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Numerical Solution of Linear HPDEs Via Bernoulli Operational Matrix of Differentiation and Comparison with Taylor Matrix Method

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Abstract: In this paper a new matrix approach for solving linear hyperbolic partial differential equations (HPDEs) is presented. The method is based on the Bernoulli expansion of two-variable functions, which consists of the matrix representation of expressions in the considered HPDE. Also, a new operational matrix of differentiation is introduced, which consists of nonzero elements under its diagonal, meanwhile the operational matrix of differentiation of other polynomial bases (such as Chebyshev, Legendre, etc.) is a strictly (upper or lower) triangular matrix. In the proposed method, HPDE together with the initial and boundary conditions are transformed into the matrix equations, which corresponds to a system of linear algebraic equations with the unknown Bernoulli coefficients. Combining these matrix equations and then solving the system yields the Bernoulli coefficients of the approximated solution. Illustrative examples are included to demonstrate the validity and applicability of the technique. All of computations are performed on a PC using several programs written in MATLAB 7.12.0.
Keywords: Bernoulli Operational Matrix of Differentiation; Hyperbolic Partial Differential Equations; Bernoulli Polynomial Solutions; Double Bernoulli Series.

1.Introduction

In applied sciences, each physical event may be modeled mathematically. So, it is very crucial to obtain the solutions of the models because these solutions provide information about the character of the modeled event. Therefore, it is very important to find high accurate approximated solutions of linear (or nonlinear) ordinary and partial differential equations (PDEs) in physics, chemistry, biology and engineering branches.

Among the above-mentioned mathematical models, hyperbolic partial differential equations (HPDEs) play a dominant role in many branches of science and engineering. These equations can model the vibrations of structures (e.g. buildings, beams and machines) and are the basis for fundamental equations of atomic physics. Wave equation and Telegraph equation are typical examples of HPDEs. Telegraph equation is commonly used in signal analysis for transmission and propagation of electrical signals and also has applications in other fields (see [15] and the references therein). On the other hand, much attention has been given in the literature to the development, analysis and implementation for the numerical solution of second-order HPDEs, especially telegraph equations (see for example, [3]-[18], [24], [25], [30]).

HPDEs can be solved in many ways. In recent years, with the developement of mathematics and computer sciences, there are great improvement in the numerical methods for solving such PDEs. Among these methods, collocation methods [1] evolved significantly during the two last decades and have been applied in many engineering problems. The concept of collocation method originate from this fact that, the basic equation must be satisfied exactly at several definite nodes which are called collocation points. After imposing these conditions, the basic equation will be changed into an (linear or nonlinear) algebraic system, which can be solved by using well-known methods such as Newton's method. However solving the obtained algebraic system is more easy than the basic equation, but there exists many challenges for proposing an effective and aapplicable algorithm to dealing with the associated algebraic system properly. Another mathematical tool for approximating the solution of HPDEs is the Galerkin approach. In this technique, the test functions are equal to the trial functions in the procedure of the solution approximation and also each of the trial functions must be satisfied in boundary conditions. The associated (linear or nonlinear) algebraic

system of this method will be provided from vanishing the inner product of the residual and all of the trial functions. As a modofiction of the Galerkin approaches, one can point out to the Tau methods, which are useful for solving non-periodic problems. One characteristic of the Tau method is that, the trial functions do not satisfy boundary conditions in relation to the supplementary conditions imposed together with the basic equation. However, Least Square method (LSM) is another classical method that frequently used for the numerical solution of these PDEs. The basic idea of LSM is the minimizing the residual norm in terms of unknown coefficients. In all of these methods (i.e., Collocation, Galerkin, Tau, and Least square) the approximated solution is written in terms of linear combination of trial functions with unknown coefficients, such that these parameters are the solution of the associated algebraic system. During the two last decades Orthogonal bases (such as Legendre, Chebyshev, etc.) have received considerable attention for making an ideal base (i.e. a set that containing trial functions) in the procedure of approximation.
Since the begining of 1994, the Taylor, Chebyshev, Legendre, Bessel, Hermite, Lagurre and Bernstein matrix methods have been used in the works [2], [7], [8], [20], [21], [22], [28], [29], [31] to solve linear differential (including HPDEs), Fredholm Volterra integro difference equations and their systems. Yet so far, to the best of our knowledge, a practical matrix method which based on Bernoulli polynomials have had few results for approximating the solution of linear HPDEs. This partially motivated our interest in such method.
In this paper, in the light of the above-mentioned methods and by means of the matrix relations between the Bernoulli polynomials and their derivatives, we develop a new approach called the Bernoulli matrix method for solving the second-order linear hyperbolic partial differential equation

$$\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial t\partial x}+\gamma\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial t}+\theta u=G(x,t), \qquad \beta^2-4\alpha\gamma>0, \tag{1}$$

with the initial conditions

$$\begin{cases} u(x,0)=f(x), & x\in[a,b], \\ \dfrac{\partial u(x,0)}{\partial t}=m(x), & x\in[a,b], \end{cases} \tag{2}$$

and boundary conditions

$$\begin{cases} u(a,t)=h(t), & 0<t\le T \\ u(b,t)=k(t), & 0<t\le T \end{cases} \tag{3}$$

where $\alpha, \beta, \gamma, \delta, \eta,$ and θ are constants. Without loss of generality and for clarity of presentation suppose that $a=b=T=1$. Note that by a simple linear transformation these substitutions could be done. Now, we assume that the solution of the considered HPDE is approximated as follows

$$u(x,t)\approx\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}B_r(x)B_s(t), \qquad a_{r,s}=\frac{1}{r!s!}\int_0^1\int_0^1 u^{(r,s)}(x,t)dxdt. \tag{4}$$

so that the Bernoulli coefficients to be determined are $a_{r,s}$ $(r,s=0,1,...,N)$.

The remainder of this paper is organized as follows: In the next section the Bernoulli polynomials together with their interesting properties are introduced. Section 3 is devoted to presenting the Bernoulli operational matrix of differentiation in (one and) two dimensions, which is the fundamental section of the proposed method. In section 4 we describe that how this matrix can be applied for solving the above-mentioned equation. Accuracy of the solution and error analysis are briefly discussed in section 5. Illustrative examples are included in section 6 for confirming the accuracy of the proposed approach. Finally conclusions are given in section 7.

2.Bernoulli Polynomials

The Bernoulli polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. Most commonly used techniques with Bernoulli polynomials have been examined in papers [9], [10], [23], [26] and references therein. In early works based on the Bernoulli polynomials, Bernoulli coefficients were determined as parameters of the approximate solutions of a delay ordinary differential equations, which is called Pantograph equation, whereas to evaluate the Bernoulli coefficients in this method only requires solving a system of

linear algebraic equations [27].
Bernoulli polynomials can be defined in several ways. One of the most well-known ways is [27]

$$\begin{cases} B_n'(x) = nB_{n-1}(x), \ \forall n \geq 1 \\ \int_0^1 B_n(x)dx = 1, \qquad \forall n \geq 1, \\ B_0(x) = 1. \end{cases} \tag{5}$$

According to the above properties, if we approximate a function $f(x)$, which belongs to $C^\infty[0,1]$, in terms of Bernoulli polynomials on the interval [0,1] with the form

$$f(x) = \sum_{n=0}^{\infty} a_n B_n(x), \tag{6}$$

then the coefficients a_n can be evaluated as follows

$$a_n = \frac{1}{n!}\int_0^1 f^{(n)}(x)dx. \tag{7}$$

Moreover, by extending the above properties in two-variable functions, one can conclude that if a (smooth enough) two-variable function $F(x,t)$ be approximated by Bernoulli polynomials in the form

$$F(x,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m} B_n(x) B_m(t), \tag{8}$$

then the coefficients $a_{m,n}$ can be evaluated as follows

$$a_{n,m} = \frac{1}{n!m!}\int_0^1\int_0^1 \frac{\partial^{n+m}F(x,t)}{\partial x^n \partial t^m}dxdt. \tag{9}$$

In the next section, by using the first property in (5), we will introduce the operational matrix of differentiation of Bernoulli polynomials and then extend this matrix for two-dimensional functions, which is called the two dimensional Bernoulli operational matrix of differentiation.

3.Two Dimensional Bernoulli Operational Matrix of Differentiation

According to the first property of (5) the follwing formula is concluded evidently

$$\underbrace{[B_0(x), B_1(x),..., B_N(x)]'}_{B'(x)} = \underbrace{[B_0(x), B_1(x), ..., B_N(x)]}_{B(x)} \underbrace{\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 2 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & N \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}}_{M} \tag{10}$$

where M is the Bernoulli operational matrix of differentiation. Trivially $B^{(k)}(x) = B(x)M^k$ for all positive integers k, where $B^{(k)}(x)$ is denoting the $k-th$ derivative of $B(x)$.
Since in this paper we deal with two-variable functions, the above-mentioned matrix must be extended to a product of two matrices. Now, suppose that

$$B(x,t) = [b_0(x,t)\; b_1(x,t) \ldots b_N(x,t)]_{1\times(N+1)^2}, \tag{11}$$

where $b_i(x,t) = [B_{i,0}(x,t)\; B_{i,1}(x,t) \ldots B_{i,N}(x,t)]$ for all $i = 0,1,...,N$ and $B_{m,n}(x,t) = B_m(x)B_n(t)$ for all $m,n = 0,1,...,N$.
By a similar procedure that was used in [1], the relations between the matrix $B(x,t)$ and its derivatives are

$$B^{(m,n)}(x,t) = B(x,t)(\bar{B})^m (B)^n, \tag{12}$$

where $m,n = 0,1,...,N$ and

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,\bar{B} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & NI \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$B = \begin{bmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

and I is the $(N+1)\times(N+1)$ identity matrix.
Now, consider the approximated solution (4), which can be rewritten in the matrix form

$$u(x,t) = B(x,t)A, \tag{13}$$

where $B(x,t)$ is introduced in (11) and

$$A = [a_{0,0}\ a_{0,1} \ldots a_{0,N}\ \ a_{1,0}\ a_{1,1} \ldots a_{1,N}\ \ \ldots a_{N,0}\ a_{N,1} \ldots a_{N,N}\ ]^T.$$

By using the relations (12) and (13), we have

$$u^{(m,n)}(x,t) = B^{(m,n)}(x,t)A = B(x,t)(\bar{B})^m (B)^n A, \qquad n = 0,1,2. \tag{14}$$

which are the fundamental relations of the proposed method. In the next section, we describe our approach which is based on (14).

4. Method of the Solution

Our aim is to investigate the approximate solution of Equation (1), under the given conditions, in the series form of Equation (4) or in the matrix form $u(x,t) = B(x,t)A$. To obtain the approximated solution, by using (14), we first reduce the terms of Equation (1) to matrix forms

$$u_{xx}(x,t) = B(x,t)(\bar{B})^2 A, \tag{15}$$

$$u_{xt}(x,t) = B(x,t)\bar{B}BA, \tag{16}$$

$$u_{tt}(x,t) = B(x,t)(B)^2 A, \tag{17}$$

$$u_x(x,t) = B(x,t)\bar{B}A, \tag{18}$$

$$u_t(x,t) = B(x,t)BA. \tag{19}$$

We can also approximate the function $G(x,t)$ in terms of Bernoulli polynomials as follows

$$G(x,t) \approx \sum_{r=0}^{N}\sum_{s=0}^{N} g_{r,s} B_r(x) B_s(t), \qquad g_{r,s} = \frac{1}{r!s!}\int_0^1\int_0^1 G^{(r,s)}(x,t)dxdt \tag{20}$$

From (20), $G(x,t)$ can be represented in the matrix form

$$G(x,t) = B(x,t)G, \tag{21}$$

where

$$G = [g_{0,0}\ g_{0,1} \cdots g_{0,N}\ g_{1,0}\ g_{1,1} \cdots g_{1,N} \cdots g_{N,0}\ g_{N,1} \cdots g_{N,N}]^T .$$

Substituting the expressions (13), (15)-(19) and (21) into the basic equation (1) and simplifying the result, we have the matrix equation

$$\{\alpha(\bar{B})^2 + \beta\bar{B}B + \gamma(B)^2 + \delta\bar{B} + \eta B + \theta I\}A = G . \tag{22}$$

Briefly, we can write Equation ((22)) in the form

$$WA = G, \tag{23}$$

where $W = [\alpha(\bar{B})^2 + \beta\bar{B}B + \gamma(B)^2 + \delta\bar{B} + \eta B + \theta I]_{(N+1)^2\times(N+1)^2}$ is the matrix coefficients.

We now present the alternative forms for $u(x,t)$ which are important to simplify matrix forms of the conditions. The simplification in conditions is done only with respect to the variable x or t. Therefore, we must use different forms for initial and boundary conditions. For initial conditions (2)

$$u(x,t) = B(x)Q(t)A, \tag{24}$$

and

$$u_t(x,t) = B(x)Q(t)BA; \tag{25}$$

for boundary conditions

$$u(x,t) = B(t)J(x)A. \tag{26}$$

The matrix representations of non-homogeneous terms of Equations (2) and (3) can be written in the forms

$$f(x) = B(x)F, \qquad F = [f_0\ f_1 \ldots f_N]^T, \qquad f_n = \frac{\int_0^1 f^{(n)}(x)dx}{n!}, \qquad n = 0,1,\ldots,N \tag{27}$$

$$m(x) = B(x)M, \qquad M = [m_0\ m_1 \ldots m_N]^T, \qquad m_n = \frac{\int_0^1 m^{(n)}(x)dx}{n!}, \qquad n = 0,1,\ldots,N \tag{28}$$

$$h(t) = B(t)H, \qquad H = [h_0\ h_1 \ldots h_N]^T, \qquad h_n = \frac{\int_0^1 h^{(n)}(t)dt}{n!}, \qquad n = 0,1,\ldots,N \tag{29}$$

$$k(t) = B(t)K, \qquad K = [k_0\ k_1 \ldots k_N]^T, \qquad k_n = \frac{\int_0^1 k^{(n)}(t)dt}{n!}, \qquad n = 0,1,\ldots,N \tag{30}$$

where

$$B(x) = [B_0(x)\ B_1(x) \ldots B_N(x)], \qquad B(t) = [B_0(t)\ B_1(t) \ldots B_N(t)],$$

$$J(x) = [B_0(x)I \quad B_1(x)I \quad \ldots \quad B_N(x)I\,]_{(N+1)\times(N+1)^2}, \qquad Q(t) = \begin{bmatrix} B(t) & 0 & \ldots & 0 \\ 0 & B(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B(t) \end{bmatrix}_{(N+1)\times(N+1)^2} .$$

By substituting relations (24)-(30) into Equations (2) and (3) and then simplifying the results, we get the matrix forms of conditions, respectively, as

$$K_1 = Q(0)A = F$$

$$K_2 = Q(0)BA = M$$

$$K_3 = J(a)A = H$$

$$K_4 = J(b)A = K.$$

To obtain the solution of Equation (1) under conditions (2) and (3), the augmented matrix is formed as follows:

$$[\overline{\overline{W}};\overline{\overline{G}}] = \begin{bmatrix} K_1 & ; & F \\ K_2 & ; & M \\ K_3 & ; & H \\ K_4 & ; & K \\ \overline{W} & ; & \overline{G} \end{bmatrix}. \tag{31}$$

The unknown Bernoulli coefficients are obtained as

$$A = \left(\overline{\overline{\overline{W}}}\right)^{-1}\overline{\overline{\overline{G}}}$$

where $[\overline{\overline{\overline{W}}};\overline{\overline{\overline{G}}}]$ is generated by first using the Gauss elimination method to $[\overline{\overline{W}};\overline{\overline{G}}]$ and then removing zero rows of the eliminated matrix. Here, $\overline{\overline{W}}$ and $\overline{\overline{G}}$ are obtained by throwing away maximum number of row vectors from W and G so that the rank of the system that defined in (31) can not be smaller than $(N+1)^2$. This process provides higher accuracy because of decreasing truncation error.

5.Accuracy of the Solution and Error Analysis

We can easily check the accuracy of the method. Since the truncated Bernoulli series (4) is an approximate solution of Equation (1), when the function $u(x,t)$ and its derivatives are substituted in Equation (1), the resulting equation must be satisfied approximately; that is, for $(x = x_p, t = t_q) \in [a,b]\times[0,T],\ p,q = 0,1,2,...$

$$E(x_p,t_q) = |\alpha u_{xx}(x_p,t_q) + \beta u_{xt}(x_p,t_q) + \gamma u_{tt}(x_p,t_q)$$
$$+\delta u_x(x_p,t_q) + \eta u_t(x_p,t_q) + \theta u(x_p,t_q) - G(x_p,t_q)| \cong 0$$

and $E(x_p,t_q) \le 10^{-k_{pq}}$ (k_{pq} is positive integer). If max $10^{-k_{pq}} = 10^{-k}$ (k is positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_p,t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}.

On the other hand, the error can be estimated by L_∞ and L_2 errors and root-mean-square error (RMS).We calculate RMS error by the following formula [7]:

$$RMS\ error = \sqrt{\frac{\sum_{i=1}^{N+1}\left(u(x_i,\tau) - \hat{u}(x_i,\tau)\right)}{N+1}}$$

where u and $\hat{u}$ are the exact and approximate solutions of the problem, respectively and τ is an arbitrary time t in $[0,T]$.

6.Illustrative Examples

In this section, three examples are given to demonstrate the applicability, efficiency and accuracy of the proposed method. Each example is modeled using the mathematical software package MATLAB 7.12.0 and all calculations are run on a Pentium 4 PC Laptop with 2 GHz of CPU and 2 GB RAM.

Example 1.[19] Let us find the solution $u = u(x,t)$ on E^2 of the Cauchy problem

$$u_{xx} - u_{tt} = 0, \quad u(x,0) = x^2, \quad \frac{\partial u(x,0)}{\partial t} = 4x^3.$$

We assume that the problem has a solution in the form

$$u(x,t)=\sum_{r=0}^{3}\sum_{s=0}^{3}a_{r,s}B_r(x)B_s(t)$$

The fundamental matrix equation of this problem and its solution, respectively, become

$$\left\{(\bar{B})^2-(B)^2\right\}A=G$$

and following the proposed method leads to

$$u(x,t)=x^2+t^2+4xt^3+4x^3t,$$

which is the exact solution.
In the following lines, the MATLAB codes of this example are provided.

```
N=3;
B=vertcat(horzcat(zeros(N,1),diag(1:N)),zeros(1,N+1));
Bh=blkdiag(B,B,B,B);
B0=blkdiag(eye(N+1),2*eye(N+1),3*eye(N+1));
Bb=vertcat(horzcat(zeros((N+1)^2-(N+1),N+1),B0),zeros(N+1,(N+1)^2));
syms x t;
for i=1:N+1
I(i)=(i-1);
end
X=x.^I;
D=zeros(N+1);
for i=1:N+1
for j=1:i
D(i,j)=(factorial(i-1)/(factorial(j-1)*factorial(i-j)))* mfun('bernoulli',i-j,0);
end
end
B=zeros(N+1,1);
B=sym(B);
for i=1:N+1
s=0;
for j=1:N+1
s=s+D(i,j).*X(j);
end
B(i)=s;
end
T=zeros(1,N+1);
for i=1:N+1
T(i)=subs(B(i),x,0);
end
Q=kron(eye(N+1),T);
G=zeros((N+1)^2,1);
f=x.^2;
m=4*x.^3;
F=zeros(N+1,1);
M=zeros(N+1,1);
for i=1:N+1
F(i)=(1/factorial(i-1))*int(diff(f,i-1),0,1);
M(i)=(1/factorial(i-1))*int(diff(m,i-1),0,1);
end
W=(Bb)^2-(Bh)^2;
WW=zeros((N+1)^2-2*(N+1),(N+1)^2);
for i=1:(N+1)^2-2*(N+1)
```

```
  for j =1:(N +1)^2
WW(i,j)=W(i,j);
  end
end
WWW=vertcat(WW,Q,Q*(Bh));
GG = zeros((N +1)^2 -2*(N +1),1);
for i =1:(N +1)^2 -2*(N +1)
GG(i)=G(i);
end
GGG=vertcat(GG,F,M);
A=inv(WWW)*(GGG);
BX=subs(B,x,x);
BT=subs(B,x,t);
U=simplify(kron(BX',BT')*A)
```

Example 2.[7] Consider linear Klein–Gordon equation, with the following boundary conditions:

$$u_{xx} - u_{tt} = t^2 x^2, \quad u(0,t) = 0, \quad u(1,t) = \frac{t^2}{2}$$

Suppose $N = 2$. The fundamental matrix equation of this problem and its solution, respectively, become

$$\left\{(\bar{B})^2 - (B)^2\right\} A = G$$

and $u(x,t) = \dfrac{x^2 t^2}{2}$, which is the exact solution. The reader can design a similar MATLAB codes like as the first example for solving this example easily.

Example 3.[7], [30] Let us consider the telegraph equation

$$u_{tt} + 4u_t + 2u = u_{xx}$$

under the initial conditions

$$u(x,0) = sin(x), \quad u_t(x,0) = -sin(x),$$

with the exact solution $u(x,t) = e^{-t} sin(x)$.

Following the procedure in Section 4, we find the fundamental matrix equation as follows:

$$\left\{(B)^2 + 4B + 2I - (\bar{B})^2\right\} A = G.$$

By taking $N = 9,11$, we solved the above problem by means of the fundamental matrix equation. The errors associated with the present method and Taylor method [7] for different values of N are compared in Table 1. From this Table one can see that the associated errors of the Taylor method usually increase during the computational interval $[0,1]$, meanwhile our Bernoulli method have a stable behavior in this interval. With the experience of the author, the Taylor method have unstable behavior to dealing with long variation functions (such as exponential or hyperbolic functions) with regard to Bernoulli method in the interval [0,1]. Moreover, outside the interval [0,1] the Bernoulli method again achieve to more accurate solutions with respect to the Taylor method. In addition, the Taylor method obtain better solutions with respect to Legendre multiwavelet method [30]; this issue support the efficiency of the proposed method.

Table 1. Error Analysis of Example 3 for $N = 9,11$

(x_r, t_r)	$N = 9$ (Taylor)	$N = 9$ (Bernoulli)	$N = 11$ (Taylor)	$N = 11$ (Bernoulli)
(0, 0)	0	9.84E-09	0	2.49E-10
(0.1, 0.1)	3.05E-16	1.15E-08	2.30E-18	1.39E-10
(0.2, 0.2)	7.01E-13	1.41E-08	7.20E-16	1.77E-10
(0.3, 0.3)	6.99E-11	1.86E-09	1.66E-13	4.68E-11
(0.4, 0.4)	1.90E-09	7.21E-08	8.12E-12	2.38E-09
(0.5, 0.5)	2.53E-08	2.08E-07	1.71E-10	7.81E-09

(0.6, 0.6)	2.15E-07	3.46E-07	2.11E-09	1.41E-08
(0.7, 0.7)	1.34E-06	3.30E-07	1.81E-09	1.44E-08
(0.8, 0.8)	6.62E-06	2.10E-08	1.18E-07	1.27E-09
(0.9, 0.9)	2.74E-05	7.08E-07	6.30E-07	3.95E-08
(1,1)	9.90E-05	9.69E-07	2.84E-06	9.81E-08
(1.1, 1.1)	3.19E-04	2.74E-06	1.12E-05	1.94E-07
(1.2, 1.2)	9.36E-04	2.36E-05	3.98E-05	5.48E-07
(1.3, 1.3)	2.54E-03	1.05E-04	1.28E-04	2.35E-06
(1.4, 1.4)	6.44E-03	3.79E-04	3.84E-04	1.04E-05
(1.5, 1.5)	1.54E-02	1.19E-03	1.07E-03	4.13E-05
(1.6, 1.6)	3.50E-02	3.43E-03	2.81E-03	1.45E-04
(1.7, 1.7)	7.61E-02	9.13E-03	7.00E-03	4.65E-04
(1.8, 1.8)	1.58E-01	2.26E-02	1.66E-02	1.36E-03
(1.9, 1.9)	3.20E-01	5.31E-02	3.79E-02	3.74E-03
(2, 2)	6.24E-01	1.18E-01	8.32E-02	9.67E-03

7. Conclusions

Obtaining the analytic solutions of second-order linear hyperbolic partial differential equations with constant coefficients are usually difficult. In many cases, it is required to approximate solutions. For this reason, a new approach which is based on the two dimensional Bernoulli operational matrix of differentiation is proposed.

The solution procedure is very simple by means of Bernoulli expansion in two variables and only few steps lead to high accurate solutions. The main goal of the presented technique was deriving an approximation to the solution of hyperbolic partial differential equations. To illustrate the method and its efficiency, three examples were provided. In the first two examples, we obtained the exact solution. In Example 3, we made a comparison between Taylor method [7] and present Bernoulli series method. In Table 1, it seems that the Bernoulli method is not good as Taylor method for small values of (x,t); but increasing (x,t), (for a fixed number N) the Bernoulli method is better than Taylor method.

It is observed that the proposed method has the best advantage when the known functions in equation can be expanded to Bernoulli series. To get the best approximation, we must take more terms from the Bernoulli expansion of functions; that is, the truncation limit N must be chosen large enough. Another considerable advantage of the method is that the Nth-order approximation gives the exact solution when the solution is polynomial of degree equal to or less than N. If the solution is not polynomial, Bernoulli series approximation converges to the exact solution as N increases. Besides, the results obtained in this paper are sometimes better than results obtained by the other methods in the references. Moreover, this method is applicable for the approximate solution of parabolic and elliptic type partial differential equations with constant or variable coefficients, and also can be extended to higher-order partial differential equations.

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