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On The Numerical Solution of Partial integro-differential equations

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Abstract: In this paper, we consider the approximate solution of the partial integro-differential equation. To solve this problem, we introduce a new nonstandard time discretization scheme. Then the fourth order finite difference and collocation method is presented for the numerical solution of this type of partial integro-differential equation (PIDE). A composite weighted trapezoidal rule is manipulated to handle the numerical integrations which results in a closed-form difference scheme. The efficiency and accuracy of the scheme is validated by its application to one test problem which have exact solutions. Numerical results show that this fourth-order scheme has the expected accuracy. The most advantages of compact finite difference method for PIDE are that it obtains high order of accuracy, while the time complexity to solve the matrix equations after we use compact finite difference method on PIDE is O(N), and it can solve very general case of PIDE.

Keywords: Compact finite difference method; PIDE; Time discretization; Partial integro-differential equations; High accuracy; Collocation method.

1. Introduction

Usually results in functional equations, e.g. partial differential equations, integral and integro-differential equation, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations. These equations arise in fluid dynamics, biological models and chemical kinetics [8, 11]. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. The principal aim of this paper is to describe an approximate solution for a parabolic partial integro-differential equation representing heat conduction in material with positive memory. Classically, a heat conduction phenomenon is represented by a parabolic partial differential equation with an infinite heat propagation speed; this is a puzzling contradiction to the physics. Indeed, the material property of the past influences on that of the present, and therefore the heat propagation can be better understood if it is represented by an integro-differential equation rather than it is modeled by the usual parabolic equations.

Solution of Integro-partial differential equations has recently attracted much attention of research. The motivation for such problems lies in different branches of physics, in rtheology, and especially in the theory of parabolic type. There are several methods for solving integro-differential equations, in (1988) E. G. Yanik and G. Fairweather use finite element methods for solving integro-differential equation of parabolic type [3]. In (1989) M. N. Leroux and V. Thomèe use Numerical solution of semilinear integro-differential equations of parabolic type with non smooth data [9]. The stability of Ritz-Volterra projections and error estimates for finite element methods for a class of integro-differential equations of parabolic type is studied by Y. Lin and T. Zhang [10]. In (992), A. K. Pani, V. Thomèe, and L.B. Wahlbin use Numerical methods for hyperbolic and parabolic integro-differential equations [1]. Global and blow-up solutions of a class of semilinear integro-differential equation, by Cui Shang-bin and Ma Yu-lan in (1994) [2]. I. H. Sloan and V. Thomèe, use Time discretization of an integro-differential equation of parabolic type [6].

Our contribution in this paper is to use the analysis of [4, 5] to introduce numerical scheme for solving partial integro-differential equations in one dimensional space with non-homogeneous Dirichlet boundary conditions, by develop a new fourth order accurate scheme. The suggested numerical scheme starts with the discretization in time by the 2-point Euler backward finite difference method. After that we deal with a combination of the compact finite difference method and the trapezoidal rule for calculating the integral term and then we use a collocation method to compute the unknown function and finally the obtained system of algebraic equations is solved by iterative methods. Then we use modified variational iteration method for solving partial integro-differential equations and make the comparison with fourth order accurate scheme. The proposed techniques are programmed using Matlab ver. 7.8.0.347 (R2009a).

The paper is organized as follows: In Section 2, we give a brief introduction to a high accurate compact finite difference formula for partial integro-differential equations with varying boundary conditions. In Section 3, the proposed scheme is directly applicable to solve one numerical example to support the efficiency of the suggested numerical scheme. Conclusions are drawn in Section 4.

2. Formulation of High-Order Compact Schemes

Compact Schemes are based on a fourth-order accurate approximation to the derivative calculated from ordinary differential equation. To developed the scheme for one-dimensional uniform Cartesian grids with spacing $\Delta x = h$, let us introduce the following notations [7]: If $u_j = u(x_j)$, then we use notations

$$\delta_+ u_j = \frac{u_{j+1} - u_j}{h} = \delta_{x_+}, \qquad \delta_- u_j = \frac{u_j - u_{j-1}}{h} = \delta_{x_-}, \tag{2.1}$$

to denote the standard forward finite difference and backward finite difference schemes for first derivative. Also,

$$\delta_0 u_j = \tfrac{1}{2}\left(\delta_+ u_j + \delta_- u_j\right) = \frac{u_{j+1} - u_{j-1}}{2h} \tag{2.2}$$

is the first-order central finite difference with respect to x. The standard second-order central finite difference is denoted as $\delta_x^2 u_j$ and is defined as

$$\delta_+ \delta_- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \frac{\delta_+ - \delta_-}{h} \tag{2.3}$$

By using the Taylor's series expansion, a fourth orders accurate finite difference for the first and second derivatives can be approximated by

$$\delta_0 u = \frac{du}{dx} + \frac{h^2}{3!}\frac{d^3u}{dx^3} = \left(1 + \frac{h^2}{6}\frac{d^2}{dx^2}\right)\frac{du}{dx} = \left(1 + \frac{h^2}{6}\delta^2\right)\frac{du}{dx} + O(h^4), \tag{2.4}$$

and

$$\delta_x^2 u = \frac{d^2u}{dx^2} + \frac{h^2}{12}\frac{d^4u}{dx^4} = \left(1 + \frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{d^2u}{dx^2} = \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + O(h^4) \tag{2.5}$$

2.1 **Compact finite difference method for solving partial integro-differential equations**

Here, we use the fourth order compact finite difference method to solve problem

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\alpha\frac{\partial u}{\partial x}=\int_0^t k(t,s)u(x,s)\,ds+f(t,x), \qquad a\le x\le b, \quad t\in(0,T) \tag{2.1.1}$$

$$u(a,t)=0, \qquad u(b,t)=0, \quad t\in(0,T) \tag{2.1.2}$$

$$u(x,0)=u_0(x), \qquad a\le x\le b. \tag{2.1.3}$$

To construct a numerical solution, we first consider the nodal points (x_j,t_j) defined in the region $[a,b]\times[0,T]$ where

$a=x_0<x_1<\cdots<x_{n-1}<x_n=b, \qquad x_{j+1}-x_j=h,$

and

$0=t_0<t_1<\cdots<t_i<\cdots<T, \qquad t_{i+1}-t_i=\tau.$

In such a case we have

$x_j=a+jh \quad j=0,1,2,\ldots,n,$ and $t_i=i\tau$ for $i=0,1,2,\ldots.$

The initial condition in equation (1.2) is approximated as follows:

$$u(x,0)=u_0=u(x,t_0), \quad a\le x\le b. \tag{2.1.4}$$

Next, the 2-point Euler backward differentiation formula is manipulated to approximate u_t, given in equation (1.1), at the time-level t_{i+1} for $i=0,1,2,\ldots.$ Therefore, we have

$$\frac{u_{i+1}(x)-u_i(x)}{\tau}-\frac{d^2u_{i+1}(x)}{dx^2}+\alpha\frac{d\,u_i(x)}{dx}=\int_0^{t_{i+1}}k_{i+1}(s)\,u(x,s)\,ds+f_{i+1}(x), \tag{2.1.5}$$

where

$f_{i+1}(x)=f(x,t_{i+1})$, $k_{i+1}(s)=k(t_{i+1},s)$

and

$u_{i+1}(x)=u(x,t_{i+1}).$

Equivalently, we can rewrite equation (2.1.5) as

$$u_{i+1}-\tau\frac{d^2u_{i+1}}{dx^2}-u_i^*=\tau\int_0^{t_{i+1}}k_{i+1}(s)\,u(x,s)\,ds+\tau\, f_{i+1}(x) \tag{2.1.6}$$

where

$$u_i^*(x)=\tilde{u}_i(x-\tau\alpha). \tag{2.1.7}$$

Equation (2.1.6), rewrite as

$$u''_{i+1}(x)=\frac{u_{i+1}(x)-u_i^*(x)}{\tau}-\int_0^{t_{i+1}}k_{i+1}(s)\,u(x,s)\,ds-f_{i+1}(x), \tag{2.1.8}$$

Putting $x=x_j, \quad j=1,\cdots,n-1$ in (2.1.3), then

$$u''_{i+1,\,j}=\frac{u_{i+1,\,j}-u^{*}_{i,\,j}}{\tau}-\int_0^{t_{i+1}}k_{i+1}(s)\,u(x_j,s)\,ds-f_{i+1,\,j},\quad j=0,\ldots,n, \tag{2.1.9}$$

where

$u''_{i+1,\,j}=u''(x_j,t_{i+1}),\quad u_{i+1,\,j}=u(x_j,t_{i+1}),\quad u^{*}_{i,\,j}=u(x_j,t_i),\quad f_{i+1,\,j}=f(x_j,t_{i+1}).$

A fourth-order accurate finite difference estimate for $u''(x)$ is,

$$\delta_x^2u_{i+1,\,j}=u''_{i+1,\,j}+\frac{h^2}{12}u^{(4)}_{i+1,\,j}=\left(1+\frac{h^2}{12}\delta^2\right)\delta^2u+O(h^4)$$

2.1.10)

Noting that $O(h^2)$ term is included in equation (2.1.10), because we want to approximate it in order to construct an $O(h^4)$ scheme. Applying δ_x^x to u''_j, we get

$$u^{(4)}_{i+1,\,j}=\delta_x^2\,u''_{i+1,\,j}+O(h^2).$$

(2.1.11)

Substituting equation (2.1.11) into (2.1.10) yields

$$\delta_x^2u_{i+1,\,j}=u''_{i+1,\,j}+\frac{h^2}{12}\left(\delta_x^2u''_{i+1,\,j}+O(h^2)\right)+O(h^4). \tag{2.1.12}$$

The fourth order accurate finite difference estimate for $u''(x)$ is used from (2.1.12) to give

$$\delta_x^2u_{i+1,\,j}=u''_{i+1,\,j}+\left(\frac{h^2}{12}\delta_x^2\right)\left(u''_{i+1,\,j}\right)+O(h^4). \tag{2.1.13}$$

Then, a compact (implicit) approximation for $u''(x)$ with fourth-order accuracy will be given as

$$u''_{i+1,\,j}=\frac{\delta_x^2u_{i+1,\,j}}{\left(1+\frac{h^2}{12}\delta_x^2\right)}+O(h^4). \tag{2.1.14}$$

Using this estimate and considering the discrete solution of equation (2.1.9) which satisfies the approximation, we get

$$\left[1-\frac{h^2}{12\,\tau}\right]\delta_x^2u_{i+1,\,j}-\frac{u_{i+1,\,j}}{\tau}+\int_0^{t\,i+1}k_{i+1}(s)\,u(x_j,s)\,ds+\frac{h^2}{12}\int_0^{t\,i+1}k_{i+1}(s)\,\delta_x^2u(x_j,s)\,ds=$$
$$=-\frac{u^{*}_{i,\,j}}{\tau}-\frac{h^2}{12\,\tau}\delta_x^2u^{*}_{i,\,j}-f_{i+1,\,j}-\frac{h^2}{12}\delta_x^2\,f_{i+1,\,j} \tag{2.1.15}$$

$$\left[\frac{1}{h^2}-\frac{1}{12\,\tau}\right]\left(u_{i+1,\,j+1}+u_{i+1,\,j-1}\right)+\left[\frac{-2}{h^2}-\frac{5}{6\,\tau}\right]u_{i+1,\,j}+\frac{5}{6}\int_0^{t\,i+1}k_{i+1}(s)u_j(s)\,ds+$$
$$+\frac{1}{12}\left[\int_0^{t\,i+1}k_{i+1}(s)u_{j+1}(s)\,ds+\int_0^{t\,i+1}k_{i+1}(s)u_{j-1}(s)\,ds\right]=$$
$$=\frac{-1}{12\,\tau}\left(u^{*}_{i,\,j+1}+u^{*}_{i,\,j-1}\right)-\frac{5}{6\,\tau}u^{*}_{i,\,j}-\frac{1}{12}\left(f_{i+1,\,j+1}+f_{i+1,\,j-1}\right)-\frac{5}{6}\,f_{i+1,\,j}. \tag{2.1.16}$$

The later integral will be handled numerically using the composite weighted trapezoidal rule given by:

$$\int_{t_0}^{t_{i+1}} f(s)\,ds \approx \tau \sum_{m=0}^{i} [w\ f(t_m) + (1-w)\ f(t_{m+1})]$$
$$= \tau \left[w\ f(t_0) + (1-w)\ f(t_{i+1}) + \sum_{m=1}^{i} f(t_m) \right] \qquad (2.1.17)$$

Using (2.1.17) we get

$$\int_{0}^{t_{i+1}} k_{i+1}(s)\, u(x,s)\, ds \approx$$
$$\approx \tau \left(w\, k_{i+1}(0)\, u_0(x) + (1-w)\, k_{i+1}(t_{i+1})\, u_{i+1}(x) + \sum_{m=1}^{i} k_{i+1}(t_m)\, u_{i+1-m}(x) \right) \qquad (2.1.18)$$

The substitutions of this equation into equation (2.1.16) yields

$$\left[\frac{1}{h^2} - \frac{1}{12\,\tau}\right]\left(u_{i+1,\,j+1} + u_{i+1,\,j-1}\right) + \left[\frac{-2}{h^2} - \frac{5}{6\,\tau}\right] u_{i+1,\,j} +$$
$$+ \frac{5\,\tau}{6}\left[w\, k_{i+1}(0)\, u_{0,\,j} + (1-w)\, k_{i+1}(t_{i+1})\, u_{i+1,\,j} + \sum_{m=1}^{i} k_{i+1}(t_m)\, u_{i+1-m,\,j} \right] +$$
$$+ \frac{\tau}{12}\left[w\, k_{i+1}(0)\, u_{0,\,j+1} + (1-w)\, k_{i+1}(t_{i+1})\, u_{i+1,\,j+1} + \sum_{m=1}^{i} k_{i+1}(t_m)\, u_{i+1-m,\,j+1} \right] +$$
$$+ \frac{\tau}{12}\left[w\, k_{i+1}(0)\, u_{0,\,j-1} + (1-w)\, k_{i+1}(t_{i+1})\, u_{i+1,\,j-1} + \sum_{m=1}^{i} k_{i+1}(t_m)\, u_{i+1-m,\,j-1} \right] =$$
$$= \frac{-1}{12\,\tau}\left(u^*_{i,\,j+1} + u^*_{i,\,j-1}\right) - \frac{5}{6\tau} u^*_{i,\,j} - \frac{1}{12}\left(f_{i+1,\,j+1} + f_{i+1,\,j-1}\right) - \frac{5}{6} f_{i+1,\,j}. \qquad (2.1.19)$$

Let $U_i(x)$ be a function that approximates $u(x_i, t_i)$ for the time-level $t_i = i\,\tau$ and is a linear combination of n+1 shape functions which is expressed as:

$$U_i(x) = \sum_{m=0}^{n} c_{mi}\ \varphi_m(x) \qquad (2.1.20)$$

Where $(c_{mi})_{m=0}^{n}$ are the unknown real coefficients, to be evaluated, and the $\varphi_m(x)$ are any knowing basis functions .

The approximate solutions $U_i(x)$ for different time-levels are determined iteratively as follows. Starting with the time-level $t_0 = 0$ the value of $u_0(x_j), u_0(x_{j+1})$ and $u_0(x_{j-1})$ for $j = 1,2,....,n-1$ are found from equation (1.2). Next, we will approximate the solution U_{i+1} for $i = 0$ in equation (2.1.16) by the shape functions U_1 as is given in equation (2.1.20). Hence equation (2.1.16) is approximated by:

$$\left(\frac{1}{h^2} - \frac{1}{12\,\tau}\right)\left(U_{1,\,j+1} + U_{1,\,j-1}\right) + \left(\frac{-2}{h^2} - \frac{5}{6\,\tau}\right) U_{1,\,j} + \frac{5\tau}{6}\left[w\, k_1(0)\, u_{0,\,j} + (1-w)\, k_1(t_1)\, U_{1,\,j}\right] +$$

$$+\frac{\tau}{12}\left[w\,k_1(0)\,u_{0,\,j+1}+(1-w)\,k_1(t_1)\,U_{1,\,j+1}\right]+\frac{\tau}{12}\left[w\,k_1(0)\,u_{0,\,j-1}+(1-w)\,k_1(t_1)\,U_{1,\,j-1}\right]=$$
$$=\frac{-1}{12\,\tau}\left(u^*_{0,\,j+1}+u^*_{0,\,j-1}\right)-\frac{5}{6\,\tau}u_{0,\,j}-\frac{1}{12}\left(f_{1,\,j+1}+f_{1,\,j-1}\right)-\frac{5}{6}\,f_{1,\,j}. \tag{2.1.21}$$

Replacing U_1 by the approximate solution given by equation (2.1.20) yields the following linear system of $n-1$ equations

$$\left[\frac{1}{h^2}-\frac{1}{12\,\tau}\right]\left(\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j+1}+\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j-1}\right)+\left[\frac{-2}{h^2}-\frac{5}{6\,\tau}\right]\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j}+$$
$$+\frac{5\tau}{6}\left[(1-w)\,k_1(t_1)\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j}\right]+\frac{\tau}{12}\left[(1-w)\,k_1(t_1)\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j+1}\right]+$$
$$+\frac{\tau}{12}\left[(1-w)\,k_1(t_1)\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j-1}\right]=\frac{-5\,\tau\,w\,k_1(0)}{6}u_{0,\,j}-\frac{5}{6\,\tau}u^*_{0,\,j}-$$
$$-\left[\frac{\tau\,w\,k_1(0)}{12}\right](u_{0,\,j+1}+u_{0,\,j-1})-\frac{1}{12\,\tau}(u^*_{0,\,j+1}+u^*_{0,\,j-1})-$$
$$-\frac{1}{12}\left(f_{1,\,j+1}+f_{1,\,j-1}\right)-\frac{5}{6}\,f_{1,\,j}, \tag{2.1.22}$$

Where,

$$\sum_{m=0}^{n}c_{m1}\,\varphi_{m\;j+1}=\sum_{m=0}^{n}c_{m1}\,\varphi_m(x_{j+1}).$$

Rewrite equation (2.1.22) as

$$\sum_{m=0}^{n}c_{m1}\left[a_1\,\varphi_{m\;j+1}+a_2\,\varphi_{m\;j}+a_1\,\varphi_{m\;j-1}\right]=$$
$$=a_3\,u_{0,\,j}+a_4\,(u_{0,\,j+1}+u_{0,\,j-1})+a_5\,u^*_{0,\,j}+a_6\,(u^*_{0,\,j+1}+u^*_{0,\,j-1})+$$
$$+a_6\,\tau\,(f_{i,\,j+1}+f_{i,\,j-1})+a_5\,\tau\,f_{i,\,j}\,,$$

(2.1.23)

where

$$a_1=\frac{1}{h^2}-\frac{1}{12\,\tau}+\frac{\tau}{12}(1-w)\,k_1(t_1)\,,\qquad a_2=\frac{-2}{h^2}-\frac{5}{6\,\tau}+\frac{5\tau}{6}(1-w)\,k_1(t_1)\,,$$
$$a_3=\frac{-5\,\tau\,w\,k_1(0)}{6}\qquad a_4=\frac{-\tau\,w\,k_1(0)}{12}\,,$$
$$a_5=\frac{-5}{6\,\tau}\,,\qquad a_6=\frac{-1}{12\,\tau}\,,$$

(2.1.24)

The system (2.1.23) consists of $(n-1)$ equation in the $(n+1)$ unknowns $(c_{m1})_{m=0}^{n}$. To get a solution of this system we need two additional conditions. These conditions are obtained from the boundary conditions (2.1.2)

$$u(a,t_i)=\sum_{m=0}^{n} c_{m1}\,\varphi_m(a)=g_1(t_i),\qquad i=0,\ldots n \tag{2.1.25}$$

$$u(b,t_i)=\sum_{m=0}^{n} c_{m1}\varphi_m(b)=g_2(t_i),\qquad i=0,\ldots n \tag{2.1.26}$$

Since f and u_0 are known at every grid point, the right hand side of equation (2.1.23) is known for all nodes. The system (2.1.23), equations (2.1.25) and (2.1.26) consist of $(n-1)$ equations in $(n+1)$ unknowns; this system is of the form

$$AC=F. \tag{2.1.27}$$

Upon solving the system (2.1.27), the function $U_1(x)$ is approximated by the sum:

$$U_1(x_j)=\sum_{m=0}^{n} c_{m1}\,\varphi_m(x_j),\quad j=0,1,2,\ldots,n.$$

(2.1.28)

Next, we find the approximate solution at time-levels $t_1, t_2,\cdots$ recursively by solving the following system for $i=1,2,\ldots.$

$$\sum_{m=0}^{n} c_{mi}\left(a_1\,\phi_{m\,j+1}+a_2\,\phi_{m\,j}+a_1\,\phi_{m\,j-1}\right)=\frac{-1}{12\,\tau}\left(u^*_{i,\,j+1}+u^*_{i,\,j-1}\right)-\frac{5}{6\,\tau}u^*_{i,\,j}$$
$$+a_3\,u_{0,\,j}+a_4\,(u_{0,\,j+1}+u_{0,\,j-1})-\frac{1}{12}(f_{i+1,\,j+1}+f_{i,\,j-1})-\frac{5}{6}f_{i+1,\,j}-\frac{5\,\tau}{6}\sum_{m=1}^{i}k_{i+1}(t_m)\,u_{i+1-m,\,j}-$$
$$-\frac{\tau\,w}{12}\sum_{m=1}^{i}k_{i+1}(t_m)\,u_{i+1-m,\,j+1}-\frac{\tau\,w}{12}\sum_{m=1}^{i}k_{i+1}(t_m)\,u_{i+1-m,\,j-1}, \tag{2.1.29}$$

where

$$a_1=\frac{1}{h^2}-\frac{1}{12\,\tau}+\frac{\tau}{12}(1-w)\,k_{i+1}(t_{i+1}),\qquad a_2=\frac{-2}{h^2}-\frac{5}{6\,\tau}+\frac{5\,\tau}{6}(1-w)\,k_{i+1}(t_{i+1}),$$
$$a_3=\frac{-5\,\tau\,w\,k_{i+1}(t_0)}{6},\qquad a_4=\frac{-\tau\,w\,k_{i+1}(t_0)}{12} \tag{2.1.30}$$

$$u(a,t_i)=\sum_{m=0}^{n} c_{mi}\,\varphi_m(a)=g_1(t_i),\qquad i=0,\ldots n \tag{2.1.31}$$

$$u(b,t_i)=\sum_{m=0}^{n} c_{mi}\,\varphi_m(b)=g_2(t_i),\qquad i=0,\ldots n. \tag{2.1.32}$$

3. Numerical Experiment

In this section, we solve the integro-differential equation (2.1.1)-(2.1.3) in $(0,1)\times(0,T)$ with

$k(x,t)=e^{-\pi^2(x-t)}$, $f(x,t)=\alpha\,\pi\,e^{-\pi^2 t}\cos(\pi\,x)-t\,e^{-\pi^2 t}\,\sin(\pi\,x)$,

$u(x,0)=\sin(\pi\,x)\quad 0\le x\le 1$,

and $g_1(t)=g_2(t)=0$.

The theoretical solution of this problem is

$u(x,t)=e^{-\pi^2 t}\sin(\pi x)$.

We employ a compact difference scheme for the space derivative so that we get a full discretization scheme with error estimation $O(h^4)+O(\tau)$ We shall compare the results obtained by the suggested approximation scheme with the exact solution.

Table 1. Comparison between exact and numerical solutions at $t=0.02,\alpha=1,\tau=0.0001$ $t=0.01,\alpha=1,\tau=0.0001$ respectively.

x	$t=0.02$, $\alpha=1$ $\tau=0.0001$			$t=0.01$, $\alpha=1$ $\tau=0.0001$		
	Exact solution	Suggested scheme	error	Exact solution	Suggested scheme	error
0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	2.5366E-001	2.5384E-001	1.8136E-004	2.7998E-001	2.8017E-001	2.0064E-004
0.2	4.8249E-001	4.8263E-001	1.3926E-004	5.3254E-001	5.3269E-001	1.5461E-004
0.3	6.6409E-001	6.6419E-001	1.0135E-004	7.3298E-001	7.3309E-001	1.1309E-004
0.4	7.8069E-001	7.8074E-001	5.1969E-005	8.6167E-001	8.6173E-001	5.8807E-005
0.5	8.2087E-001	8.2087E-001	2.3644E-006	9.0602E-001	9.0602E-001	1.0883E-006
0.6	7.8069E-001	7.8064E-001	5.6463E-005	8.6167E-001	8.6161E-001	6.0876E-005
0.7	6.6409E-001	6.6399E-001	1.0516E-004	7.3298E-001	7.3287E-001	1.1485E-004
0.8	4.8249E-001	4.8235E-001	1.4202E-004	5.3254E-001	5.3239E-001	1.5587E-004
0.9	2.5366E-001	2.5348E-001	1.8278E-004	2.7998E-001	2.7977E-001	2.0129E-004
1	0.00	0.00	0.00	0.00	0.00	0.00

Table 2. Comparison between exact and numerical solutions at $t=0.1,\alpha=1,\tau=0.00001$, and $t=0.5,\alpha=1,\tau=0.01$, respectively.

x	$t=0.1$, $\alpha=1$, $\tau=0.00001$			$t=0.5$, $\alpha=1$, $\tau=0.01$		
	Exact solution	Suggested scheme	error	Exact solution	Suggested scheme	error
0	0	0	0	0	0	0
0.1	1.1517E-001	1.1518E-001	8.1966E-006	2.2224E-003	2.2839E-003	6.1571E-005
0.2	2.1907E-001	2.1907E-001	6.0145E-006	4.2273E-003	4.2925E-003	6.5201E-005
0.3	3.0153E-001	3.0153E-001	4.1968E-006	5.8184E-003	5.8618E-003	4.3436E-005
0.4	3.5447E-001	3.5447E-001	1.8736E-006	6.8399E-003	6.8504E-003	1.0561E-005
0.5	3.7271E-001	3.7271E-001	6.2373E-007	7.1919E-003	7.1663E-003	2.5542E-005
0.6	3.5446E-001	3.5446E-001	3.0599E-006	6.8399E-003	6.7808E-003	5.9095E-005
0.7	3.0152E-001	3.0152E-001	5.2060E-006	5.8184E-003	5.7337E-003	8.4579E-005
0.8	2.1907E-001	2.1907E-001	6.7475E-006	4.2273E-003	4.1324E-003	9.4883E-005
0.9	1.1517E-001	1.1516E-001	8.5817E-006	2.2224E-003	2.1454E-003	7.7000E-005
1	0	0	0	0	0	0

Table 3. Comparison between exact and numerical solutions at $t=0.3,\alpha=0.4,\tau=0.00005$, , and $t=0.7,\alpha=3,\tau=0.04$ respectively.

x	$t=0.3$, $\alpha=0.4$, $\tau=0.00005$			$t=0.7$, $\alpha=3$, $\tau=0.04$		
	Exact solution	Suggested scheme	error	Exact solution	Suggested scheme	error
0	0	0	0	0	0	0
0.1	1.5998E-002	1.6001E-002	1.9143E-006	3.0872E-004	3.1712E-004	8.4025E-006
0.2	3.0432E-002	3.0432E-002	1.0104E-006	5.8722E-004	5.8988E-004	2.6648E-006
0.3	4.1885E-002	4.1885E-002	2.5407E-007	8.0823E-004	7.9708E-004	1.1148E-005
0.4	4.9239E-002	4.9238E-002	5.5018E-007	9.5014E-004	9.2196E-004	2.8178E-005
0.5	5.1773E-002	5.1772E-002	1.2984E-006	9.9903E-004	9.5468E-004	4.4351E-005
0.6	4.9239E-002	4.9237E-002	1.9195E-006	9.5014E-004	8.9397E-004	5.6169E-005
0.7	4.1885E-002	4.1883E-002	2.3548E-006	8.0823E-004	7.4767E-004	6.0564E-005
0.8	3.0432E-002	3.0429E-002	2.5363E-006	5.8722E-004	5.3249E-004	5.4717E-005
0.9	1.5998E-002	1.5996E-002	2.7161E-006	3.0872E-004	2.7298E-004	3.5731E-005
1	0	0	0	0	0	0

4. Table 4. Comparison between exact and numerical solutions at $t=0.008, \alpha=0.2, \tau=0.00002$, , and $t=0.03, , \alpha=0.1, \tau=0.0005$, respectively.

x	$t=0.008$, $\alpha=0.2$, $\tau=0.00002$			$t=0.03$, $\alpha=0.1$, $\tau=0.0005$		
	Exact solution	Suggested scheme	error	Exact solution	Suggested scheme	error
0	0	0	0	0	0	0
0.1	2.8555E-001	2.8556E-001	8.2374E-006	2.2982E-001	2.2989E-001	7.4749E-005
0.2	5.4316E-001	5.4316E-001	6.1979E-006	4.3714E-001	4.3720E-001	5.8002E-005
0.3	7.4759E-001	7.4760E-001	4.4805E-006	6.0168E-001	6.0172E-001	3.8594E-005
0.4	8.7885E-001	8.7885E-001	2.2335E-006	7.0732E-001	7.0733E-001	1.5313E-005
0.5	9.2407E-001	9.2407E-001	2.2325E-007	7.4372E-001	7.4371E-001	9.4272E-006
0.6	8.7885E-001	8.7884E-001	2.6581E-006	7.0732E-001	7.0728E-001	3.3165E-005
0.7	7.4759E-001	7.4759E-001	4.8416E-006	6.0168E-001	6.0163E-001	5.3509E-005
0.8	5.4316E-001	5.4315E-001	6.4599E-006	4.3714E-001	4.3708E-001	6.8301E-005
0.9	2.8555E-001	2.8554E-001	8.3746E-006	2.2982E-001	2.2974E-001	7.9473E-005
1	0	0	0	0	0	0

8. Conclusion

A fourth-order accurate compact finite difference scheme for partial integro-differential problems was developed. The method reduces the underlying problem to linear system of algebraic equations, which can be solved successively to obtain a numerical solution at varied time-levels. Numerical experiments which shown in the above scheme are good agreement with the exact ones. Moreover, the results in tables (1-4) confirm that the numerical solutions can be refined when the time-step τ is reduced, or the number of nodes is increased.

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