Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.12785/jant/030104

On Some I-Convergent Sequence Spaces Over N-Normed Spaces

Kuldip Raj[1] *and Ayhan Esi*[2,*]

[1] School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India
[2] Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey

Received: 5 Jul. 2014, Revised: 20 Oct. 2014, Accepted: 23 Oct. 2014
Published online: 1 Jan. 2015

Abstract: In the present paper we introduce some strongly almost summable difference sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$ in n-normed spaces and examine some properties of the resulting sequence spaces.

Keywords: paranorm space, I-convergence, difference sequence spaces, Orlicz function, Musielak-Orlicz function, n-normed spaces

1 Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler[5] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[15]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7,8]) and Gunawan and Mashadi [9] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot,\cdots,\cdot||$ on X^n satisfying the following four conditions:

1. $||x_1,x_2,\cdots,x_n|| = 0$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent in X;
2. $||x_1,x_2,\cdots,x_n||$ is invariant under permutation;
3. $||\alpha x_1,x_2,\cdots,x_n|| = |\alpha|\ ||x_1,x_2,\cdots,x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x+x',x_2,\cdots,x_n|| \leq ||x,x_2,\cdots,x_n|| + ||x',x_2,\cdots,x_n||$

is called a n-norm on X, and the pair $(X,||\cdot,\cdots,\cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1,x_2,\cdots,x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1,x_2,\cdots,x_n$ which may be given explicitly by the formula

$$||x_1,x_2,\cdots,x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1},x_{i2},\cdots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots,n$. Let $(X,||\cdot,\cdots,\cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1,a_2,\cdots,a_n\}$ be linearly independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1,x_2,\cdots,x_{n-1}||_\infty = \max\{||x_1,x_2,\cdots,x_{n-1},a_i|| : i = 1,2,\cdots,n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\cdots,a_n\}$.

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notion of difference sequence spaces was introduced by Kızmaz [10], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r be

* Corresponding author e-mail: aesi23@hotmail.com

non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}.$$

Taking $r = 1$, we get the spaces which were introduced and studied by Kızmaz [10].

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

Also, it was shown in [13] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \le K(L)M(u)$ for all values of $u \ge 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \ge 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathscr{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([14,18]). A sequence $\mathscr{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \ge 0\},\ k = 1, 2, \cdots$$

is called the complementary function of the Musielak-Orlicz function $\mathscr{M}$. For a given Musielak-Orlicz function $\mathscr{M}$, the Musielak-Orlicz sequence space $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows

$$t_{\mathscr{M}} = \Big\{x \in w : I_{\mathscr{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathscr{M}} = \Big\{x \in w : I_{\mathscr{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathscr{M}}.$$

We consider $t_{\mathscr{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathscr{M}}\Big(\frac{x}{k}\Big) \le 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathscr{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \le KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \le a$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \ge 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [24, Theorem 10.4.2, pp. 183]). For more details about sequence spaces (see [17,19,20,21,22]) and reference therein.

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ and for all $k \in \mathbb{N}$.

The notion of ideal convergence was introduced first by P. Kostyrko [11] as a generalization of statistical convergence which was further studied in topological spaces (see [2]). More applications of ideals can be seen in ([2,3]).

A linear functional $\mathscr{L}$ on ℓ_∞ is said to be a Banach limit see [1] if it has the properties :

1. $\mathscr{L}(x) \ge 0$ if $x \ge 0 (i.e. x_n \ge 0$ for all $n)$,
2. $\mathscr{L}(e) = 1$, where $e = (1, 1, \cdots)$,
3. $\mathscr{L}(Dx) = \mathscr{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.

Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathscr{L}(x) = L$

for all $\mathscr{L} \in \mathfrak{B}$. Lorentz [12] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \quad \text{uniformly in } m.$$

Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces see ([6,23]). In particular Sahiner [23] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm.

We continue in this direction and introduce some I-convergent generalized sequence spaces using Musielak-Orlicz function over n-normed spaces.

Let $(X,||.||)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\varepsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \varepsilon\Big\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathscr{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

1. $\phi \in \mathscr{I}$;
2. $A, B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$;
3. $A \in \mathscr{I}$, $B \subset A$ imply $B \in \mathscr{I}$, while an admissible ideal $\mathscr{I}$ of Y further satisfies $\{x\} \in \mathscr{I}$ for each $x \in Y$ (see [6]).

Given $\mathscr{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \varepsilon\Big\}$ belongs to $\mathscr{I}$ (see [11]).

Let I be an admissible ideal of $\mathbb{N}$, $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $(X,||\cdot,\cdots,\cdot||)$ be a n-normed space. Let $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X,||\cdot,\cdots,\cdot||)$. We define the following sequence spaces in this paper:

$$\hat{w}^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||) = \Big\{x = (x_k) \in S(n-X) : \forall\, \varepsilon > 0, \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k - L)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I$$
$$\text{for some } \rho > 0,\ L \in X \text{ and } z_1,\cdots,z_{n-1} \in X\Big\},$$

$$\hat{w}_0^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||) = \Big\{x = (x_k) \in S(n-X) : \forall\, \varepsilon > 0, \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I$$
$$\text{for some } \rho > 0, \text{ and } z_1,\cdots,z_{n-1} \in X\Big\},$$

$$\hat{w}_\infty(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||) = \Big\{x = (x_k) \in S(n-X) : \exists\, K > 0 \text{ such that } \sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \leq K$$
$$\text{for some } \rho > 0, \text{ and } z_1,\cdots,z_{n-1} \in X\Big\},$$

$$\hat{w}_\infty^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||) = \Big\{x = (x_k) \in S(n-X) : \exists\, K > 0 \text{ such that } \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\} \in I$$
$$\text{for some } \rho > 0, \text{ and } z_1,\cdots,z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The main aim of this paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2 Main Results

Theorem 2.1. Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be any sequence of strictly positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $\hat{w}^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$, $\hat{w}_0^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$, $\hat{w}_\infty(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ and $\hat{w}_\infty^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ are linear spaces over the complex field $\mathbb{C}$.

Proof. Let $x = (x_k), y = (y_k) \in \hat{w}^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ and $\alpha, \beta \in \mathbb{C}$. So

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k - L)}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I$$
$$\text{for some } \rho_1 > 0,\ L \in X \text{ and } z_1,\cdots,z_{n-1} \in X\Big\}$$

and

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k - L)}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I$$
$$\text{for some } \rho_2 > 0,\ L \in X \text{ and } z_1,\cdots,z_{n-1} \in X\Big\}.$$

Since $||\cdot,\cdots,\cdot||$ is a n-norm, $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and so by using inequality (1), we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r(\alpha x_k+\beta y_k)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$\leq D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$+D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$\leq DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$+DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k},$$

where $F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get

$$\Big\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r(\alpha x_k+\beta y_k)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\varepsilon\Big\}$$
$$\subseteq\Big\{n\in\mathbb{N}: DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\varepsilon}{2}\Big\}$$
$$\cup\Big\{n\in\mathbb{N}: DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\varepsilon}{2}\Big\}.$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $\hat{w}_0^I(\mathcal{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$, $\hat{w}_\infty(\mathcal{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ and $\hat{w}_\infty^I(\mathcal{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ are linear spaces.

Theorem 2.2. Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of strictly positive real numbers. For any fixed $n\in\mathbb{N}$, $\hat{w}_\infty(\mathcal{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ is a paranormed space with

$$g_n(x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\rho>0 \text{ is such that } \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq1, \forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Proof. It is clear that $g_n(x)=g_n(-x)$. Since $M_k(0)=0$, we get $\inf\{\rho^{\frac{p_n}{H}}\}=0$ for $x=0$ therefore, $g_n(0)=0$. Let us take $x=(x_k)$ and $y=(y_k)$ in $\hat{w}_\infty(\mathcal{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$. Let

$$B(x)=\Big\{\rho^{\frac{p_n}{H}}:\rho>0, \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq1, \forall z_1,\cdots,z_{n-1}\in X\Big\},$$

$$B(y)=\Big\{\rho^{\frac{p_n}{H}}:\rho>0, \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq1, \forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Let $\rho_1\in B(x)$ and $\rho_2\in B(y)$. If $\rho=\rho_1+\rho_2$, then we have
$$\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r(x_k+y_k))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]$$
$$\leq\frac{\rho_1}{\rho_1+\rho_2}\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]$$
$$+\frac{\rho_2}{\rho_1+\rho_2}\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r y_k)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big].$$

Thus,
$$\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r(x_k+y_k))}{\rho_1+\rho_2},z_1,\cdots,z_{n-1}||\Big)^{p_k}\leq1$$
and

$$g_n(x+y)\leq\inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}}:\rho_1\in B(x),\ \rho_2\in B(y)\Big\}$$
$$\leq\inf\Big\{\rho_1^{\frac{p_n}{H}}:\rho_1\in B(x)\Big\}+\inf\Big\{\rho_2^{\frac{p_n}{H}}:\rho_2\in B(y)\Big\}$$
$$=g_n(x)+g_n(y).$$

Let $\sigma^m\to\sigma$ where $\sigma,\sigma^m\in\mathbb{C}$ and let $g_n(x^m-x)\to0$ as $m\to\infty$. We have to show that $g_n(\sigma^m x^m-\sigma_x)\to0$ as $m\to\infty$. Let

$$B(x^m)=\Big\{\rho_m^{\frac{p_n}{H}}:\rho_m>0, \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k^m)}{\rho_m},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq1, \forall z_1,\cdots,z_{n-1}\in X\Big\},$$

$$B(x^m-x)=\Big\{\rho_m'^{\frac{p_n}{H}}:\rho_m'>0, \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r(x_k^m-x_k))}{\rho_m'},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq1, \forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

If $\rho_m \in B(x^m)$ and $\rho_m' \in B(x^m - x)$ then we observe that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k||\frac{t_{km}(u_k\sigma^m\Delta^r x_k^m - u_k\sigma\Delta^r x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}, z_1,\cdots,z_{n-1}||$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\sigma^m\Delta^r x_k^m - u_k\sigma\Delta^r x_k^m)}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}, z_1,\cdots,z_{n-1}||$$
$$+||\frac{t_{km}(u_k\sigma\Delta^r x_k^m - u_k\sigma\Delta^r x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}, z_1,\cdots,z_{n-1}||\Big)\Big]$$
$$\leq \frac{|\sigma^m-\sigma|\rho_m}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k^m)}{\rho_m}, z_1,\cdots,z_{n-1}||\Big)\Big]$$
$$+\frac{|\sigma|\rho_m'}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r x_k^m - \Delta^r x_k)}{\rho_m'}, z_1,\cdots,z_{n-1}||\Big)\Big].$$

From the above inequality, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\sigma^m\Delta^r x_k^m - \sigma\Delta^r x_k)}{\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \leq 1$$

and consequently,

$$g_n(\sigma^m x^m - \sigma x) \leq \inf\Big\{\Big(\rho_m|\sigma^m-\sigma|+\rho_m'|\sigma|\Big)^{\frac{p_n}{H}} : \rho_m \in B(x^m), \rho_m' \in B(x^m - x)\Big\}$$
$$\leq (|\sigma^m-\sigma|)^{\frac{p_n}{H}} \inf\Big\{\rho^{\frac{p_n}{H}} : \rho_m \in B(x^m)\Big\}$$
$$+(|\sigma|)^{\frac{p_n}{H}} \inf\Big\{(\rho_m')^{\frac{p_n}{H}} : \rho_m' \in B(x^m - x)\Big\} \to 0 \; as \; m \to \infty.$$

This completes the proof.

Theorem 2.3. Let $\mathscr{M}, \mathscr{M}', \mathscr{M}''$ are Musielak-Orlicz functions. Then we have
(i) $\hat{w}_0^I(\mathscr{M}', u, p, \Delta^r, ||\cdot,\cdots,\cdot||) \subseteq \hat{w}_0^I(\mathscr{M}\circ\mathscr{M}', u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$ provided (p_k) is such that $H_0 = \inf p_k > 0$.
(ii) $\hat{w}_0^I(\mathscr{M}', u, p, \Delta^r, ||\cdot,\cdots,\cdot||) \cap \hat{w}_0^I(\mathscr{M}'', u, p, \Delta^r, ||\cdot,\cdots,\cdot||) \subseteq \hat{w}_0^I(\mathscr{M}'+\mathscr{M}'', u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$.

Proof. (i) For given $\varepsilon > 0$, first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Since M_k is continuous for each k, choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_k(t) < \varepsilon_0$. Let $(x_k) \in \hat{w}_0^I(\mathscr{M}, u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$. Now from the definition

$$B(\delta) = \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} \geq \delta^H\Big\} \in I.$$

Thus if $n \notin B(\delta)$ then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} < \delta^H$$

$$\Longrightarrow \sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} < n\delta^H$$

$$\Longrightarrow \sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} < \delta^H \quad \text{for all } k, m = 1, 2, 3, \cdots$$

$$\Longrightarrow \sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big] < \delta \quad \text{for all } k, m = 1, 2, 3, \cdots.$$

Hence from above and using the continuity of $\mathscr{M} = (M_k)$ we must have

$$\Big[M_k\Big(M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big) < \varepsilon_0, \quad \forall\; k, m = 1, 2, 3, \cdots$$

which consequently implies that

$$\sum_{k=1}^{n}\Big[M_k\Big(M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$< \max\{\varepsilon_0^H, \varepsilon_0^{H_0}\}$$
$$< \varepsilon.$$

Thus $\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} < \varepsilon.$

This shows that

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} \geq \varepsilon\Big\} \subset B(\delta)$$

and so belongs to I. This proves the result.

(ii) Let $(x_k) \in \hat{w}_0^I(\mathscr{M}', u, p, \Delta^r, ||\cdot,\cdots,\cdot||) \cap \hat{w}_0^I(\mathscr{M}'', u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$. Then the fact

$$\frac{1}{n}\Big[(M_k'+M_k'')\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$\leq D\frac{1}{n}\Big[M_k'\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$
$$+D\frac{1}{n}\Big[M_k''\Big(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$

gives the result.

Theorem 2.4. The sequence spaces $\hat{w}_0^I(\mathscr{M}, u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$ and $\hat{w}_\infty^I(\mathscr{M}', u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$ are solid.

Proof. Let $(x_k) \in \hat{w}_0^I(\mathscr{M}, u, p, \Delta^r, ||\cdot,\cdots,\cdot||)$, let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(u_k\Delta^r\alpha_k x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big\} \subset$$

$$\left\{n\in\mathbb{N}:\frac{C}{n}\sum_{k=1}^{n}\left[M_k\left(||\frac{t_{km}(u_k\Delta^r x_k)}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_k}\geq\varepsilon\right\}\in I,$$

where $C = \max\{1,|\alpha_k|^H\}$. Hence $(\alpha_k x_k)\in\hat{w}_0^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ for all sequences of scalars α_k with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$ whenever $(x_k)\in\hat{w}_0^I(\mathscr{M},u,p,\Delta^r,||\cdot,\cdots,\cdot||)$.

Similarly, we can prove that $\hat{w}_\infty^I(\mathscr{M}',u,p,\Delta^r,||\cdot,\cdots,\cdot||)$ is also solid.

References

[1] S. Banach, Theorie operations linearies, Chelsea Publishing Co., New York, (1955).
[2] P. Das, P. Kostyrko, W. Wilczynski and P. Malik, Math. Slovaca **58**, 605-620 (2008).
[3] P. Das and P.Malik, Real Anal. Exchange **33 (2)**, 351-364 (2007-2008).
[4] M. Et and R. Çolak, oochow J. Math. **21**, 377-386 (1995).
[5] S. Gähler, Math. Nachr. **28**,1-43 (1965).
[6] M. Gurdal, and S. Pehlivan, Southeast Asian Bull. Math. **33 (2)**, 257-264 (2009).
[7] H. Gunawan, Scientiae Mathematicae Japonicae **5**, 47-54 (2001).
[8] H. Gunawan, Bull. Aust. Math. Soc. **64**, 137-147 (2001).
[9] H. Gunawan and M. Mashadi, Int. J. Math. Math. Sci. **27**, 631-639 (2001).
[10] H. Kızmaz, Canad. Math-Bull. **24** , pp. 169-176(1981).
[11] P. Kostyrko, T. Salat and W. Wilczynski, Real Anal. Exchange **26 (2)**, 669-686 (2000).
[12] G. G. Lorentz, Acta Math. **80**, 167-190 (1948).
[13] J. Lindenstrauss and L. Tzafriri, Israel J. Math. **10**, 345-355 (1971).
[14] L. Maligranda, Seminars in Mathematics 5, Polish Academy of Science, 1989.
[15] A. Misiak, Math. Nachr. **140**, 299-319 (1989).
[16] M. Mursaleen, Houston J. Math. **9**, 505-509 (1983).
[17] M. Mursaleen, Quart. J. Math. Oxford (2) **34**, 77-86 (1983).
[18] J. Musielak, Lecture Notes in Mathematics, 1034 (1983).
[19] K. Raj, A. K. Sharma and S. K. Sharma, Int. J. Pure Appl. Math. **67**, 475-484 (2011).
[20] K. Raj, S. K. Sharma and A. K. Sharma, Armenian J. Math. **3**, 127-141 (2010).
[21] K. Raj and S. K. Sharma, Acta Univ. Sapientiae Math. **3**, 97-109 (2011).
[22] W. Raymond, Y. Freese, and J. Cho, N. Y. Nova Science Publishers, Huntington, 2001.
[23] A. Sahiner, M. Gurdal, S. Saltan and H. Gunawan, Taiwanese J. Math. **11(5)**, 1477-1484 (2007).
[24] A. Wilansky, Summability through Functional Analysis, North- Holland Math. Stud. **85** (1984).

Kuldip Raj was born in Jammu, India on September 16,1968. He got his B.Sc(1990), M.Sc(1992) and Ph.D(1999) from University of Jammu, Jammu, India. His research interests include Functional Analysis, Operator Theory, Sequence , Series and Summability Theory. In 1998, Kuldip Raj was appointed in Education Department of Jammu and Kashmir Govt. In 2007, Kuldip Raj was appointed as Assistant Professor in the School of Mathematics, Shri Mata Vaishno Devi University, Katra J&K India. He has been Published more than 120 papers in reputed International Journals of Mathematics.

Ayhan Esi was born in Istanbul, Turkey, on March 5, 1965. Ayhan Esi got his B.Sc. from Inonu University in 1987 and M. Sc. and Ph.D. degree in pure mathematics from Elazig University, Turkey in 1990 and 1995, respectively. His research interests include Summability Theory, Sequences and Series in Analysis and Functional Analysis. In 2000, Esi was appointed to Education Faculty in Gaziantep University. In 2002, Esi was appointed as the head of Department of Mathematics in Science and Art Faculty in Adiyaman of the Inonu University. In 2006, Esi joined the Department of Mathematics of Adiyaman University. He is married and has 2 children.