologies in Agriculture Engineering 586-589, (2010).

[8] S.Malik, and R.Singh, An improved estimator using two auxiliary attributes. *Applied mathematics and computation***219** 10983-10986, (2013).

[9] J.Shabbirand S. Gupta: On improvement in variance estimation using auxiliary information. *Commun. Statist.*

Theor. Meth. **36(12)**, 2177–2185, (2007).

[10] H.P.Singh, and R. Singh: Improved ratio-type estimator for variance using auxiliary information. *J. Indian Soc. Agric. Stat.* **54(3)**, 276–287, (2001).

[11] H. P.Singh, and R.Singh, Estimation of variance through regression approach in two phase sampling. *Aligarh Journal of Statistics*, **23**, 13-30, (2003).

[12] H. P.Singh, and R.S. Solanki: A new procedure for variance estimation in simple random sampling using auxiliary information. *Stat. Paps.*DOI 10.1007/s00362-012-0445-2, (2012).

[13] R. Singh, P. Chauhan, N.Sawan: On linear combination of Ratio-producttype exponential estimator for estimating finite population mean. *Statistics in Transition* **9(1)**, 115, (2008).

[14] R.Singh,P. Chauhan,N.Sawanand F.Smarandache: Almost unbiased ratio and product type estimator of finite population variance using the knowledge of kurtosis of an auxiliary variable in sample surveys. *Octogon Mathematical Journal*, Vol. **16**, No. 1, 123-130, (2008).

[15] R. Singh,M.Kumar, F.Smarandache:Almost Unbiased Estimator for Estimating Population Mean Using Known Value of Some Population Parameter(s);*Pak.j.stat.oper.res.***76 (2)**, 63-76,(2008).

[16] R. Singh,P. Chauhan,N.Sawan, and F.Smarandache: Improved Exponential Estimator for Population Variance Using Two Auxiliary Variables. *Italian Jour. Of Pure and Appld. Math.*, **28**, 103-110,(2011).

[17] V. K. Singh, R. Singh and S.Florentin: Some improved estimators for population variance using two auxiliary variables in double sampling. On improvement in estimating population parameter(s) using auxiliary information. Educational Publishing & Journal of Matter Regularity (Beijing), (2013).

[18] R. Singh, V.K. Singh,M.Khoshnevisan:A Difference Type Estimator for Estimating Population Variance with Possible Applications to Random Stock and Dividend Growth.*American Journal of Applied Mathematics*; **2(3)**, 92-95,(2014).