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Variational Iteration Technique and Some Methods for the Approximate Solution of Nonlinear Equations

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Abstract: In this paper, we suggest and analyze a new recurrence relation which generates the iterative methods of higher order for solving nonlinear equation $f(x) = 0$. This general recurrence relation is obtained by using variational iteration technique. We purpose some new iterative methods for solving nonlinear equations. We also test different examples to illustrate the efficiency of these new methods. Comparison with other similar methods is also given. These new methods can also be considered as alternative to the existing methods. This technique can be applied to suggest a wide class of new iterative methods for solving nonlinear equations.

Keywords: Variational iteration technique; Iterative method; Convergence; Newton's method; Taylor series; Examples

1 Introduction

Iterative methods for finding the approximate solutions of the nonlinear equation $f(x) = 0$, are being developed by using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques. See [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17] and the references therein.

Let us consider the nonlinear equation of the type

$$f(x) = 0. \tag{1}$$

We can rewrite (1), in the following equivalent form as:

$$x = H(x) \tag{2}$$

which is a fixed point problem. This alternative equivalent formulation plays an important and fundamental part in developing various iterative methods for solving nonlinear equation. We use the fixed point formulation (2) to suggest the following iterative methods.

For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1} = H(x_n), \qquad n = 0,1,2,3,\cdots. \tag{3}$$

Such type of iterative methods are called the explicit method, see [16].
In a similar way, we can use the fixed point formulation (2) to suggest the following iterative method.

$$x_{n+1} = H(x_{n+1}), \qquad n = 0,1,2,3,\cdots. \tag{4}$$

Such types of iterative methods are called the implicit methods. We remark that, to implement these iterative methods, one usually uses the predictor-corrector technique. Some well known implicit and explicit type of iterative methods are given as:
Algorithm1.1.[16]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)}, \qquad n = 0,1,2,\cdots.$$

Algorithm 1.1 is known as Newton method and has second-order convergence [16].

Algorithm1.2.[16]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

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$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f(x_n)]^2 - f(x_n)f''(x_n)}, \qquad n = 0,1,2,\cdots.$$

Algorithm 1.2 is known as Halley method which has third-order convergence, see [16].

Algorithm1.3.[16]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)} - \frac{(x_{n+1}-x_n)^2}{2!}\frac{f''(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\cdots.$$

Algorithm 1.3 is an implicit method. To implement the Algorithm 2.3, we use the predictor-corrector technique. We use Algorithm 2.1 as predictor and Algorithm 1.3 as corrector step. Consequently, we have the following two-step iterative method.

Algorithm1.4.[16]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{(y_n - x_n)^2}{2!}\frac{f''(x_n)}{f'(x_n)}, \qquad n = 0,1,2,3\cdots.$$

We would like to mention that Algorithm 1.4 can be written in the equivalent form as:

Algorithm1.5.[16]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)} - \frac{1}{2}\left[\frac{f(x_n)}{\acute{f}(x_n)}\right]^2 \frac{f''(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\cdots.$$

Algorithm 1.5 is known as Householder method. It has also third-order convergence [16].

These well known iterative methods can be derived by using different techniques such as Taylor series , quadrature formula, homotopy perturbation method, Adomian decomposition method [1,2,3,4,5,6,7,8,9,10, 11,12,13,14,15,16,17].

In this paper, we use the variational iteration technique to suggest and analyze some new iterative methods for solving the nonlinear equations, the origin of which can be traced back to Inokuti et al [7]. However it is He [6] who realized the potential of this method for solving a wide class of both linear and nonlinear problems arising in various branches of pure and applied sciences. Noor[9] suggested and analyzed a class of iterative methods for solving the nonlinear equations. Later on Noor and Shah[12] suggested the scheme for accelerated order of convergence. Noor and Shah [12]used the main scheme by involving the auxiliary function $g(x)$. Now, we modify the main relation and arrange in some different way. New suggested relation also involves the auxiliary function and the methods generated by this main relation are more efficient and robust.

2 Construction of iterative methods

In this section, we use the variational iteration technique to derive some new iterative methods. These methods are multi-step methods consisting predictor and corrector steps. The convergence of the methods is better than the one step methods. We use the variational iteration technique to obtain some new iterative methods of order $p+1$. where $p \geq 1$, is the order of convergence of the predictor iteration function $\phi(x)$, which is the essential part of the main scheme.

Let $\phi(x)$ be an iteration function of order $p \geq 1$, and $g(x)$ be an auxiliary arbitrary function. We consider the function $H(x)$ defined as:

$$H(x) = \phi(x) + \lambda[f(x)]^p g(x), \tag{5}$$

where λ is the Lagrange's multiplier.

Using the optimality criteria from (5), we obtain the value of λ as:

$$\lambda = -\frac{\phi'(x)}{p[f(x)]^{p-1}f'(x)g(x) + [f(x)]^p g'(x)}. \tag{6}$$

From (5) and (6), we obtain

$$H(x) = \phi(x) - \frac{\phi'(x)f(x)g(x)}{pf'(x)g(x) + [f(x)]g'(x)}. \tag{7}$$

Now combining (2) and (7), we obtain

$$x = H(x) = \phi(x) - \frac{\phi'(x)f(x)g(x)}{pf'(x)g(x) + [f(x)]g'(x)} \tag{8}$$

This is another fixed point problem. We use this fixed point formulation to suggest the following iterative scheme as:

Algorithm2.1. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1}=\phi(x_n)-\frac{\phi'(x_n)f(x_n)g(x_n)}{pf'(x_n)g(x_n)+f(x_n)g'(x_n)}.$$

This is the main recurrence relation involving the iteration function $\phi(x)$. This scheme generates the iterative methods of order $p+1$. We select $\phi(x_n)$ as predictor having the order of convergence p. We also note that, if we take $\phi(x_n)=I$ and $p=1$. then Algorithm 2.1 collapses to the recurrence relation which has been suggested by Noor [9].

For simplicity, we first consider the well known Newton method as a predictor such that

$$\phi(x)=x-\frac{f(x)}{\acute{f}(x)}. \tag{9}$$

Thus

$$\phi'(x)=\frac{f(x)f''(x)}{[\acute{f}(x)]^2}. \tag{10}$$

Using (9) and (10) in (8), we have

$$x=x-\frac{f(x)}{\acute{f}(x)}-\frac{[f(x)]^2g(x)f''(x)}{[f(x)]^2[2f'(x)g(x)+f(x)g'(x)]}, \tag{11}$$

which is another fixed point formulation. This fixed point formula enables us to suggest the iterative method for solving nonlinear equations as:

Algorithm2.2. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1}=x_n-\frac{f(x_n)}{\acute{f}(x_n)}-\frac{[f(x_n)]^2g(x_n)f''(x_n)}{[f'(x_n)]^2[2f'(x_n)g(x_n)+f(x_n)g'(x_n)]},$$

For different values of the auxiliary function $g(x)$, we can obtain several Householder type iterative methods for solving nonlinear equations.

1. Let $g(x_n)=e^{-\alpha x_n}$. Then from Algorithm 2.2, we obtain the following iterative method for solving the nonlinear equation (1).

Algorithm2.3. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1}=x_n-\frac{f(x_n)}{\acute{f}(x_n)}-\frac{[f(x_n)]^2f''(x_n)}{[f'(x_n)]^2[2f'(x_n)-\alpha f(x_n)]},$$

If $\alpha=0$, then Algorithm 2.3 reduces to the well known Householder method [16].

2. Let $g(x_n)=e^{-\frac{\alpha}{f'(x_n)}}$, and $g'(x_n)=-e^{-\frac{\alpha}{\acute{f}(x_n)}}(\frac{\alpha f''(x_n)}{[f'(x_n)]^2})$ Then from Algorithm 2.2, we obtain the following iterative method for solving nonlinear equations.

Algorithm2.4. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1}=x_n-\frac{f(x_n)}{\acute{f}(x_n)}-\frac{[f(x_n)]^2f''(x_n)}{f'(x_n)[2f'(x_n)^2-\alpha f(x_n)f''(x_n)]},$$

If $\alpha=0$, then Algorithm 2.4 reduces to the well known Householder method [16].

3. Let $g(x_n)=e^{-\frac{\alpha f(x_n)}{\acute{f}(x_n)}}$. Then from Algorithm 2.2, we have the following iterative method for solving the nonlinear equation (1).

Algorithm2.5. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$x_{n+1}=x_n-\frac{f(x_n)}{\acute{f}(x_n)}-\frac{[f(x_n)]^2f''(x_n)}{[2f'(x_n)^3-\alpha f(x_n)(f'(x_n)^2-f''(x_n)f(x_n))]},$$

If $\alpha=0$, then Algorithm 2.5, reduces to the well known Householder method [16].

We replace the approximation of second derivative in Algorithms 2.3-2.5 by a suitable substitution involving only the first derivative [12] and obtain predictor-corrector type iterative methods.

$$f(y)\approx f(x)+(y-x)f'(x)+\frac{(y-x)^2}{2}f''(x)=\frac{[f(x)]^2f''(x)}{2[f'(x)]^2} \tag{12}$$

where

$$y=x-\frac{f(x)}{\acute{f}(x)},$$

From which, we have after simplifying

$$f''(x)\approx\frac{2[f'(x)]^2f(y)}{[f(x)]^2}. \tag{13}$$

Replacing the value of $f''(x)$, in Algorithm 2.3-2.5 for all values of n, we obtain the following iterative methods as:

Algorithm2.6. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$y_n=x_n-\frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1}=y_n-\frac{2f(y_n)}{2f'(x_n)-\alpha f(x_n)},n=0,1,2\cdots.$$

If $\alpha = 0$, then Algorithm 2.6 reduces to the following method.

Algorithm2.7.[**10**]. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}, \quad n = 0,1,2,\cdots.$$

This method is suggested by Noor [9] and has third order convergence.

Algorithm2.8. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)}{f'(x_n)f(x_n) - \alpha f(y_n)}, n = 0,1,2,\cdots.$$

If $\alpha = 0$, then Algorithm 2.8 reduces to the Algorithm 2.7.

Algorithm2.9. For a given x_0, find the approximation solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)}{2f'(x_n) - \alpha[f(x_n) - 2f(y_n)]}, n = 0,1,2,\cdots.$$

If $\alpha = 0$, then Algorithm 2.9 reduces to the Algorithm 2.7.

It is important to say that never choose such a value of α which makes the denominator zero. It is necessary that sign of α should be chosen so as to keep the denominator largest in magnitude in above Algorithms.

3 Convergence analysis

In this section, we consider the convergence criteria of the main iterative scheme Algorithm 2.2 developed in section 2 and Algorithm 2.6 derived after substitution the auxiliary function and approximation of the second derivative of $f(x)$.

Theorem 1.*Assume that the function $f : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}$ for an open interval in $\mathscr{D}$ with simple root $r \in \mathscr{D}$. Let $f(x)$ be a smooth sufficiently in some neighborhood of root and then Algorithm 2.2 has third order convergence.*

*Proof.*Let r ba a simple root of the nonlinear equation $f(x)$. Since f is sufficiently differentiable. Expanding $f(x)$ and $\acute{f}(x)$ in Taylor's series at r, we obtain

$$f(x_n) = \acute{f}(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)]. \tag{14}$$

and

$$\acute{f}(x_n) = \acute{f}(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^7)]. \tag{15}$$

where

$$e_n = x_n - r, c_k = \frac{f^k(r)}{k!\acute{f}(r)} \text{ and } k = 2,3,\cdots$$

From Eq.(14) and Eq.(15), we get

$$\begin{aligned}\frac{f(x_n)}{\acute{f}(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 \\ &+ (8c_2^4 - 20c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (13c_2c_5 - 28c_2^2c_4 \\ &- 5c_6 - 16c_2^5 + 52c_2^3c_3 + 17c_3c_4 - 33c_2c_3^2)e_n^6 + O(e_n^7).\end{aligned} \tag{16}$$

Similarly, we can obtain

$$f''(x_n) = \acute{f}(r)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + O(e_n^5)]. \tag{17}$$

Using Eq. (14)and Eq. (17), we get

$$\begin{aligned}f''(x_n)f(x_n^2)g(x_n) &= [f'(r)]^3[2c_2g(r)e_n^2 + (6c_3g(r) + 2c_2(g'(r) + \\ &4c_2^2g(r))e_n^3 + O(e_n^4)].\end{aligned} \tag{18}$$

Using Eq.(15), we obtain

$$\begin{aligned}f(x)g'(x) &= f'(r)[g'(r)e_n + (c_2g'(r) + g''(r))e_n^2 + (c_3g'(r) \\ &+ \frac{1}{2}g'''(r) + c_2g''(r))e_n^3 + O(e_n^4)].\end{aligned} \tag{19}$$

Using Eq. (18) and Eq. (19),we get

$$\begin{aligned}f'(x)g(x) &= f'(r)[g(r) + ((g'(r) + 2c_2g(r))e_n + (\frac{1}{2}g''(r) + 2c_2g'(r) \\ &+ 3c_3g(r))e_n^2 + (\frac{1}{2}g''(r) + 2c_2g'(r) + 3c_3g(r))e_n^2 + O(e_n^3)].\end{aligned} \tag{20}$$

Now from Eq.(18) and Eq.(21), we get

$$\begin{aligned}2f'(x)g(x) + f(x)g'(x) &= f'(r)[2g(r) + (3g'(r) + 4c_2g(r))e_n \\ &+ (2g''(r) + 5c_2g'(r) + 6c_3g(r)e_n^2) + [\frac{5}{6}g''(r)\end{aligned}$$

$$+8c_4g(r)+3c_2g''(r)+7c_3g'(r)]e_n^3+O(e_n^4). \quad (21)$$

From Eq.(15), Eq.(18) and Eq.(21), we obatain

$$\frac{[(f)(x)]^2g(x)f''(x)}{[(f')(x)]^2(2f'(x)g(x)+f(x)g'(x))}$$

$$=c_2e_n^2-\frac{1}{2}(8c_2^2-6c_3+c_2)e_n^3+O(e_n^4). \quad (22)$$

Now with the help of Eq.(16) and Eq.(22), we get

$$x_{n+1}=p+\frac{1}{2}(-2c_3+4c_2^2+c_2\frac{g'(r)}{g(r)})e_n^3+O(e_n^4). \quad (23)$$

Finally, the error equation is

$$e_{n+1}=\frac{1}{2}[-2c_3+4c_2^2+c_2\frac{g'(r)}{g(r)}]e_n^3+O(e_n^4), \quad (24)$$

which shows that the main recurrence relation Algoritm 2.2 has 3^{rd} order convergence. Methods derived from this main scheme has also 3^{rd} order convergence.

Theorem 2. *Assume that the function $f:\mathscr{D}\subset\mathbb{R}\to\mathbb{R}$ for an open interval in $\mathscr{D}$ with simple root $p\in\mathscr{D}$. Let $f(x)$ be a smooth sufficiently in some neighborhood of root and then Algorithm 2.6 has third order convergence with the following error equation*

$$e_{n+1}=2c_2^2e_n^3+[-\frac{1}{2}c_2(-14c_3+18c_2^2+\alpha c_2)]e_n^4+O(e_n^5). \quad (25)$$

Similarly, one can check the order of convergence of all other newly derived methods.

4 Numerical results

We now present some examples to illustrate the efficiency of the new developed two-step iterative methods (see Tables 4.1-4.12). We compare the Newton method (NM) [16], Noor's method (KN) [10], Algorithm 2.7, Algorithm 2.8 and Algorithm 2.9, which are introduced here in this paper. We also note that these methods do not require the computation of second derivative to carry out the iterations. All computations are done using the MAPLE using 60 digits floating point arithmetics (Digits: =60). We will use $\varepsilon=10^{-32}$. The following stopping criteria are used for computer programs.

$$(i)\ |x_{n+1}-x_n|\leq\varepsilon,\ (ii)\ |f(x_n)|\leq\varepsilon.$$

The computational order of convergence p approximated for all the examples in Tables 4.1-4.12, (see [17]) by means of

$$\rho=\frac{ln(|x_{n+1}-x_n|/|x_n-x_{n-1}|)}{ln(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)}$$

along with the total number of functional evaluations (TNFE) as required for the iterations during the computation.

Table 4.1 Comparison of example 4.1 for $\alpha=1$.

Method	IT	TNFE	x_n	δ	ρ
NM	7	14	1.404916	7.33e-26	2.00003
KN	17	51	1.404916	4.41e-21	2.87384
Alg 2.7	6	18	1.404916	1.57e-40	3.01692
Alg 2.8	5	15	1.404916	2.09e-22	2.90662
Alg 2.9	5	15	1.404916	3.43e-29	3.04833

Table 4.2 Comparison of example 4.1 for $\alpha=0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	7	14	1.404916	7.33e-26	2.00003
KN	17	51	1.404916	4.41e-21	2.87384
Alg 2.7	6	18	1.404916	1.14e-23	3.12188
Alg 2.8	5	15	1.404916	3.43e-16	3.34476
Alg 2.9	5	15	1.404916	4.37e-17	3.30201

Example 4.1. We consider the nonlinear equation

$$f(x)=sin^2x-x^2+1.$$

We consider $\alpha=1$ and $\alpha=0.5$ for all the methods to compare the numerical results in Table 4.1 and Table 4.2 respectively.

Table 4.1 depicts the numerical results of example 4.1. We use the initial guess $x_0=1$ for the computer program for $\alpha=1$.

Table 4.2 depicts the numerical results of example 4.1. We use the initial guess $x_0=1$, for the computer program for $\alpha=0.5$.

Example 4.2. We consider the nonlinear equation $f_2(x)=x^2-e^{-x}-3x+2$.
We consider $\alpha=1$ and $\alpha=0.5$ for all the methods to compare the numerical results in Table 4.3 and Table 4.4 respectively.

Table 4.3 Comparison of example 4.2 for $\alpha = 1$.

Method	IT	TNFE	x_n	δ	ρ
NM	7	14	0.25730	9.10e-28	2.00050
KN	5	15	0.25730	3.23e-42	2.91743
Alg 2.7	5	15	0.25730	3.18e-34	3.01615
Alg 2.8	5	15	0.25730	4.49e-41	2.95762
Alg 2.9	4	12	0.25730	1.81e-39	2.98436

Table 4.4 Comparison of example 4.2 for $\alpha = 0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	7	14	0.257530	9.10e-28	2.00050
KN	5	15	0.257530	3.23e-42	2.91743
Alg 2.7	5	15	0.257530	3.18e-34	3.01615
Alg 2.8	5	15	0.257530	4.49e-41	2.95762
Alg 2.9	5	15	0.257530	1.81e-39	2.98436

Table 4.3 shows the efficiency of the methods for example 4.2. We use the initial guess $x_0 = 2$, for the computer program for $\alpha = 1$. Number of iterations and computational order of convergence gives us an idea about the better performance of the new methods.

Table 4.4 shows the efficiency of the methods for example 4.2. We use the initial guess $x_0 = 2$, for the computer program for $\alpha = 0.5$. Number of iterations and computational order of convergence gives us an idea about the better performance of the new methods.

Example 4.3. We consider the nonlinear equation $f_3(x) = (x-1)^2 - 1$.
We consider $\alpha = 1$ and $\alpha = 0.5$ for all the methods to compare the numerical results in Table 4.5 and Table 4.6 respectively.

In Table 4.5, the numerical results for example 4.3 are described. We use the initial guess $x_0 = 3.5$ for the computer program for $\alpha = 1$. We observe that all the methods approach to the approximate solution after equal number of iterations but the computational order of convergence has little bit difference.

In Table 4.6, the numerical results for example 4.3 are described. We use the initial guess $x_0 = 3.5$ for the computer program for $\alpha = 0.5$.. We observe that all the methods approach to the approximate solution after equal number of iterations but the computational order of convergence has little bit difference.

Table 4.5 Comparison of example 4.3 for $\alpha = 1$.

Method	IT	TNFE	x_n	δ	ρ
NM	9	18	2.00000	8.28e-22	2.00025
KN	6	18	2.00000	1.84e-28	2.95752
Alg 2.7	6	18	2.00000	1.11e-40	3.08772
Alg 2.8	6	18	2.00000	7.57e-30	3.01953
Alg 2.9	6	18	2.00000	1.64e-33	3.99885

Table 4.6 Comparison of example 4.3 for $\alpha = 0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	9	18	2.0000	8.22e-22	2.00125
KN	6	18	2.0000	1.04e-28	2.95652
Alg 2.7	6	18	2.0000	1.31e-40	2.98872
Alg 2.8	6	18	2.0000	2.57e-30	2.96983
Alg 2.9	6	18	2.0000	2.64e-33	2.99885

Table 4.7 Comparison of example 4.4 for $\alpha = 1$.

Method	IT	TNFE	x_n	δ	ρ
NM	8	16	2.15443	5.64e-28	2.00003
KN	6	18	2.15443	1.35e-32	3.03016
Alg 2.7	5	15	2.15443	1.46e-27	3.07947
Alg 2.8	6	18	2.15443	2.73e-43	3.00842
Alg 2.9	4	12	2.154438	2.01e-19	3.29320

number of iterations but the computational order of convergence has little bit difference.

Example 4.4. We consider the nonlinear equation $f_4(x) = x^3 - 10$.
We consider $\alpha = 1$ and $\alpha = 0.5$ for all the methods to compare the numerical results in Table 4.7 and Table 4.8 respectively.

Table 4.7 shows the numerical results for example 4.4. For the computer program we use the initial guess $x_0 = 1.5$, and $\alpha = 1$. We note that the new derived methods have better computational order of convergence and approach to the desired result in less number of iterations.

Table 4.8 Comparison of example 4.4 for $\alpha = 0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	8	16	2.15443	3.64e-28	2.00003
KN	6	18	2.15443	1.35e-32	3.00017
Alg 2.7	5	15	2.15443	1.43e-27	3.07989
Alg 2.8	6	18	2.15443	2.44e-43	3.00876
Alg 2.9	4	12	2.15443	2.01e-19	2.99321

Table 4.9 Comparison of example 4.5 for $\alpha = 1$.

Method	IT	TNFE	x_n	δ	ρ
NM	10	20	-1.20764	2.73e-21	2.00085
KN	7	21	-1.20764	3.34e-35	2.97222
Alg 2.7	6	18	-1.20764	1.51e-31	2.98705
Alg 2.8	6	18	-1.20764	1.92e-35	2.99293
Alg 2.9	7	21	-1.20764	2.87e-33	2.98710

Table 4.8 shows the numerical results for example 4.4. For the computer program we use the initial guess $x_0 = 1.5$, and $\alpha = 0.5$. We note that the new derived methods have better computational order of convergence and approach to the desired result in less number of iterations.

Example 4.5. We consider the nonlinear equation

$$f_5(x) = xe^{x^2} - sin^2 x + 3cosx + 5.$$

We consider $\alpha = 1$ and $\alpha = 0.5$ for all the methods to compare the numerical results in Table 4.9 and Table 4.10 respectively.

In Table 4.9, we show the numerical results for the example 4.5. We use the initial guess $x_0 = -2$, and $\alpha = 1$, for the computer program. We observe that the new methods approach to the desired approximate solution in equal or less number of iterations. We calculate the computational order of convergence for all the methods which verify the rate of convergence and efficiency of the methods. In Table 4.10, we show the numerical results for the example 4.5. We use the initial guess $x_0 = -2$, and $\alpha = 0.5$, for the computer program. We observe that the new methods approach to the desired approximate solution in equal or less number of iterations. We calculate the computational order of convergence for all the methods which verify the rate of convergence and efficiency of the methods.

Table 4.10 Comparison of example 4.5 for $\alpha = 0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	10	20	-1.20764	2.78e-21	2.00055
KN	7	21	-1.20764	2.34e-35	2.99722
Alg 2.7	6	18	-1.20764	1.11e-31	2.98708
Alg 2.8	7	21	-1.20764	1.92e-35	2.99793
Alg 2.9	6	18	-1.20764	2.87e-33	2.99910

Table 4.11 Comparison of example 4.6 for $\alpha = 1$.

Method	IT	TNFE	x_n	δ	ρ
NM	8	16	1.365230	4.74e-27	2.00003
KN	6	18	1.365230	2.38e-21	3.14948
Alg 2.7	5	15	1.365230	8.18e-24	3.13992
Alg 2.8	6	18	1.365230	2.31e-33	3.02907
Alg 2.9	4	12	1.365230	3.17e-17	2.81408

Table 4.12 Comparison of example 4.6 for $\alpha = 0.5$.

Method	IT	TNFE	x_n	δ	ρ
NM	8	16	1.365230	4.45e-27	2.00003
KN	6	18	1.365230	2.56e-21	3.14398
Alg 2.7	5	15	1.365230	8.10e-24	3.10092
Alg 2.8	4	12	1.365230	2.38e-33	3.00007
Alg 2.9	4	12	1.365230	3.07e-17	2.98408

Example 4.6. We consider the nonlinear equation

$$f_6(x) = x^3 - 4x^2 - 10.$$

We consider $\alpha = 1$ and $\alpha = 0.5$ for all the methods to compare the numerical results in Table 4.11 and Table 4.12 respectively.

In Table 4.11, we show the numerical results for the example 4.6. We use the initial guess $x_0 = 0.75$ and $\alpha = 1$. for the computer program. We observe that the new methods approach to the desired approximate solution in less number of iterations.

In Table 4.12, we show the numerical results for the example 4.6. We use the initial guess $x_0 = 0.75$ and $\alpha =$

0.5. for the computer program. We observe that the new methods perform in better way and approach to the desired approximate solution in less number of iterations.

5 Conclusion

In this paper, we have presented the main recurrence relation by using the variational iteration technique. This main recurrence relation generates the iterative methods for solving nonlinear equations. For illustration, we have applied this technique to obtain third-order convergent methods by using Newton method as auxiliary predictor functions. This technique also generates the Halley-like and Householder type iterative methods. Using appropriate substitutions, we modified the methods and obtained the second derivative-free predictor-corrector type of iterative methods. Per iteration methods require three computations of the given function and its derivative. These all methods are also free from second derivative. If we consider the definition of efficiency index [16] as $p^{\frac{1}{m}}$, where p is the order of convergence of the method and m is the number of functional evaluations per iteration required by the method. We have that all of the methods obtained of third order convergence have the efficiency index equal to $3^{\frac{1}{3}} \approx 1.442$, which is better than the one of Newton's method $2^{\frac{1}{2}} \approx 1.414$. Obtained methods are also compared in their performance and efficiency to some of other known methods. The presented approach can also be applied further to obtain higher order convergent methods.

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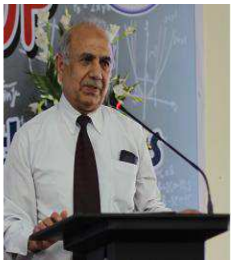

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