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Certain Properties of Modified Laguerre Polynomials Via Lie Algebra

K. B. Kachhia[1,*], *J. C. Prajapati*[2] *and S. D. Purohit*[3]

[1] Department of Mathematical Sciences, Faculty of Applied Sciences, Charotar University of Science and Technology (Charusat), Changa, Anand-388421, Gujarat, India.
[2] Department of Mathematics, Faculty of Technology and Engineering, Marwadi Education Foundation Group of Institutions (MEFGI), Rajkot- 360003, Gujarat, India.
[3] Department of Mathematics, University College of Engineering, Rajasthan Technical University, Kota- 324010, Rajasthan, India.

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Abstract: The aim of present paper is to discuss some operators defined on a Lie algebra for the purpose of deriving some properties of modified Laguerre polynomials.

Keywords: Lie algebra, Modified Laguerre polynomials and Differential equation.

1 Introduction

Many important classical differential equations has connection with Lie theory. The interplay between differential equations, special functions and Lie theory is particularly play important role in mathematical physics. Radulescu [1] has discussed some properties of Hermite and Laguerre polynomials [2] using some operators defined on a Lie algebra. Further Mandel [3] obtained some properties of simple Bessel polynomials considered by Krall and Frink [4]. Pathan and Khan [5] discussed some properties of two variable Laguerre polynomials studied by Dattoli and Torre [6,7].

The Modified Laguerre polynomials (McBride [8]), defined as

$$f_n^{\beta}(x) = \frac{(\beta)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n;\\ 1-\beta-n;\end{matrix}\,x\right] = (-1)^n\,L_n^{-\beta-n}(x) \quad (1)$$

Then $f_n^{\beta}(x)$ satisfies the two independent differential recurrence relations

$$\frac{d}{dx}(f_n^{\beta}(x)) = f_{n-1}^{\beta}(x) \quad (2)$$

and

$$x\frac{d}{dx}(f_n^{\beta}(x)) = (x+n+\beta)\,f_n^{\beta}(x) - (n+1)\,f_{n+1}^{\beta}(x) \quad (3)$$

Also (5) and (6) determine the ordinary differential equation

$$x\frac{d^2}{dx^2}(f_n^{\beta}(x)) + (1-\beta-n-x)\frac{d}{dx}f_n^{\beta}(x) + n f_n^{\beta}(x) = 0 \quad (4)$$

2 Main Result

Let $End\ V$ be the Lie algebra of endomorphisms of a vector space V, endowed with the Lie bracket $[\cdot,\cdot]$ defined by $[A,B] = AB - BA$, for every $A,B \in End\ V$. The main result of the paper is as follows.

Theorem 1.*Let $A,B \in End\ V$ be such that $[A,B]y_n = -y_n$, where the sequence $(y_n)_n \subset V$ is defined as follows: $Ay_0 = 0$ and $By_n = -(n+1)y_{n+1}$, for every $n \geq 1$. Then $Ay_n = y_{n-1}$ and y_n is an eigenvector of eigenvalue $-n$ for BA, for every $n \geq 1$.*

Proof: First, we shall prove

$$Ay_n = y_{n-1}, \text{ for every } n \geq 1.$$

For $n = 1$, this equality is evident, because

$$[A,B]y_0 = -y_0,$$

* Corresponding author e-mail: krunalmaths@hotmail.com

$$A(By_0) - B(Ay_0) = -y_0,$$

also $Ay_0 = 0$ and $By_0 = -y_1$ and therefore,

$$Ay_1 = y_0$$

Now, suppose that $Ay_n = y_{n-1}$, then we have

$$[A,B]y_n = -y_n,$$

$$\text{i. e. } A(By_n) - B(Ay_n) = -y_n,$$

$$\text{i. e. } A(-(n+1)y_{n+1}) - B(y_{n-1}) = -y_n,$$

$$\text{i. e. } -(n+1)A(y_{n+1}) + ny_n = -y_n,$$

$$\text{i. e. } A(y_{n+1}) = \frac{-(n+1)}{-(n+1)}y_n,$$

$$\text{i. e. } A(y_{n+1}) = y_n.$$

Therefore by mathematical induction $Ay_n = y_{n-1}$, for every $n \geq 1$. It immidiately follows that $BAy_n = -ny_n$. Hence, y_n is an eigenvector of eigenvalue $-n$ for BA, for every $n \geq 1$.

3 A Concrete Application

Let $V = C^\infty(R \times R)$, we define the operators $A, B \in End\ V$ as

$$Au(x,y) = y^{-1}\frac{\partial u}{\partial x} \tag{5}$$

$$Bu(x,y) = xy\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} - (x+\beta)yu \tag{6}$$

for $(x,y) \in R \times R$.

We claim that the operators (5) and (6) obey the commutation relation $[A,B]y_n = -y_n$

Indeed,

$$[A,B]u(x,y) = A(Bu(x,y)) - B(Au(x,y)) \tag{7}$$

which gives

$$[A,B]u(x,y) = \left(y^{-1}\frac{\partial}{\partial x}\right)\left(xy\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} - (x+\beta)yu\right) - \left(xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (x+\beta)y\right)\left(y^{-1}\frac{\partial u}{\partial x}\right) = -u, \tag{8}$$

i.e.

$$[A,B]u(x,y) = -u(x,y).$$

Now, if $u(x,y)$ assumes the form $y_n(x,y) = f_n(x)y^n \in C^\infty(R \times R)$, then we have

$$[A,B](f_n(x)y^n) = -f_n(x)y^n,$$

and our claim is justified.

Now, the relation $By_n = -(n+1)y_{n+1}$ gives

$$\left(xy\tfrac{\partial}{\partial x} - y^2\tfrac{\partial}{\partial y} - (x+\beta)y\right)(f_n(x)y^n) = -(n+1)f_{n+1}(x)y^{n+1}$$

i.e.

$$x\frac{\partial}{\partial x}(f_n(x)) = (x+n+\beta)f_n(x) - (n+1)f_{n+1}(x) \tag{9}$$

Again, the relation $Ay_n = y_{n-1}$ gives

$$\left(y^{-1}\frac{\partial}{\partial x}\right)(f_n(x)y^n) = f_{n-1}(x)y^{n-1}$$

i.e.

$$\frac{\partial}{\partial x}(f_n(x)) = f_{n-1}(x) \tag{10}$$

Finally, the relation $BAy_n = -ny_n$ gives

$$\left(xy\tfrac{\partial}{\partial x} - y^2\tfrac{\partial}{\partial y} - (x+\beta)y\right)\left(y^{-1}\tfrac{\partial}{\partial x}\right)(f_n(x)y^n) = -nf_n(x)y^n$$

i.e.

$$x\frac{\partial^2}{\partial x^2}(f_n(x)) + (1-\beta-n-x)\frac{\partial}{\partial x}(f_n(x)) + nf_n(x) = 0 \tag{11}$$

Now, we observe that modified Laguerre polynomials $f_n^\beta(x)$ is a solution of the differential equation (11). Further we note that the relations (9) and (10) are differential recurrence relations satisfied by modified Laguerre polynomials $f_n^\beta(x)$.

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Krunal B. Kachhia, Assistant Professor, Department of Mathematical Sciences, Faculty of Applied Sciences, Charotar University of Science and Technology (Charusat), Changa, Anand-388421, Gujarat, India. He completed B.Sc. (Mathematics) in 2008, M.Sc. (Mathematics) in 2010 from The Sardar Patel University, V. V. Nagar, Gujarat, M.Phil. (Mathematics) in 2012 from The Sardar Patel University, V.V.Nagar, Gujarat and currently pursuing Ph.D. from The Charotar University of Science and Technology (Charusat), Changa, Gujarat. He has more than 3 years of academic experience. His research field is Special Functions and Fractional Calculus. He has published 5 research papers in the national and international journal of repute.

Jyotindra C. Prajapati, (Ph.D.), Professor and Head, Department of Mathematics, Faculty of Technology and Engineering, Marwadi Education Foundation's Group of Institutions (MEFGI), Rajkot-360003, Gujarat, India. He completed B.Sc. (Statistics) in 1992, B.Sc. (Mathematics) in 1993, M.Sc. (Mathematics) in 1995 from The M. S. University of Baroda, Vadodara, Gujarat. M.Phil. (Mathematics) in 1997 from The S. P. University, V.V.Nagar, Gujarat and Ph.D. from The S. V. National Institute of Technology, Surat, Gujarat. He has more than 15 years of academic experience. His research field is Special Functions, Integral Transforms and Fractional Calculus. He has published more than 50 research papers in the national and international journal of repute. He associated with more than 50 National/International journals' Editorial board. He presented several papers, delivered several expert lectures and conducted various roll in academic events (judge, session chair, coordinator) in National/International conferences, Seminars, STTP and SDP etc. He is life member of 16 professional body including IMS, RMS, CMS, ISCA, SSFA, AMTI, ISHM etc. He is guiding 4 research scholars for Ph.D. degree. He published 5 books on Engineering Mathematics with Pearson Education, one research level book entitled "The Mittag-Leffler function: Generalization and Applications", LAP-Lambert Academic Publishing, Germany, 2010, ISBN: 978-3-8383-9976-8 and also published one book chapter entitled "Pseudo Regularity in Commutative Banach Algebras", Advances in Mathematics, Research, Volume 18, Chapter 5, pp. 187-222, Editor: Albert R. Baswell, NOVA SCIENCE PUBLISHERS, INC. USA 2013, ISBN: 978-1-62417-930-3.

Sunil Dutt Purohit, (Ph.D.), Associate Professor of Mathematics, Department of Mathematics, Rajasthan Technology University, Kota-324010, Rajasthan India. He is working as Associate Professor of Mathematics, Department of Mathematics, Rajasthan Technology University, Kota, India. His research interest includes Special functions, Fractional Calculus, Integral transforms, Basic Hypergeometric Series, Geometric Function Theory and Mathematical Physics. He has published more than 70 research papers in international esteemed journals. He is reviewer for Mathematical Reviews, USA (American Mathematical Society) and Zentralblatt MATH, Berlin since last seven years. He is member, Editorial Board for number of international mathematical and interdisciplinary journals.