Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/040102

Uni-soft Substructures of Rings and Modules

Aslıhan Sezgin Sezer[1,*], *Akın Osman Atagün*[2] *and Naim Çağman*[3]

[1] Department of Mathematics, Amasya University, 05100 Amasya, Turkey
[2] Department of Mathematics, Bozok University, Yozgat, Turkey
[3] Department of Mathematics, Gaziosmanpaşa University, Tokat, Turkey

Received: 1 Jul. 2014, Revised: 12 Oct. 2014, Accepted: 16 Oct. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, we introduce union soft subrings and union soft ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower α-inclusion of soft sets. We also obtain significant relation between soft subrings and union soft subrings, soft ideals and union soft ideals, soft submodules and union soft submodules.

Keywords: Soft sets, union soft subrings (ideals), union soft submodules, anti image, α-inclusion.

1 Introduction

The notion of soft set was introduced in 1999 by Molodtsov [28] as a new mathematical tool for dealing with uncertainties. Since its inception, it has received much attention in the mean of algebraic structures such as groups [2], semirings [11], rings [1], BCK/BCI-algebras [16,17,18], d-algebras [19], ordered semigroups [20], BL-algebras [33], BCH-algebras [22] and near-rings [31]. Moreover, Xiao et al. [32] proposed the notion of exclusive disjunctive soft sets and studied some of its operations and Gong et al. [15] studied the bijective soft set with its operations. Atagün and Sezgin defined the concepts of soft subrings and ideals of a ring, soft subfields of a field and soft submodules of a module [4] and studied their related properties with respect to soft set operations. Çağman et al. defined two new soft groups, soft int-groups [8] and soft uni-groups [9], which are based on the inclusion relation and the intersection of sets and union of sets, respectively.

Algebraic structures of soft sets have been studied by some authors. Maji et al. [25] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [3] introduced several operations of soft sets and Sezgin and Atagün [30] studied on soft set operations as well. Soft set relations and functions [5] and soft mappings [27] were proposed and many related concepts were discussed. Moreover, the theory of soft set has gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making as in the following studies: [6,7,12,13,14,26,29,34].

In [4], Atagün and Sezgin defined the notions of soft subrings and soft ideals of a ring, soft subfields of a field, soft submodules of a module. They studied their properties especially with respect to soft set operations in more detail. In this paper, first we extend Atagün and Sezgin's study [4] by focusing on soft subrings and ideals of a ring and soft submodules of a module with respect to image, preimage and upper α-inclusion of soft sets. We then introduce union soft subrings and ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower α-inclusion of soft sets. Moreover, we obtain relations between soft subrings and union soft subrings, soft ideals and union soft ideals and soft submodules and union soft submodules. The union soft set theory (in a few algebraic structures) is also studied in the following papers [21,23,24].

2 Preliminaries

Throughout this paper, R will always denote a ring with zero 0_R, M a left R-module with identity 0_M and N a left submodule of M. Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

* Corresponding author e-mail: aslihan.sezgin@amasya.edu.tr

Definition 1.[28] *If F is a mapping given by $F: A \to P(U)$, then the set $F_A = \{(x, F(x)) : x \in A\}$ is called a soft set over U.*

Definition 2.[3] *The relative complement of a soft set F_A over U is denoted by F_A^r, where $F_A^r : A \to P(U)$ is a mapping given as $F_A^r(\alpha) = U \setminus F_A(\alpha)$ for all $\alpha \in A$.*

Definition 3.[8,9] *Let F_A and G_B be soft sets over U and Ψ be a function from A to B. Image of F_A under Ψ and anti image of F_A under Ψ are the soft sets $\Psi(F_A)$ and $\Psi^\star(F_A)$, where $\Psi(F_A) : B \to P(U)$ and $\Psi^\star(F_A) : B \to P(U)$ are set-valued functions defined as if $\Psi^{-1}(b) \neq \emptyset$, then $\Psi(F_A)(b) = \bigcup\{F(a) \mid a \in A \text{ and } \Psi(a) = b\}$, otherwise $\Psi(F_A)(b) = \emptyset$ and if $\Psi^{-1}(b) \neq \emptyset$, then $\Psi^*(F_A)(b) = \bigcap\{F(a) \mid a \in A \text{ and } \Psi(a) = b\}$, otherwise $\Psi^*(F_A)(b) = \emptyset$ for all $b \in B$, respectively. Preimage (or inverse image) of G_B under Ψ is the soft set $\Psi^{-1}(G_B)$, where $\Psi^{-1}(G_B) : A \to P(U)$ is a set-valued function defined by $\Psi^{-1}(G_B)(a) = G(\Psi(a))$ for all $a \in A$.*

Definition 4.[3] *Let F_A and G_B be two soft sets over U such that $A \cap B \neq \emptyset$. The restricted union of F_A and G_B is denoted by $F_A \cup_{\mathscr{R}} G_B$, and is defined as $F_A \cup_{\mathscr{R}} G_B = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.*

Theorem 1.[9] *Let F_H and T_K be soft sets over U, F_H^r, T_K^r be their relative soft sets, respectively and Ψ be a function from H to K. Then, i) $\Psi^{-1}(T_K^r) = (\Psi^{-1}(T_K))^r$, ii) $\Psi(F_H^r) = (\Psi^\star(F_H))^r$ and $\Psi^\star(F_H^r) = (\Psi(F_H))^r$.*

Definition 5.[10] *Let F_A be a soft set over U and α be a subset of U. Then, upper α-inclusion of F_A, denoted by $F_A^{\supseteq \alpha}$ and lower α-inclusion of F_A, denoted by $F_A^{\subseteq \alpha}$ are defined as $F_A^{\supseteq \alpha} = \{x \in A \mid F(x) \supseteq \alpha\}$, $F_A^{\subseteq \alpha} = \{x \in A \mid F(x) \subseteq \alpha\}$, respectively.*

Definition 6.[4] *Let S be a subring of R and let F_S be a soft set over R. Then, F_S is called a soft subring of R, denoted by $F_S \tilde{<} R$, if for all $x, y \in S$, $F(x - y) \supseteq F(x) \cap F(y)$ and $F(xy) \supseteq F(x) \cap F(y)$.*

Definition 7.[4] *Let I be an ideal of R and let F_I be a soft set over R. Then, F_I is called a soft ideal of R, denoted by simply $F_I \tilde{\lhd} R$, if for all $x, y \in I$ and $r \in R$, $F(x - y) \supseteq F(x) \cap F(y)$, $F(rx) \supseteq F(x)$ and $F(xr) \supseteq F(x)$.*

Definition 8.[4] *Let N be a submodule of M and F_N be a soft set over M. Then, F_N is called a soft submodule of M, denoted by simply $F_N \tilde{<} M$, if for all $x, y \in N$ and $r \in R$, $F(x - y) \supseteq F(x) \cap F(y)$ and $F(rx) \supseteq F(x)$.*

3 Some characterizations for soft subrings and soft ideals

In this section, we obtain some significant characterizations for soft subrings and soft ideals of a ring with respect to image, preimage and upper α-inclusion of soft sets.

Theorem 2.*Let F_S be a soft set over R and α be a subset of R such that $F(0_R) \supseteq \alpha$. If F_S is a soft subring of R, then $F_S^{\supseteq \alpha}$ is a subring of R.*

*Proof.*Since $F(0_R) \supseteq \alpha$, then $0_R \in F_S^{\supseteq \alpha}$ and $\emptyset \neq F_S^{\supseteq \alpha} \subseteq R$. Assume $x, y \in F_S^{\supseteq \alpha}$, then $F(x) \supseteq \alpha$ and $F(y) \supseteq \alpha$. We need to show that $x - y \in F_S^{\supseteq \alpha}$ and $xy \in F_S^{\supseteq \alpha}$ for all $x, y \in F_S^{\supseteq \alpha}$. Since F_S is a soft subring of R, it follows that $F(x - y) \supseteq F(x) \cap F(y) \supseteq \alpha \cap \alpha = \alpha$. Furthermore, $F(xy) \supseteq F(x) \cap F(y) \supseteq \alpha$, which completes the proof.

Theorem 3.*Let F_S and G_T be soft sets over R, where S and T are subrings of R and Ψ be a ring isomorphism from S to T. If F_S is a soft subring of R, then so is $\Psi(F_S)$.*

*Proof.*Let $t_1, t_2 \in T$. Since Ψ is surjective, there exists $s_1, s_2 \in S$ such that $\Psi(s_1) = t_1$ and $\Psi(s_2) = t_2$. Then, $(\Psi(F_S))(t_1 - t_2) = \bigcup\{F(s) : s \in S, \Psi(s) = t_1 - t_2\} = \bigcup\{F(s) : s \in S, s = \Psi^{-1}(t_1 - t_2)\} = \bigcup\{F(s) : s \in S, s = \Psi^{-1}(\Psi(s_1 - s_2)) = s_1 - s_2\} = \bigcup\{F(s_1 - s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1, 2\} \supseteq \bigcup\{F(s_1) \cap F(s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1, 2\} = (\bigcup\{F(s_1) : t_1 \in S, \Psi(s_1) = t_1\}) \cap (\bigcup\{F(s_2) : t_2 \in S, \Psi(s_2) = t_2\}) = (\Psi(F_S))(t_1) \cap (\Psi(F_S))(t_2)$. Similarly, one can show that $(\Psi(F_S))(t_1 t_2) \supseteq (\Psi(F_S))(t_1) \cap (\Psi(F_S))(t_2)$. Hence, $\Psi(F_S)$ is a soft subring of R.

Theorem 4.*Let F_S and G_T be soft sets over R, where S and T are subrings of R and Ψ be a ring homomorphism from S to T. If G_T is a soft subring of R, then so is $\Psi^{-1}(G_T)$.*

*Proof.*Let $s_1, s_2 \in S$. Then, $(\Psi^{-1}(G_T))(s_1 - s_2) = G(\Psi(s_1 - s_2)) = G(\Psi(s_1) - \Psi(s_2)) \supseteq G(\Psi(s_1)) \cap G(\Psi(s_2)) = (\Psi^{-1}(G_T))(s_1) \cap (\Psi^{-1}(G_T))(s_2)$ and similarly $(\Psi^{-1}(G_T))(s_1 s_2) \supseteq (\Psi^{-1}(G_T))(s_1) \cap (\Psi^{-1}(G_T))(s_2)$. Hence, $\Psi^{-1}(G_T)$ is a soft subring of R.

Theorem 5.*Let F_I be a soft set over R and α be a subset of R such that $F(0_R) \supseteq \alpha$. If F_I is a soft ideal of R, then $F_I^{\supseteq \alpha}$ is an ideal of R.*

*Proof.*Since $F(0_R) \supseteq \alpha$, then $0_R \in F_I^{\supseteq \alpha}$ and $\emptyset \neq F_I^{\supseteq \alpha} \subseteq R$. Assume $x, y \in F_I^{\supseteq \alpha}$ and $r \in R$. Then, $F(x) \supseteq \alpha$ and $F(y) \supseteq \alpha$. We need to show that $x - y \in F_I^{\supseteq \alpha}$, $rx \in F_I^{\supseteq \alpha}$ and $xr \in F_I^{\supseteq \alpha}$ for all $x, y \in F_I^{\supseteq \alpha}$. Since F_I is a soft ideal of R, it follows that $F(x - y) \supseteq F(x) \cap F(y) \supseteq \alpha \cap \alpha = \alpha$. Furthermore, $F(rx) \supseteq F(x) \supseteq \alpha$ and $F(xr) \supseteq F(x) \supseteq \alpha$, which completes the proof.

Theorem 6.*Let F_I and G_J be soft sets over R, where I and J are ideals of R and Ψ be a ring isomorphism from I to J. If F_I is a soft ideal of R, then so is $\Psi(F_I)$.*

*Proof.*Let $j_1, j_2 \in J$ and $r \in R$. Then, $(\Psi(F_I))(j_1 - j_2) \supseteq (\Psi(F_I))(j_1) \cap (\Psi(F_I))(j_2)$ is satisfied as in the case of Theorem 3. Now, let $r \in R$ and $j \in J$. Since Ψ is surjective, there exists $\tilde{i} \in I$ such that $\Psi(\tilde{i}) = j$.

Then,
$(\Psi(F_I))(rj) = \bigcup\{F(i) : i \in I, \Psi(i) = rj\} = \bigcup\{F(i) : i \in I, i = \Psi^{-1}(rj)\} = \bigcup\{F(i) : i \in I, i = \Psi^{-1}(r\Psi(\tilde{i}))\} = \bigcup\{F(i) : i \in I, i = \Psi^{-1}(\Psi(r\tilde{i})) = r\tilde{i}\} = \bigcup\{F(r\tilde{i}) : r\tilde{i} \in I, \Psi(\tilde{i}) = j\} \supseteq \bigcup\{F(\tilde{i}) : \tilde{i} \in I, \Psi(\tilde{i}) = j\} = (\Psi(F_I))(j)$. Similarly, one can show that $(\Psi(F_I))(jr) \supseteq (\Psi(F_I))(j)$ for all $r \in R$ and $j \in J$. Hence, $\Psi(F_I)$ is a a soft ideal of R.

Theorem 7.*Let F_I and G_J be soft sets over R, where I and J are ideals of R and Ψ be a ring epimorphism from I to J. If G_J is a soft ideal of R, then so is $\Psi^{-1}(G_J)$.*

*Proof.*Let $i_1, i_2 \in I$, then $(\Psi^{-1}(G_J))(i_1 - i_2) \supseteq (\Psi^{-1}(G_J))(i_1) \cap (\Psi^{-1}(G_J))(i_2)$ is satisfied as shown in Theorem 4. Now, let $r \in R$ and $i \in I$. Since Ψ is surjective, there exists $j \in J$ such that $\Psi(i) = j$. Then, $(\Psi^{-1}(G_J))(ri) = G(\Psi(ri)) = G(\Psi(r)\Psi(i)) \supseteq G(\Psi(i)) = (\Psi^{-1}(G_J))(i)$ and $(\Psi^{-1}(G_J))(ir) = G(\Psi(ir)) = G(\Psi(i)\Psi(r)) \supseteq G(\Psi(i)) = (\Psi^{-1}(G_J))(i)$. Hence, $\Psi^{-1}(G_J)$ is a soft ideal of R.

4 Union soft subrings and union soft ideals

In this section, we introduce union soft subrings and union soft ideals of a ring, investigate their basic properties and establish the relation between soft subrings and union soft subrings as well as soft ideals and union soft ideals.

Definition 9.*Let S be a subring of R and F_S be a soft set over R. F_S is called a union soft subring of R, denoted $F_S \widetilde{<}_u R$, if $F(x - y) \subseteq F(x) \cup F(y)$ and $F(xy) \subseteq F(x) \cup F(y)$ for all $x, y \in S$.*

*Example 1.*Given the ring $R = (\mathbb{Z}_8, +, .)$, $S_1 = \{0, 2, 4, 6\} < R$ and the soft set F_{S_1} over R, where $F : S_1 \to P(R)$ is a set-valued function defined by $F(x) = \{y \in \mathbb{Z}_8 : y \in < x >\}$ for all $x \in S_1$. Here, $F(0) = \{0\}$, $F(2) = F(6) = \{0, 2, 4, 6\}$ and $F(4) = \{0, 4\}$. Then one can easily show that $F_{S_1} \widetilde{<}_u R$. Now, the subring of R be given as $S_2 = \{0, 4\}$ and the soft set G_{S_2} over R, where $G : S_2 \to P(R)$ is a set-valued function defined by $G(0) = \{0, 1, 3, 4, 5\}$ and $G(4) = \{0, 1, 3\}$. Then, $G(4 \cdot 4) = G(0) = \{0, 1, 3, 4, 5\} \nsubseteq G(4) \cup G(4) = \{0, 1, 3\}$. It follows that G_{S_2} is not a union soft subring of R.

*Example 2.*Given the ring $R = M_2(\mathbb{Z}_6)$, i.e. 2×2 matrices with $\mathbb{Z}_6$ terms, with the operations addition and multiplication of matrices. Let $S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right\}$. It is obvious that S is a subring of R. Let the soft set T_S over R, where $T : S \to P(R)$ is a set-valued function defined by

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \right\} \text{ and } T\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}.$$

Then, one can easily show that $T_S \widetilde{<}_u R$. However, if we define a soft set H_S over R such that

$$H\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\} \text{ and } H\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix} \right\}$$

then,

$$H\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = H\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \nsubseteq H\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) \cup H\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right).$$

Thus, H_S is not a union soft subring of R.

In [4], Atagün and Sezgin showed that the restricted intersection, the sum and the product of two soft subrings of R is a soft subring of R. Here, we show that the restricted union of two union soft subrings of R is a union soft subring of R.

Theorem 8.*If $F_{S_1} \widetilde{<}_u R$ and $G_{S_2} \widetilde{<}_u R$, then $F_{S_1} \cup_{\mathscr{R}} G_{S_2} \widetilde{<}_u R$.*

*Proof.*Since S_1 and S_2 are subrings of R, then $S_1 \cap S_2$ is a subring of R. By Definition 4, let $F_{S_1} \cup_{\mathscr{R}} G_{S_2} = (F, S_1) \cup_{\mathscr{R}} (G, S_2) = (H, S_1 \cap S_2)$, where $H(x) = F(x) \cup G(x)$ for all $x \in S_1 \cap S_2 \neq \emptyset$. Then, for all $x, y \in S_1 \cap S_2$, $H(x - y) = F(x - y) \cup G(x - y) \subseteq (F(x) \cup F(y)) \cup (G(x) \cup G(y)) = (F(x) \cup G(x)) \cup (F(y) \cup G(y)) = H(x) \cup H(y)$ and similarly $H(xy) \subseteq H(x) \cup H(y)$. Therefore, $F_{S_1} \cup_{\mathscr{R}} G_{S_2} = H_{S_1 \cap S_2} \widetilde{<}_u R$.

Theorem 9.*If $F_S \widetilde{<}_u R$, then $F(0_R) \subseteq F(x)$ for all $x \in S$.*

*Proof.*Since F_S is a union soft subring of R, then $F(0_R) = F(x - x) \subseteq F(x) \cup F(x) = F(x)$ for all $x \in S$.

Theorem 10.*If $F_S \widetilde{<}_u R$, then $S_F = \{x \in S \mid F(x) = F(0_R)\}$ is a subring of S.*

*Proof.*It is obvious that $0_R \in S_F$ and $\emptyset \neq S_F \subseteq S$. We need to show that $x - y \in S_F$ and $xy \in S_F$ for all $x, y \in S_F$, which means that $F(x - y) = F(0_R)$ and $F(xy) = F(R)$ have to be satisfied. Since $x, y \in S_F$, then $F(x) = F(y) = F(0_R)$. By Theorem 9, $F(0_R) \subseteq F(x - y)$ and $F(0_R) \subseteq F(xy)$ for all $x, y \in S_F$. Since F_S is a union soft subring of R, then $F(x - y) \subseteq F(x) \cup F(y) = F(0_R)$ and $F(xy) \subseteq F(x) \cup F(y) = F(0_R)$ for all $x, y \in S_F$. Therefore, S_F is a subring of S.

Theorem 11.*Let F_S be a soft set over R and α be a subset of R such that $F(0_R) \subseteq \alpha$. If F_S is a union soft subring of R, then $F_S^{\subseteq \alpha}$ is a subring of R.*

*Proof.*Since $F(0_R) \subseteq \alpha$, then $0_R \in F_S^{\subseteq \alpha}$ and $\emptyset \neq F_S^{\subseteq \alpha} \subseteq R$. Let $x, y \in F_S^{\subseteq \alpha}$, then $F(x) \subseteq \alpha$ and $F(y) \subseteq \alpha$. We need to show that $x - y \in F_S^{\subseteq \alpha}$ and $xy \in F_S^{\subseteq \alpha}$ for all $x, y \in F_S^{\subseteq \alpha}$. Since F_S is a union soft subring of R, it follows that $F(x - y) \subseteq F(x) \cup F(y) \subseteq \alpha \cup \alpha = \alpha$. Furthermore, $F(xy) \subseteq F(x) \cup F(y) \subseteq \alpha$, which completes the proof.

The following theorem gives the relation between soft subrings and union soft subrings of a ring.

Theorem 12.*Let F_S be a soft set over R. Then, F_S is a union soft subring of R iff F_S^r is a soft subring of R.*

Proof. Let F_S be a union soft subring of R. Then for all $x,y\in R$, $F^r(x-y)=R\setminus F(x-y)\supseteq R\setminus(F(x)\cup F(y))=(R\setminus F(x))\cap(R\setminus F(y))=F^r(x)\cap F^r(y)$ and $F^r(xy)=R\setminus F(xy)\supseteq R\setminus(F(x)\cup F(y))=(R\setminus F(x))\cap(R\setminus F(y))=F^r(x)\cap F^r(y)$. Thus, F^r_S is a soft subring of R. The converse can be proved similarly.

Theorem 13. *Let F_S and G_T be soft sets over R, where S and T are subrings of R and Ψ be a ring homomorphism from S to T. If G_T is a union soft subring of R, then so is $\Psi^{-1}(G_T)$.*

Proof. Let G_T be a union soft subring of R. Then, G^r_T is a soft subring of R by Theorem 12 and $\Psi^{-1}(G^r_T)$ is a soft subring of R by Theorem 4. Thus, $\Psi^{-1}(G^r_T)=(\Psi^{-1}(G_T))^r$ is a soft subring of R by Theorem 1 (i). Therefore, $\Psi^{-1}(G_T)$ a union soft subring of R by Theorem 12.

Theorem 14. *Let F_S and G_T be soft sets over R, where S and T are subrings of R and Ψ be a ring isomorphism from S to T. If F_S is a union soft subring of R, then so is $\Psi^\star(F_S)$.*

Proof. Let F_S be a union soft subring of R. Then, F^r_S is a soft subring of R by Theorem 12 and $\Psi(F^r_S)$ is a soft subring of R by Theorem 3. Thus, $\Psi(F^r_S)=(\Psi^\star(F_S))^r$ is a soft subring of R by Theorem 1 (ii). So, $\Psi^\star(F_S)$ is a union soft subring of R by Theorem 12.

Theorem 15. *Let R_1 and R_2 be two rings and $F_{S_1}\widetilde{<}_uR_1$, $H_{S_2}\widetilde{<}_uR_2$. If $f:S_1\to S_2$ is a ring homomorphism, then i) $H_{f(S_1)}\widetilde{<}_uR_2$ and $F_{Kerf}\widetilde{<}_uR_1$, ii) If f is an epimorphism, $F_{f^{-1}(S_2)}\widetilde{<}_uR_1$.*

Proof.i) Since $S_1<R_1$, $S_2<R_2$ and $f:S_1\to S_2$ is a ring homomorphism, then $f(S_1)<R_2$ and as $f(S_1)\subseteq S_2$, the result is obvious by Definition 9. Moreover, since $Kerf<R_1$ and $Kerf\subseteq S_1$, the rest of the proof is clear by Definition 9. *ii)* Since $S_1<R_1$, $S_2<R_2$ and $f:S_1\to S_2$ is a ring epimorphism, then it is clear that $f^{-1}(S_2)<R_1$. Since $F_{S_1}\widetilde{<}_uR_1$ and $f^{-1}(S_2)\subseteq S_1$, $F_1(x-y)\subseteq F_1(x)\cup F_1(y)$ and $F_1(xy)\subseteq F_1(x)\cup F_1(y)$ for all $x,y\in f^{-1}(S_2)$. This completes the proof.

Corollary 1. *Let $F_{S_1}\widetilde{<}_uR_1$, $H_{S_2}\widetilde{<}_uR_2$ and $f:S_1\to S_2$ is a ring homomorphism, then $H_{\{0_{S_2}\}}\widetilde{<}_uR_2$.*

Definition 10. *Let I be an ideal of R and let F_I be a soft set over R. Then, F_I is called a union soft ideal of R, denoted by $F_I\widetilde{\lhd}_uR$, if $F(x-y)\subseteq F(x)\cup F(y)$, $F(rx)\subseteq F(x)$ and $F(xr)\subseteq F(x)$ for all $x,y\in I$ and $r\in R$.*

Example 3. Consider the ring $R=(\mathbb{Z}_{16},+,.)$, the ideal of R as $I_1=\{0,8\}$ and the soft set F_{I_1} over R, where $F:I_1\to P(R)$ is a set-valued function defined by $F(0)=\{0,3,15\}$ and $F(8)=\{0,3,6,9,12,15\}$. It can be easily shown that $F_{I_1}\widetilde{\lhd}_uR$. Now, let the ideal of R be $I_2=\{0,4,8,12\}$ and the soft set G_{I_2} over R, where $G:I_2\to P(R)$ is a set-valued function defined by $G(0)=\{0,4,9,12\}$, $G(4)=G(12)=\{0,4,6,9,15\}$ and $G(8)=\{0,4,6,12\}$. Then, $G(2\cdot 8)=G(0)=\{0,4,9,12\}\nsubseteq G(8)=\{0,4,6,12\}$. It follows that G_{I_2} is not a union soft ideal of R.

Theorem 16. *If $F_{I_1}\widetilde{\lhd}_uR$ and $G_{I_2}\lhd_uR$, then $F_{I_1}\cup_{\mathscr{R}}G_{I_2}\widetilde{\lhd}_uR$.*

Proof. Since $I_1,I_2\lhd R$, then $I_1\cap I_2\lhd R$. By Definition 4, $F_{I_1}\cup_{\mathscr{R}}G_{I_2}=H_{I_1\cap I_2}$, where $H(x)=F(x)\cup G(x)$ for all $x\in I_1\cap I_2\neq\emptyset$. Then for all $x,y\in I_1\cap I_2$ and $r\in R$, $H(x-y)=F(x-y)\cup G(x-y)\subseteq(F(x)\cup F(y))\cup(G(x)\cup G(y))=(F(x)\cup G(x))\cup(F(y)\cup G(y))=H(x)\cup H(y)$, $H(rx)=F(rx)\cup G(rx)\subseteq F(x)\cup G(x)=H(x)$ and $H(xr)=F(xr)\cup G(xr)\subseteq F(x)\cup G(x)=H(x)$. This completes the proof.

Theorem 17. *If $F_I\widetilde{\lhd}R$, then $I_F=\{x\in I\mid F(x)=F(0_R)\}$ is an ideal of R.*

Proof. The proof follows from Theorem 10 and Definition 10.

Theorem 18. *Let F_I be a soft set over R and α be a subset of R such that $F(0_R)\subseteq\alpha$. If F_I is a union soft ideal of R, then $F_I^{\subseteq\alpha}$ is an ideal of R.*

Theorem 19. *Let F_I be a soft set over R. Then, F_I is a union soft ideal of R iff F^r_I is a soft ideal of R.*

Proof. Let F_I be a union soft ideal of R, $x,y\in I$ and $r\in R$. Then, for all $x,y\in I$ and $r\in R$, $F^r(x-y)=R\setminus F(x-y)\supseteq R\setminus(F(x)\cup F(y))=(R\setminus F(x))\cap(R\setminus F(y))=F^r(x)\cap F^r(y)$. Moreover, $F^r(xr)=R\setminus F(xr)\supseteq R\setminus F(x)=F^r(x)$ and $F^r(rx)=R\setminus F(rx)\supseteq R\setminus F(x)=F^r(x)$. Thus, F^r_I is a soft ideal of R. The converse can be proved similarly.

Theorem 20. *Let F_I and G_J be soft sets over R, where I and J are ideals of R and Ψ be a ring epimorphism from I to J. If G_J is a union soft ideal of R, then so is $\Psi^{-1}(G_J)$.*

Proof. Follows from Theorem 1 (i), 7 and 19.

Theorem 21. *Let F_I and G_J be soft sets over R, where I and J are ideals of R and Ψ be a ring isomorphism from I to J. If F_I is a union soft ideal of R, then so is $\Psi^\star(F_I)$.*

Proof. Follows from Theorem 1 (ii), 6 and 19.

5 Some characterizations for soft submodules

In this section, we obtain some characterizations for soft submodules of a module with respect to image, preimage and upper α-inclusion of soft sets.

Theorem 22. *Let F_N be a soft set over M and α be a subset of M such that $F(0_M)\supseteq\alpha$. If F_N is a soft submodule of M, then $F_N^{\supseteq\alpha}$ is a submodule of M.*

Theorem 23. *Let F_N and G_K be soft sets over M, where N and K are submodules of M and Ψ be a module isomorphism from N to K. If F_N is a soft submodule of M, then so is $\Psi(F_N)$.*

Proof. Let $k_1, k_2 \in K$. Since Ψ is surjective, there exists $n_1, n_2 \in N$ such that $\Psi(n_1) = k_1$ and $\Psi(n_2) = k_2$. Thus, as in the case of Theorem 3, $(\Psi(F_N))(k_1 - k_2) \supseteq (\Psi(F_N))(k_1) \cap (\Psi(F_N))(k_2)$ is satisfied. Now, let $r \in R$ and $k \in K$. Since Ψ is surjective, there exists $\tilde{n} \in N$ such that $\Psi(\tilde{n}) = k$. Then, $(\Psi(F_N))(rk) = \bigcup\{F(n) : n \in N, \Psi(n) = rk\} = \bigcup\{F(n) : n \in N, n = \Psi^{-1}(rk)\} = \bigcup\{F(n) : n \in N, i = \Psi^{-1}(r\Psi(\tilde{n}))\} = \bigcup\{F(n) : n \in N, i = \Psi^{-1}(\Psi(r\tilde{n})) = r\tilde{n}\} = \bigcup\{F(r\tilde{n}) : r\tilde{n} \in N, \Psi(\tilde{n}) = k\} \supseteq \bigcup\{F(\tilde{n}) : \tilde{n} \in N, \Psi(\tilde{n}) = k\} = (\Psi(F_N))(k)$. Hence, $\Psi(F_N)$ is a soft submodule of M.

Theorem 24. *Let F_N and G_K be soft sets over M, where N and K are submodules of M and Ψ be a module homomorphism from N to K. If G_K is a soft submodule of M, then so is $\Psi^{-1}(G_K)$.*

Proof. Let $n_1, n_2 \in N$. As in the case of Theorem 4, $(\Psi^{-1}(G_K))(n_1 - n_2) \supseteq (\Psi^{-1}(G_K))(n_1) \cap (\Psi^{-1}(G_K))(n_2)$ is satisfied. Now let $r \in R$ and $n \in N$. Then, $(\Psi^{-1}(G_K))(rn) = G(\Psi(rn)) = G(r\Psi(n)) \supseteq G(\Psi(n)) = (\Psi^{-1}(G_K))(n)$. Hence, $\Psi^{-1}(G_K)$ is a soft submodule of M.

6 Union soft submodules

In this section, we introduce union soft submodules of a module, investigate its basic properties and establish the relation between soft submodules and union soft submodules.

Definition 11. *Let N be a submodule of M and F_N be a soft set over M. Then, F_N is called a union soft submodule of M, denoted by $(F,N)\widetilde{<}_u M$ or simply $F_N \widetilde{<}_u M$, if $F(x-y) \subseteq F(x) \cup F(y)$ and $F(rx) \subseteq F(x)$ for all $x, y \in N$ and $r \in R$.*

Example 4. Consider the ring $R = (\mathbb{Z}_{12}, +, .)$, the left R-module $M = (\mathbb{Z}_{12}, +)$ with natural operation and the submodule $N_1 = \{0, 6\}$ of M. Let the soft set F_{N_1} over M, where $F : N_1 \to P(M)$ is a set valued function defined by $F(0) = \{0, 4, 9\}$ and $F(6) = \{0, 3, 4, 9, 11\}$. Then, it can be easily seen that $(F, N_1)\widetilde{<}M$. Now, let the submodule of M be $N_2 = \{0, 4, 8\}$ and the soft set G_{N_2} over M, where $G : N_2 \to P(M)$ is a set valued function defined by $G(0) = \{0, 3, 9\}$ and $G(4) = \{0, 3, 5, 8, 11\}$ and $G(8) = \{0, 3, 5, 8, 9, 11\}$. Then, $G(2 \cdot 4) = G(8) = \{0, 3, 5, 8, 9, 11\} \nsubseteq G(4) = \{0, 3, 5, 8, 11\}$. Therefore, G_{N_2} is not a union soft submodule of M.

The following theorems are given without their proofs, since one can easily show them in view of Section 5.

Theorem 25. *If F_{N_1} is a union soft submodule of M and G_{N_2} is a union soft submodule of M, then so is $F_{N_1} \cup_{\mathscr{R}} G_{N_2}$.*

Theorem 26. *If $F_N \widetilde{<}_u M$, then $F(0_M) \subseteq F(x)$ for all $x \in N$.*

Theorem 27. *If $F_N \widetilde{<}_u M$, then $N_F = \{x \in N \mid F(x) = F(0_M)\}$ is a submodule of N.*

Theorem 28. *Let F_N be a soft set over M and α be a subset of M such that $F(0_M) \subseteq \alpha$. If F_N is a union soft submodule of M, then $F_N^{\subseteq \alpha}$ is a submodule of M.*

Theorem 29. *Let F_N be a soft set over M. Then, F_N is a union soft submodule of M if and only if F_N^r is a soft submodule of M.*

Theorem 30. *Let F_N and G_K be soft sets over M, where N and K are submodules of M and Ψ be a module homomorphism from N to K. If G_K is a union soft submodule of M, then so is $\Psi^{-1}(G_K)$.*

Theorem 31. *Let F_N and G_K be soft sets over R, where N and K are submodules of M and Ψ be a module isomorphism from N to K. If F_N is a union soft submodule of M, then so is $\Psi^{\star}(F_N)$.*

Theorem 32. *Let M_1 and M_2 be two R-modules, $F_{N_1}\widetilde{<}_u M_1$, $H_{N_2}\widetilde{<}_u M_2$. If $f : N_1 \to N_2$ is a module homomorphism, then i) $H_{f(N_1)}\widetilde{<}_u M_2$ and $F_{Ker f}\widetilde{<}_u M_1$, ii) If f is an epimorphism, $F_{f^{-1}(N_2)}\widetilde{<}_u M_1$.*

Corollary 2. *Let $F_N\widetilde{<}_u M_1$, $H_{N_2}\widetilde{<}_u M_2$ and $f : N_1 \to N_2$ is a module homomorphism, then $H_{\{0_{N_2}\}}\widetilde{<}_u M_2$.*

7 Conclusion

Atagün and Sezgin in [4] defined soft subrings and soft ideals of a ring, soft subfields of a field and soft submodule of a left module. In this paper, we have introduced union soft subrings and union soft ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower α-inclusion of soft sets. We also obtain significant relations between soft subrings and union soft subrings, soft ideals and union soft ideals of a ring and soft submodules and union soft submodules of a left module. To extend this work, one could study the union soft substructures of different algebras.

References

[1] Acar, U., Koyuncu, F., Tanay, B.: Soft sets and soft rings. Comput. Math. Appl. **59**, 3458-3463 (2010)

[2] Aktaş, H., Çağman, N.: Soft sets and soft groups, Inform. Sci. **177**, 2726-2735 (2007)

[3] Ali, M.I., Feng, F, Liu, X., Min, W.K., Shabir, M.: On some new operations in soft set theory, Comput. Math. Appl. **57**, 1547-1553 (2009)
[4] Atagün, A.O., Sezgin, A.: Soft substructures of rings, fields and modules, Comput. Math. Appl. **61** (3), 592-601 (2011)
[5] Babitha K.V. and Sunil J.J.: Soft set relations and functions, Comput. Math. Appl. **60** (7), 1840-1849 (2010).
[6] Çağman, N., Enginoğlu, S.: Soft matrix theory and its decision making, Comput. Math. Appl. **59**, 3308-3314 (2010)
[7] Çağman, N., Enginoğlu, S.: Soft set theory and uni-int decision making, Eur. J. Op. Res. **207**, 848-855 (2010)
[8] Çağman, Çıtak F. and Aktaş H.: Soft int-groups and its applications to group theory, Neural Comput. Appl., DOI: 10.1007/s00521-011-0752-x.
[9] Çağman N., Sezer A.S., Atagün A.O.: Soft union groups and their applications, (submitted).
[10] Çağman N., Sezer A.S., Atagün A.O.: α-inclusions of a soft set and their applications to group theory, (submitted).
[11] Feng, F., Jun, Y.B., Zhao, X.: Soft semirings, Comput. Math. Appl. **56**, 2621-2628 (2008)
[12] Feng, F., Jun, Y.B., Liu X.Y., Li L.F.: An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. **234**, 10-20 (2010)
[13] Feng F., Li C., Davvas B., Ali, M.I.: Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. **14** (6), 899-911 (2010)
[14] Feng F., Liu X.Y., Leoreanu-Fotea V., Jun Y.B.: Soft sets and soft rough sets, Inform. Sci. **181**, 1125-1137 (2011)
[15] Gong K., Xiao Z. and Zhang X., The bijective soft set with its operations, Comput. Math. Appl. **60** (8), 2270-2278 (2010)
[16] Jun, Y.B.: Soft BCK/BCI-algebras, Comput. Math. Appl. **56**, 1408-1413 (2008)
[17] Jun, Y.B., Park, C.H.: Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. **178**, 2466-2475 (2008)
[18] Jun, Y.B., Lee, K.J., Zhan, J.: Soft p-ideals of soft BCI-algebras, Comput. Math Appl. **58**, 2060-2068 (2009)
[19] Jun Y.B., Lee K.J., Park C.H.: Soft set theory applied to ideals in d-algebras, Comput. Math. Appl. **57** (3), 367-378 (2009)
[20] Jun Y.B., Lee K.J., Khan A., Soft ordered semigroups, Math. Logic Q. **56** (1), 42-50 (2010)
[21] Jun Y.B.: Union-soft sets with applications in BCK-BCI-algebras, Bull. Korean Math. Soc. **50** 6, 1937-1956 (2013)
[22] KazancıO., Yılmaz Ş. and Yamak S., Soft sets and soft BCH-algebras, Hacet. J. Math. Stat. **39** (2), 205-217 (2010)
[23] Kim C.S., Kang J.G. and Kim J.S.: Uni-soft (Quasi) Ideals of Semigroups, Applied Mathematical Sciences, **7** 50, 2455-2468 (2013)
[24] K. J. Lee, Union-Soft Filters of CI-Algebras, Applied Mathematical Sciences, **7** 117, 5831-5838 (2013)
[25] Maji, P.K., Biswas, R., Roy, A.R.: Soft set theory, Comput. Math. Appl. **45**, 555-562 (2003)
[26] Maji, P.K.,Roy, A.R. and Biswas R.: An application of soft sets in a decision making problem, Comput. Math. Appl. **44**, 1077-1083 (2002)
[27] Majumdar P. and Samanta S.K.: On soft mappings, Comput. Math. Appl. **60** (9), 2666-2672 (2010)
[28] Molodtsov, D.: Soft set theory-first results, Comput. Math. Appl. **37**, 19-31 (1999)
[29] Molodtsov D.A., Leonov V.Y. and Kovkov D.V.: Soft sets technique and its application, Nechetkie Sistemi Myakie Vychisleniya 1 **(1)**, 8-39 (2006)
[30] Sezgin, A., Atagün, A.O.: On operations of soft sets. Comput. Math. Appl. **61**(5), 1457-1467 (2011)
[31] Sezgin,. A, Atagün, A.O., Aygün, E.: A note on soft near-rings and idealistic soft near-rings. Filomat **25** (1), 53-68 (2011)
[32] Xiao Z., Gong K., Xia S., Zou Y., Exclusive disjunctive soft sets, Comput. Math. Appl. **59** (6), 2128-2137 (2010)
[33] Zhan, J., Jun, Y. B.: Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. **59** (6), 2037-2046 (2010)
[34] Zou Y. and Xiao Z., Data analysis approaches of soft sets under incomplete information, Knowledge Based Systems **21**, 941-945 (2008)

Aslıhan Sezgin Sezer is Assistant Professor Doctor of Mathematics at Amasya University. She received the PhD degree in Mathematics at Gaziosmanpaşa University (Turkey). Her research interests are in the areas of soft set and its algebraic applications and general near-ring theory.

Akın Osman Atagün is Associate Professor of Mathematics at Bozok University. He received the PhD degree in Mathematics at Erciyes University (Turkey). His main research interests are: Near-ring, Prime ideals of near-rings and near-ring modules, Soft set and its algebraic applications.

Naim Çağman is Associate Professor of Mathematics at Gaziosmanpaşa University. He received the PhD degree in Mathematics at University of Leeds (UK). His main research interests are: Mathematical Logic, Fuzzy Logic, Soft Set Theory, Game Development, Mathematics Education.