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Common Fixed Point Theorems in Ordered Complex Valued Generalized Metric Spaces

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Abstract: In 2013, Radenovic et al. [4] introduced a notion of complex valued generalized metric space and obtained common fixed point result for mappings in such spaces. In this paper, we establish some common fixed point results using some rational inequalities.

Keywords: Weakly increasing map, complex valued generalized metric spaces, partially ordered set, common fixed point

1 Introduction

The concept of a complex valued metric space which is a generalization of the classical metric space was recently introduced by Azam, Fisher and Khan [2]. An ordinary metric d is a real-valued function from a set $X \times X$ into $\mathbb{R}$, where X is a nonempty set. That is, $d : X \times X \to \mathbb{R}$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $\mathrm{Re}(z)$ and second coordinate is called $\mathrm{Im}(z)$. Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex numbers. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$, that is $z_1 \precsim z_2$, if one of the following holds:

(C1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
(C2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
(C3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
(C4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 < z_2$ if only (C4) is satisfied.

Remark 1.1. We note that the following statements hold:

(i). $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz\ \forall z \in \mathbb{C}$,
(ii). $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,
(iii). $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

In 2011, Azam et al. [2] defined the complex-valued metric space (X, d) in the following way:

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i). $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(ii). $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii). $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = 2i|z_1 - z_2|, \text{ for all } z_1, z_2 \in X.$$

Then (X, d) is a complex valued metric space.

In 2013, Radenovic et. al. [4] introduced the concept of complex valued generalized metric space as follows:

Definition 1.4. Let X be a nonempty set. If a mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i). $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(ii). $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii). $d(x,y) \precsim d(x,u) + d(u,v) + d(v,y)$, for all $x, y \in X$ and all distinct $u, v \in X$ each one is different from x and y.

Then d is called a complex valued generalized metric on X and (X, d) is called a complex valued generalized metric space.

Example 1.5. Let $X = \{1, -1, i, -i\}$. Define $d : X \times X \to \mathbb{C}$ as follows:

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$d(1,1)=d(-1,-1)=d(i,i)=d(-i,-i)=0,$
$d(-1,i)=d(i,-1)=d(1,i)=d(i,1)=2e^{i\theta},$
$d(1,-1)=d(-1,1)=7e^{i\theta},$
$d(1,-i)=d(-i,1)=d(-1,-i)=d(-i,-1)=d(i,-i)=d(-i,i)=4e^{i\theta}.$

It is easy to verify that (X,d) is a complex valued generalized metric space when $\theta\in[0,\pi/2]$. Note that

$$7e^{i\theta}=d(1,-1)>d(1,i)+d(i,-1)=4e^{i\theta}.$$

So, (X,d) is not a complex valued metric space.

Definition 1.6. Let (X,d) be a complex valued generalized metric space. A sequence $\{x_n\}$ in X is said to be

(i). Convergent to x, if for every $c\in\mathbb{C}$, with $0\prec c$ there is $k\in\mathbb{N}$ such that for all $n>k$, $d(x_n,x)\prec c$. We denote this by $\{x_n\}\to x$ as $n\to\infty$ or $\lim_{n\to\infty}x_n\to x$.
(ii). Cauchy sequence, if for every $c\in\mathbb{C}$, with $0\prec c$ there is $k\in\mathbb{N}$ such that for all $n,m>k$, $d(x_n,x_m)\prec c$.
(iii). Complete, if every Cauchy sequence in X converges in X.

Lemma 1.7. Let (X,d) be a complex valued generalized metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.

Lemma 1.8. Let (X,d) be a complex valued generalized metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_m)|\to 0$ as $n,m\to\infty$.

In 2010, Altun et. al. [1] introduced the concept of weakly increasing map as follows:

Definition 1.9. Let $(X,\lesssim)$ be a partially ordered set. A pair (f,g) of self-maps of X is said to be weakly increasing if $fx\lesssim gfx$ and $gx\lesssim fgx$ for all $x\in X$. If $f=g$, then we have $fx\lesssim f^2x$ for all x in X and in this case, we say that f is a weakly increasing map.

Definition 1.10. A non-empty subset W of a partially ordered set X is said to be totally ordered if every two elements of W are comparable.

2 Main Results

Theorem 2.1. Let f and g be weakly increasing self-mappings of a complete complex valued generalized metric space (X,d). Suppose that, for every comparable $x,y\in X$, we have either

$$d(fx,gy)\lesssim k\left[\frac{d(x,fx)d(x,gy)+\{d(x,y)\}^2+d(x,fx)d(x,y)}{d(x,fx)+d(x,y)+d(x,gy)}\right], \tag{1}$$

in case $d(x,fx)+d(x,y)+d(x,gy)\neq 0$, $0<k<1$, or

$$d(fx,gy)=0,\ \text{if } d(x,fx)+d(x,y)+d(x,gy)=0. \tag{2}$$

If f or g is continuous or for any non-decreasing sequence $\{x_n\}$ with $x_n\to z$ in X we necessarily have $x_n\lesssim z$ for all $n\in\mathbb{N}$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is totally ordered if and only if f and g have unique common fixed point.

Proof. First, we shall show that if f or g has a fixed point, then it is a common fixed point of f and g. Let u be a fixed point of f. Then from (1) with $x=y=u$, we have

$$\begin{aligned}d(u,gu)&=d(fu,gu)\\&\lesssim k\left[\frac{d(u,fu)d(u,gu)+\{d(u,u)\}^2+d(u,fu)d(u,u)}{d(u,fu)+d(u,u)+d(u,gu)}\right]\\&=0,\quad\text{which implies that, } u=gu.\end{aligned}$$

So, u is the common fixed point of f and g. Similarly, if u is a fixed point of g, then it is also fixed point of f.

Now, let x_0 be an arbitrary point of X. If $fx_0=x_0$, then the proof is finished.

Assume that $fx_0\neq x_0$. Construct a sequence $\{x_n\}$ in X as follows:

$$x_1=fx_0\lesssim gfx_0=gx_1=x_2,\ \text{and}$$
$$x_2=gx_1\lesssim fgx_1=fx_2=x_3.$$

Continuing in this way, we have $x_1\lesssim x_2\lesssim\ldots\lesssim x_n\lesssim x_{n+1}\lesssim\ldots$. Assume that $d(x_{2n},x_{2n+1})\succ 0$, for every $n\in\mathbb{N}$. If not, then $x_{2n}=x_{2n+1}$ for some n. For all those n, $x_{2n}=x_{2n+1}=fx_{2n}$ and the proof is finished.

Assume that $d(x_{2n},x_{2n+1})\succ 0$ for $n=0,1,2,\ldots$. As x_{2n} and x_{2n+1} are comparable, so we have

$$\begin{aligned}d(x_{2n+1},x_{2n+2})&=d(fx_{2n},gx_{2n+1})\\&\lesssim k\Big[d(x_{2n},fx_{2n})d(x_{2n},gx_{2n+1})+\{d(x_{2n},x_{2n+1})\}^2\\&\qquad+d(x_{2n},fx_{2n})d(x_{2n},x_{2n+1})\Big]\times\\&\qquad[d(x_{2n},fx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},gx_{2n+1})]^{-1}\\&=k\Big[d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+\{d(x_{2n},x_{2n+1})\}^2\\&\qquad+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1})]\times\\&\qquad[d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})]^{-1}\\&=kd(x_{2n},x_{2n+1})\,[d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})\\&\qquad+d(x_{2n},x_{2n+1})]\times\\&\qquad[d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})]^{-1}\\&=kd(x_{2n},x_{2n+1}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}d(x_{2n},x_{2n+1})&=d(fx_{2n},gx_{2n-1})\\&\lesssim k\Big[d(x_{2n},fx_{2n})d(x_{2n},gx_{2n-1})+\{d(x_{2n},x_{2n-1})\}^2\\&\qquad+d(x_{2n},fx_{2n})d(x_{2n},x_{2n-1})]\\&\qquad[d(x_{2n},fx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},gx_{2n-1})]^{-1}\\&=k\Big[d(x_{2n},x_{2n+1})d(x_{2n},x_{2n})+\{d(x_{2n},x_{2n-1})\}^2\\&\qquad+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n-1})]\\&\qquad[d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n})]^{-1}\\&=kd(x_{2n},x_{2n-1})\left[\frac{d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n+1})}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n-1})}\right]\\&=kd(x_{2n},x_{2n-1})=kd(x_{2n-1},x_{2n}).\end{aligned}$$

Hence, for all $n \geq 0$, we have $d(x_{n+1},x_{n+2}) \precsim hd(x_n,x_{n+1})$.

Consequently, we have

$d(x_{n+1},x_{n+2}) \precsim hd(x_n,x_{n+1}) \precsim h^2 d(x_n,x_{n+1}) \precsim \ldots \precsim h^{n+1} d(x_n,x_{n+1})$, for all $n \geq 0$. Now, for all $m > n$, we have

$$\begin{aligned} d(x_m,x_n) &\precsim d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_m,x_{m-1}) \\ &\precsim h^n d(x_0,x_1) + h^{n+1} d(x_0,x_1) + \ldots + h^{m-1} d(x_0,x_1) \\ &\precsim \frac{h^n}{1-h} d(x_0,x_1). \end{aligned}$$

Therefore, we have

$$|d(x_m,x_n)| = \frac{h^n}{1-h} |d(x_0,x_1)|.$$

Hence,

$$lim_{n\to\infty} |d(x_m,x_n)| = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. But X is complete metric space, so $\{x_n\}$ is convergent to some point, say u, in X, i.e., $x_n \to u$ as $n \to \infty$.

If f or g is continuous, then it is clear that $fu = u = gu$. If neither f, nor g is continuous, then by given assumption $x_n \precsim u$ for all $n \in \mathbb{N}$. We claim that u is a fixed point of g. If not, then $d(u, fu) = z > 0$. From (1), we obtain

$$\begin{aligned} z &\precsim d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2}, fu) \\ &= d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(fu,gx_{n+1}) \\ &\precsim d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) \\ &\quad +k\left[d(u, fu)d(u,gx_{n+1}) + \{d(u,x_{n+1})\}^2 + d(u, fu)d(u,x_{n+1})\right] \\ &\qquad \left[d(u, fu) + d(u,x_{n+1}) + d(u,gx_{n+1})\right]^{-1} \\ &= d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) \\ &\quad +k\left[d(u, fu)d(u,x_{n+2}) + \{d(u,x_{n+1})\}^2 + d(u, fu)d(u,x_{n+1})\right] \\ &\quad \left[d(u, fu) + d(u,x_{n+1}) + d(u,x_{n+2})\right]^{-1}, \text{ and so,} \\ |z| &\leq |d(u,x_{n+1})| + |d(x_{n+1},x_{n+2})| \\ &\quad +k\Big|d(u, fu)d(u,x_{n+2}) + d(u,x_{n+1})^2 + d(u, fu)d(u,x_{n+1}) \\ &\qquad \left[d(u, fu) + d(u,x_(n+1)) + d(u,x_(n+2))\right]^{-1}\Big|. \end{aligned}$$

Letting $n \to \infty$, we get $|z| \leq 0$, a contradiction, and so $u = fu$. Therefore $fu = gu = u$.

Now, suppose that the set of common fixed points of f and g is totally ordered. We prove that common fixed point of f and g is unique. Let, if possible, u and v are distinct common fixed points of f and g.

Putting $x = u, y = v$ in (1), we get

$$\begin{aligned} d(u,v) &= d(fu,gv) \\ &\precsim k\left[\frac{d(u, fu)d(u,gv) + \{d(u,v)\}^2 + d(u, fu)d(u,v)}{d(u, fu) + d(u,v) + d(u,gv)}\right] \\ &\precsim k\left[\frac{d(u,u)d(u,v) + \{d(u,v)\}^2 + d(u,u)d(u,v)}{d(u,u) + d(u,v) + d(u,v)}\right] \\ &= kd(u,v), \text{ which implies that,} \end{aligned}$$

$|d(u,v)| = k|d(u,v)|$, a contradiction, Hence $u = v$.

Conversely, if f and g have only one common fixed point then the set of common fixed point of f and g being singleton is totally ordered.

Corollary 2.2. Let f be a weakly increasing self-mapping of a complex valued generalized metric space (X,d). Suppose that, for every comparable $x,y \in X$, we have either

$$d(fx, fy) \precsim k\left[\frac{d(x, fx)d(x, fy) + \{d(x,y)\}^2 + d(x, fx)d(x,y)}{d(x, fx) + d(x,y) + d(x, fy)}\right], \quad (3)$$

in case $d(x, fx) + d(x,y) + d(x, fy) \neq 0$, $0 < k < 1$, or

$$d(fx, fy) = 0, \text{ if } d(x, fx) + d(x,y) + d(x, fy) = 0. \quad (4)$$

If f is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \precsim z$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of common fixed points of f is totally ordered if and only if f has unique fixed point.

Example 2.3. Let $X = \{1,3,5,7\}$ be endowed with order $x \precsim y$ if and only if $y \precsim x$. Then $\precsim$ is a partial order in X. Define a generalized metric $d : X \times X \to \mathbb{C}$ as follows:

(x,y)	d(x,y)
$(1,1),(3,3),(5,5),(7,7)$	0
$(1,3),(3,1),(3,5),(5,3)$	$2e^{i\alpha}$
$(1,5),(5,1)$	$5e^{i\alpha}$
$(1,7),(7,1),(3,7),(7,3),(5,7),(7,5)$	$7e^{i\alpha}$

Clearly (X,d) is a complex valued generalized metric space for $\alpha \in [0,\pi/2]$. Here, we note that $d(1,5) = 5e^{i\alpha} \not\precsim 4e^{i\alpha} = d(1,3) + d(3,5)$, so (X,d) is not a complex valued metric space.

We define $f,g : X \to X$ by

$$fx = 1 \text{ for } x \in X \text{ and}$$

$$gx = \begin{cases} 1, & \text{for } x \in \{1,3,5\} \\ 3, & \text{for } x = 7. \end{cases}$$

Note that $fx \precsim fgx$ and $gx \precsim gfx$ for all x in X.

For $k = 1/2$, we consider the following cases:

(i).If $x \in X$ and $y \in X\backslash 7$, then we have $fx = gx = 1$ and so $d(fx,gy) = 0$ and (1) is satisfied obviously.

(ii).When $x \in X$ and $y = 7$, then we have $fx = 1$, $gy = 3$ and $d(fx,gy) = d(1,3) = 2e^{i\alpha}$.

Also,

$$\begin{aligned} &\frac{d(x, fx)d(x,gy) + \{d(x,y)\}^2 + d(x, fx)d(x,y)}{d(x, fx) + d(x,y) + d(x,gy)} \\ &= \frac{d(1,1)d(1,3) + \{d(1,7)\}^2 + d(1,1)d(1,7)}{d(1,1) + d(1,7) + d(1,3)} \\ &= \frac{49e^{2i\alpha}}{7e^{i\alpha} + 2e^{i\alpha}} = \frac{49}{9}e^{i\alpha} \end{aligned}$$

Clearly, we have

$$d(fx,gy) \precsim k\left[\frac{d(x, fx)d(x,gy) + \{d(x,y)\}^2 + d(x, fx)d(x,y)}{d(x, fx) + d(x,y) + d(x,gy)}\right].$$

Thus, the conditions of Theorem 2.1 are satisfied for $k = 1/2$. Moreover, 1 is the unique common fixed point of f and g.

Theorem 2.4. Let f and g be weakly increasing self-mappings of a complex valued generalized metric space (X,d). Suppose that, for every comparable $x,y \in X$, we have either

$$d(fx,gy) \precsim \frac{k}{d(x,fx)+d(y,gy)} \max\Big[\{d(x,y)\}^2, \{d(x,fx)\}^2, \{d(y,gy)\}^2, \frac{1}{2}\{d(x,gy)\}^2, \frac{1}{2}d(y,fx)^2\Big], \quad (5)$$

in case $d(x,fx)+d(y,gy) \neq 0$, $0<k<1$, or

$$d(fx,gy) = 0, \text{ if } d(x,fx)+d(y,gy) = 0. \quad (6)$$

If f or g is continuous or for any non-decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \precsim z$ for all $n \in \mathbb{N}$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is totally ordered if and only if f and g have unique common fixed point.

Proof. First, we shall show that if f or g has a fixed point, then it is a common fixed point of f and g. Let u be a fixed point of f. Then from (5) with $x=y=u$, we have for $u \neq gu$,

$$\begin{aligned} d(u,gu) &= d(fu,gu) \\ &\precsim \frac{k}{d(u,fu)+d(u,gu)} \max\Big[\{d(u,u)\}^2, \{d(u,fu)\}^2, \{d(u,gu)\}^2, \frac{1}{2}\{d(u,gu)\}^2, \frac{1}{2}\{d(u,fu)\}^2\Big] \\ &= \frac{k \max\left[0,0,\{d(u,gu)\}^2, \frac{1}{2}\{d(u,gu)\}^2, 0\right]}{0+d(u,gu)} \\ &= kd(u,gu), \quad \text{which implies that} \end{aligned}$$

$|d(u,gu)| \leq k|d(u,gu)|$, a contradiction and so $gu=u=fu$. Therefore u is the common fixed point of f and g. Similarly, if u is a fixed point of g, then it is also fixed point of f.

Now, let x_0 be an arbitrary point of X. If $fx_0 = x_0$, then the proof is finished.

Assume that $fx_0 \neq x_0$. Construct a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned} x_1 &= fx_0 \precsim gfx_0 = gx_1 = x_2, \quad \text{and} \\ x_2 &= gx_1 \precsim fgx_1 = fx_2 = x_3. \end{aligned}$$

Continuing in this way, we have $x_1 \precsim x_2 \precsim \ldots \precsim x_n \precsim xn+1 \precsim \ldots$. Assume that $d(x_{2n},x_{2n+1}) \succ 0$, for every $n \in \mathbb{N}$. If not, then $x_{2n} = x_{2n+1}$ for some n. For all those n, $x_{2n} = x_{2n+1} = fx_{2n}$ and the proof is finished.

Assume that $d(x_{2n},x_{2n+1}) \succ 0$ for $n=0,1,2,\ldots$. As x_{2n} and x_{2n+1} are comparable, so we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(fx_{2n},gx_{2n+1}) \\ &\precsim \frac{k}{d(x_{2n},fx_{2n})+d(x_{2n+1},gx_{2n+1})} \max\Big[\{d(x_{2n},x_{2n+1})\}^2, \{d(x_{2n},fx_{2n})\}^2, \{d(x_{2n+1},gx_{2n+1})\}^2, \frac{1}{2}\{d(x_{2n},gx_{2n+1})\}^2, \frac{1}{2}\{d(x_{2n+1},fx_{2n})\}^2\Big] \\ &= \frac{k}{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})} \max\Big[\{d(x_{2n},x_{2n+1})\}^2, \{d(x_{2n},x_{2n+1})\}^2, \{d(x_{2n+1},x_{2n+2})\}^2, \frac{1}{2}\{d(x_{2n},x_{2n+2})\}^2, \frac{1}{2}\{d(x_{2n+1},x_{2n+1})\}^2\Big] \\ &= \frac{k}{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})} \max\Big[\{d(x_{2n},x_{2n+1})\}^2, \{d(x_{2n+1},x_{2n+2})\}^2, \frac{1}{2}\{d(x_{2n},x_{2n+2})\}^2\Big]. \end{aligned} \quad (7)$$

If

$$\max\Big[\{d(x_{2n},x_{2n+1})\}^2, \{d(x_{2n+1},x_{2n+2})\}^2, \frac{1}{2}\{d(x_{2n},x_{2n+2})\}^2\Big] = \{d(x_{2n},x_{2n+1})\}^2 \quad (8)$$

then from (7), we have

$$d(x_{2n+1},x_{2n+2}) \precsim k\frac{\{d(x_{2n},x_{2n+1})\}^2}{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}, \text{ that is,}$$

$$\{d(x_{2n+1},x_{2n+2})\}^2 + d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2}) - k\{d(x_{2n},x_{2n+1})\}^2 = 0.$$

The positive root of the quadratic equation $t^2+tk=0$ is $q = \frac{1}{2}\left[(1+4k)^{1/2}-1\right]$, and since $k<1$, it follows that, $q<1$.

Thus, we have

$$d(x_{2n+1},x_{2n+2}) \precsim qd(x_{2n},x_{2n+1}), \text{ for } n=0,1,2,\ldots. \quad (9)$$

If max of the three numbers in the braces on the left side in (8) is $\{d(x_{2n+1},x_{2n+2})\}^2$, then from (7), we have $d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}) \precsim kd(x_{2n+1},x_{2n+2})$, which is impossible, since $k<1$.

If max of the three numbers in the braces on the left side in (8) is $\frac{1}{2}\{d(x_{2n},x_{2n+2})\}^2$, then from (7), we have $d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}) \precsim kd(x_{2n+1},x_{2n+2})$, which is impossible, since $k<1$.

If max of the three numbers in the braces on the left side in (8) is $\frac{1}{2}\{d(x_{2n},x_{2n+2})\}^2$, then from (7), we have

$$(2-k)\{d(x_{2n+1},x_{2n+2})\}^2 + (2-2k)d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2}) - k\{d(x_{2n},x_{2n+1})\}^2 = 0.$$

The positive root of the quadratic equation $(2-k)p^2 + (2-2k)pk = 0$ is $r = \frac{k}{(2-k)}$, and since $k<1$, it follows that $r<1$. Thus, we get

$$d(x_{2n+1},x_{2n+2}) \precsim rd(x_{2n},x_{2n+1}), \text{ for } n=0,1,2,\ldots. \quad (10)$$

Hence by putting $h = \max\{q,r\}$, we have from (9) and (10)

$$d(x_{2n+1},x_{2n+2}) \precsim hd(x_{2n},x_{2n+1}), \text{ for } n = 0,1,2,\ldots. \quad (11)$$

Hence for all $n = 0$, we have $d(x_{n+1},x_{n+2}) \precsim hd(x_n,x_{n+1})$.

Consequently, we have
$d(x_{n+1},x_{n+2}) \precsim hd(x_n,x_{n+1}) \precsim h^2 d(x_n,x_{n+1}) \precsim \ldots \precsim h^{n+1}d(x_n,x_{n+1})$,
for all $n = 0$. Now, for all $m > n$, we have

$$\begin{aligned} d(x_m,x_n) &\precsim d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_m,x_{m-1}) \\ &\precsim h^n d(x_0,x_1) + h^{n+1}d(x_0,x_1) + \ldots + h^{m-1}d(x_0,x_1) \\ &\precsim \frac{h^n}{(1-h)} d(x_0,x_1). \end{aligned}$$

Therefore, we have
$|d(x_m,x_n)| = \dfrac{h^n}{(1-h)}|d(x_0,x_1)|$.
Hence,
$\lim_{n\to\infty} |d(x_m,x_n)| = 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in X. But X is complete metric space, so $\{x_n\}$ is convergent to some point, say u, in X, i.e., $x_n \to u$ as $n \to \infty$.

If f or g is continuous, then it is clear that $fu = u = gu$. If neither f, nor g is continuous, then by given assumption $x_n \precsim u$ for all $n \in \mathbb{N}$. We claim that u is a fixed point of g. If not, then
$d(u, fu) = z > 0$.
From (5), we obtain

$$\begin{aligned} z &\precsim d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2}, fu) \\ &= d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(fu,gx_{n+1}) \\ &\precsim d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) \\ &\quad + \frac{k}{d(u, fu) + d(x_{n+1},gx_{n+1})} \max\Big[\{d(u,x_{n+1})\}^2, \\ &\quad \{d(u, fu)\}^2, \{d(x_{n+1},gx_{n+1})\}^2, \\ &\quad \frac{1}{2}\{d(u,gx_{n+1})\}^2, \frac{1}{2}\{d(x_{n+1}, fu)\}^2\Big] \\ &\precsim d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) \\ &\quad + \frac{k}{d(u, fu) + d(x_{n+1},x_{n+2})} \max\Big[\{d(u,x_{n+1})\}^2, \\ &\quad \{d(u, fu)\}^2, \{d(x_{n+1},x_{n+2})\}^2, \\ &\quad \frac{1}{2}\{d(u,x_{n+2})\}^2, \frac{1}{2}\{d(x_{n+1}, fu)\}^2\Big] \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} z &\precsim d(u,u) + d(u,u) + \frac{k}{d(u, fu) + d(u,u)} \times \\ &\quad \max\Big[\{d(u,u)\}^2, \{d(u, fu)\}^2, \{d(u,u)\}^2, \\ &\quad \frac{1}{2}\{d(u,u)\}^2, \frac{1}{2}\{d(u, fu)\}^2\Big] \\ &= kd(u, fu), \text{ that is,} \end{aligned}$$

$|z| \le k|d(u, fu)| = k|z|$, which is a contradiction, and so, $u = fu$. Therefore $fu = gu = u$.

Now, suppose that the set of common fixed points of f and g is totally ordered. We prove that common fixed point of f and g is unique. Let, if possible, u and v are distinct common fixed points of f and g.

Putting $x = u, y = v$ in (5), we get

$$\begin{aligned} d(u,v) &= d(fu,gv) \\ &\precsim \frac{k}{d(u, fu) + d(v,gv)} \max\Big[\{d(u,v)\}^2, \{d(u, fu)\}^2, \\ &\quad \{d(v,gv)\}^2, \frac{1}{2}\{d(u,gv)\}^2, \frac{1}{2}\{d(v, fu)\}^2\Big] \\ &= \frac{k}{d(u,u) + d(v,v)} \max\Big[\{d(u,v)\}^2, \{d(u,u)\}^2, \{d(v,v)\}^2, \\ &\quad \frac{1}{2}\{d(u,v)\}^2, \frac{1}{2}\{d(v,u)\}^2\Big], \text{ which implies that,} \end{aligned}$$

$d(u,v) = 0$, a contradiction. Hence $u = v$.

Conversely, if f and g have only one common fixed point then the set of common fixed point of f and g being singleton is totally ordered.

Remark 2.5. We note that the conclusion of Theorem 2.4 remains valid if we replace the condition (5) by the condition of the type

$$d(fx,gy) \precsim k \max\left[\frac{\{d(x, fx)\}^2 + \{d(y,gy)\}^2}{d(x, fx) + d(y,gy)}, \frac{1}{2}\frac{\{d(x,gy)\}^2 + \{d(y, fx)\}^2}{d(x, fx) + d(y,gy)}\right]. \quad (12)$$

Corollary 2.6. Let f be a weakly increasing self-mapping of a complex valued generalized metric space (X,d). Suppose that, for every comparable $x,y \in X$, we have either

$$d(fx, fy) \precsim \frac{k}{d(x, fx) + d(y, fy)} \max\Big[\{d(x,y)\}^2, \{d(x, fx)\}^2, \{d(y, fy)\}^2, \frac{1}{2}\{d(x, fy)\}^2, \frac{1}{2}\{d(y, fx)\}^2\Big] \quad (13)$$

in case $d(x, fx) + d(y, fy) \neq 0$, $0 < k < 1$, or

$$d(fx, fy) = 0, if d(x, fx) + d(y, fy) = 0. \quad (14)$$

If f is continuous or for any non-decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \precsim z$ for all $n \in \mathbb{N}$,

3 Periodic point result

A fixed point u of f is also a fixed point of f^n for every natural n. However, the converse is not true. For example, consider $X = [-1,0]$ and define f by $fx = -1-x$. Then f has a unique fixed point $-\frac{1}{2}$ and every even iterate of f is the identity map, which has every point of $[-1,0]$ as a fixed point. On the other hand if $X = [0,\pi]$, $fx = \sin x$, then every iterate of f has the same fixed point as f.

If a map f satisfies $F(f) = F(f^n)$ for every natural n, where $F(f)$ is the set of fixed points of f, then it is said to have property P[3]. The set $O(x,\infty) = \{x, fx, f^2x, \dots\}$ is called the orbit of x.

Theorem 3.1. Let f be a self map of a complete complex valued generalized metric space (X,d) satisfying the conditions as in Corollary 2.2. If $O(x,\infty)$ is totally ordered, then f has property P.

Proof. From Corollary 2.2, f has a fixed point. Let $u \in F(f^n)$.

From (4), we have

$$\begin{aligned} d(fu,u) &= d(f(f^n u), f(f^{n-1}u)) \\ &\lesssim k\big[d(f^n u, f(f^n u))d(f^n u, f(f^{n-1}u))+ \\ &\quad \{d(f^n u, f^{n-1}u)\}^2 + d(f^n u, f(f^n u))d(f^n u, f^{n-1}u)\big] \times \\ &\quad \big[d(f^n u, f(f^n u)) + d(f^n u, f^{n-1}u) + d(f^n u, f(f^{n-1}u))\big]^{-1} \\ &= k\big[d(f^n u, f^{n+1}u)d(f^n u, f^n u)+ \\ &\quad \{d(f^n u, f^{n-1}u)\}^2 + d(f^n u, f^{n+1}u)d(f^n u, f^{n-1}u)\big] \times \\ &\quad \big[d(f^n u, f^{n+1}u) + d(f^n u, f^{n-1}u) + d(f^n u, f^n u)\big]^{-1} \\ &= k\left[\frac{\{d(u, f^{n-1}u)\}^2 + d(u, fu)d(u, f^{n-1}u)}{d(u, fu) + d(u, f^{n-1}u)}\right] \\ &= kd(u, f^{n-1}u). \end{aligned}$$

Obviously, we have

$$\begin{aligned} d(u, fu) &= d(fu, f^n u) \\ &\lesssim kd(f^{n-1}u, f^n u) \lesssim k^2 d(f^{n-2}u, f^{n-1}u) \lesssim \\ &\quad \dots \lesssim k^n d(u, fu). \end{aligned}$$

Since $0 < k < 1$, implies, $d(u, fu) = 0$ and $u = fu$.

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