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Intersection Soft Subspaces and Union Soft Subspaces with their Applications

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Abstract: In this paper, we first introduce two kinds of subspaces of a vector space with respect to soft structures, which are intersection-soft subspace (IS-subspace) and union-soft subspace (US-subspace). These new concepts shows how a soft set affects on a subspace of a vector space in the mean of intersection, union and inclusion of sets and thus, they can be regarded as a bridge among classical sets, soft sets and vector spaces. We then investigate their related properties with respect to soft set operations, soft image, soft preimage, soft anti image, α-inclusion of soft sets and linear transformations of the vector spaces. Furthermore, we obtain the relation between IS-subspaces and US-subspaces and give the applications of these new subspaces on vector spaces.

Keywords: Soft set, IS-subspace, US-subspace, soft image, soft anti image, α-inclusion

1 Introduction

Soft set theory was introduced by Molodtsov [26] for modeling vagueness and uncertainty and it has received much attention since Maji et al. [23], Ali et al. [6] and Sezgin and Atagün [29] introduced and studied operations of soft sets. Soft set theory has also potential applications especially in decision making as in [10,11, 24,32]. This theory has started to progress in the mean of algebraic structures, since Aktaş and Çağman [5] defined and studied soft groups. Since then, soft semirings [15], soft BCK/BCI-algebras [18], soft p-ideals [19], soft BCH-algebras [20], soft rings [4], soft near-rings [30], soft set relations and functions [9], soft mappings [25], soft substructures of rings, fields and modules [8], union soft substructures of near-rings and near-ring modules [28], normalistic soft groups [27] are defined and studied in detailed.Soft set has also been studied in the following papers [1,2,3,21,22,31].

In this paper, we first introduce intersection soft subspace of a vector space that is abbreviated by IS-subspace and investigate its related properties with respect to soft set operations. We then give the application of soft image, soft preimage, upper α-inclusion of soft sets, linear transformations of vector spaces on vector spaces in the mean of IS-subspaces. Then, we introduce union soft subspace of a vector space that is abbreviated by US-subspace and investigate its related properties and obtain a significant relation between IS-subspaces and US-subspaces. Moreover, we apply soft preimage, soft anti-image, lower α-inclusion of soft sets, linear transformations of vector spaces on this soft subspace. This study is of great importance since SI-subspaces and SU-subspaces show how a soft set affect on a subspace of a vector space in the mean of intersection, union and inclusion of sets, so it functions as a bridge among classical sets, soft sets and vector spaces.

2 Preliminaries

Let U be a universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1.*[26] A pair (F,A) is called a* soft set *over U, where F is a mapping given by*

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U.

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Note that a soft set (F,A) can be denoted by F_A. In this case, when we define more then one soft set in some subsets A, B, C of parameters E, the soft sets will be denoted by F_A, F_B, F_C, respectively. On the other case, when we define more then one soft set in a subset A of the set of parameters E, the soft sets will be denoted by F_A, G_A, H_A, respectively. For more details, we refer to [11,16, 17,23,26,7].

Definition 2.*[6] The relative complement of the soft set F_A over U is denoted by F_A^r, where $F_A^r : A \to P(U)$ is a mapping given as $F_A^r(\alpha) = U \setminus F_A(\alpha)$, for all $\alpha \in A$.*

Definition 3.*[6] Let F_A and G_B be two soft sets over U such that $A \cap B \neq \emptyset$. The restricted intersection of F_A and G_B is denoted by $F_A \Cap G_B$, and is defined as $F_A \Cap G_B = (H,C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.*

Definition 4.*[6] Let F_A and G_B be two soft sets over U such that $A \cap B \neq \emptyset$. The restricted union of F_A and G_B is denoted by $F_A \cup_{\mathscr{R}} G_B$, and is defined as $F_A \cup_{\mathscr{R}} G_B = (H,C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.*

Definition 5.*[12] Let F_A and G_B be soft sets over the common universe U and Ψ be a function from A to B. Then we can define the soft set $\Psi(F_A)$ over U, where $\Psi(F_A) : B \to P(U)$ is a set valued function defined by*

$$\Psi(F_A)(b) = \begin{cases} \bigcup\{F(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise \end{cases}$$

for all $b \in B$. Here, $\Psi(F_A)$ is called the soft image of F_A under Ψ. Moreover we can define a soft set $\Psi^{-1}(G_B)$ over U, where $\Psi^{-1}(G_B) : A \to P(U)$ is a set-valued function defined by $\Psi^{-1}(G_B)(a) = G(\Psi(a))$ for all $a \in A$. Then, $\Psi^{-1}(G_B)$ is called the soft preimage (or inverse image) of G_B under Ψ.

Definition 6.*[13] Let F_A and G_B be soft sets over the common universe U and Ψ be a function from A to B. Then we can define the soft set $\Psi^\star(F_A)$ over U, where $\Psi^\star(F_A) : B \to P(U)$ is a set-valued function defined by*

$$\Psi^\star(F_A)(b) = \begin{cases} \bigcap\{F(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise \end{cases}$$

for all $b \in B$. Here, $\Psi^\star(F_A)$ is called the soft anti image of F_A under Ψ.

Theorem 21*[13] Let F_H and T_K be soft sets over U, F_H^r, T_K^r be their relative soft sets, respectively and Ψ be a function from H to K. Then,*

i)$\Psi^{-1}(T_K^r) = (\Psi^{-1}(T_K))^r$.
ii)$\Psi(F_H^r) = (\Psi^\star(F_H))^r$ and $\Psi^\star(F_H^r) = (\Psi(F_H))^r$.

Definition 7.*[14] Let F_A be a soft set over U and α be a subset of U. Then upper α-inclusion of F_A, denoted by $F_A^{\supseteq\alpha}$, is defined as*

$$F_A^{\supseteq\alpha} = \{x \in A \mid F(x) \supseteq \alpha\}.$$

Similarly,

$$F_A^{\subseteq\alpha} = \{x \in A \mid F(x) \subseteq \alpha\}$$

is called the lower α-inclusion of F_A.

A nonempty subset U of a vector space V is called a *subspace* of V if U is a vector space on F. From now on, V denotes a vector space over F and if U is a subspace of V, then it is denoted by $U < V$.

3 IS-subspaces

In this section, we first define intersection-soft subspace of a vector space, abbreviated as IS-subspace. We then investigate its related properties with respect to soft set operations.

Definition 8.*Let U be a subspace of V and G_U be a soft set over V. Then G_U is called an IS-subspace of V, denoted by $G_U \widetilde{<}_i V$, if the following properties are satisfied:*

s1)$G(x+y) \supseteq G(x) \cap G(y)$ and
s2)$G(\alpha x) \supseteq G(x)$

for all $x, y \in U$ and $\alpha \in F$.

Example 3.1*Let the vector space over $\mathbb{Z}_2$ be*

$$V = \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}$$

and the subspace U of V be V itself.
Let the soft set G_U over V, where $G : U \to P(V)$ is a set-valued function defined by

$$\begin{aligned} G\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}, \\ G\left(\begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}, \\ G\left(\begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}, \\ G\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix} \right\}. \end{aligned}$$

Then, one can show that $G_U \widetilde{<}_i V$. However, if we define the soft set H_U over V such that

$$\begin{aligned} H\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix} \right\}, \\ H\left(\begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}, \\ H\left(\begin{bmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix} \right\}, \\ H\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix}\right) &= \left\{ \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{bmatrix} \right\} \end{aligned}$$

then,

$$H\left(\overline{0}.\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix}\right) = H\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}\right) \not\supseteq H\left(\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{bmatrix}\right).$$

Thus, H_U is not an IS-subspace of V.

It is easy to see that if we take the subspace of V as $U = \{0_V\}$, where 0_V is the zero element of V, then it is obvious that G_U is an IS-subspace of V no matter how G is defined. Thus, every vector space has at least one IS-subspace.

Proposition 3.2 *If $G_U \widetilde{<}_i V$, then $G(0_V) \supseteq G(x)$ for all $x \in U$.*

Proof. Since G_U is an IS-subspace of V, then

$$G(x+y) \supseteq G(x) \cap G(y).$$

for all $x,y \in U$ and since $(U,+)$ is a group, if we take $y = -x$ then, for all $x \in U$,

$$G(x-x) = G(0_V) \supseteq G(x) \cap G(x) = G(x).$$

Theorem 31 *If $G_{U_1} \widetilde{<}_i V$ and $H_{U_2} \widetilde{<}_i V$, then $G_{U_1} \Cap H_{U_2} \widetilde{<}_i V$.*

Proof. Since U_1 and U_2 are subspaces of V, then $U_1 \cap U_2$ is a subspace of V. By Definition 3, let $G_{U_1} \Cap H_{U_2} = (G,U_1) \Cap (H,U_2) = (T, U_1 \cap U_2)$, where $T(x) = G(x) \cap H(x)$ for all $x \in U_1 \cap U_2 \neq \emptyset$. Then for all $x,y \in U_1 \cap U_2$ and $\alpha \in F$,

s1) $T(x+y) = G(x+y) \cap H(x+y) \supseteq (G(x) \cap G(y)) \cap (H(x) \cap H(y)) = (G(x) \cap H(x)) \cap (G(y) \cap H(y)) = T(x) \cap T(y)$,
s2) $T(\alpha x) = G(\alpha x) \cap H(\alpha x) \supseteq G(x) \cap H(x) = T(x)$.

Therefore $G_{U_1} \Cap H_{U_2} = T_{U_1 \cap U_2} \widetilde{<}_i V$.

Definition 9. *Let (G,U_1) and (H,U_2) be two IS-subspaces of V_1 and V_2, respectively. The product of IS-subspaces (G,U_1) and (H,U_2) is defined as $(G,U_1) \times (H,U_2) = (Q, U_1 \times U_2)$, where $Q(x,y) = G(x) \times H(y)$ for all $(x,y) \in U_1 \times U_2$.*

Theorem 32 *If $G_{U_1} \widetilde{<}_i V_1$ and $H_{U_2} \widetilde{<}_i V_2$, then $G_{U_1} \times H_{U_2} \widetilde{<}_i V_1 \times V_2$.*

Proof. Since U_1 and U_2 are subspaces of V_1 and V_2, respectively, then $U_1 \times U_2$ is a subspace of $V_1 \times V_2$. By Definition 9, let $G_{U_1} \times H_{U_2} = (G,U_1) \times (H,U_2) = (Q, U_1 \times U_2)$, where $Q(x,y) = G(x) \times H(y)$ for all $(x,y) \in U_1 \times U_2$. Then for all $(x_1,y_1),(x_2,y_2) \in U_1 \times U_2$ and $(\alpha_1,\alpha_2) \in F \times F$,

s1) $Q((x_1,y_1)+(x_2,y_2)) = Q(x_1+x_2, y_1+y_2) = G(x_1+x_2) \times H(y_1+y_2) \supseteq (G(x_1) \cap G(x_2)) \times (H(y_1) \cap H(y_2)) = (G(x_1) \times H(y_1)) \cap (G(x_2) \times H(y_2)) = Q(x_1,y_1) \cap Q(x_2,y_2)$,
s2) $Q((\alpha_1,\alpha_2)(x_1,y_1)) = Q(\alpha_1 x_1, \alpha_2 y_1) = G(\alpha_1 x_1) \times H(\alpha_2 y_1) \supseteq G(x_1) \times H(y_1) = Q(x_1,y_1)$.

Hence $G_{U_1} \times H_{U_2} = Q_{U_1 \times U_2} \widetilde{<}_i V_1 \times V_2$.

Definition 10. *Let G_{U_1} and H_{U_2} be two IS-subspaces of V. If $U_1 \cap U_2 = \{0_V\}$, then the sum of IS-subspaces G_{U_1} and H_{U_2} is defined as $G_{U_1} + H_{U_2} = T_{U_1+U_2}$, where $T(x+y) = G(x) + H(y)$ for all $x+y \in U_1+U_2$.*

Theorem 33 *If $G_{U_1} \widetilde{<}_i V$ and $H_{U_2} \widetilde{<}_i V$ where $U_1 \cap U_2 = \{0_V\}$, then $G_{U_1} + H_{U_2} \widetilde{<}_i V$.*

Proof. Since U_1 and U_2 are subspaces of V, then U_1+U_2 is a subspace of V. By Definition 10, let $G_{U_1} + H_{U_2} = (G,U_1) + (H,U_2) = (T, U_1+U_2)$, where $T(x+y) = G(x) + H(y)$ for all $x+y \in U_1+U_2$. It is obvious that since $U_1 \cap U_2 = \{0_V\}$, then the sum is well defined. Then for all $x_1+y_1, x_2+y_2 \in U_1+U_2$ and $\alpha \in F$,

$$\begin{aligned} T((x_1+y_1)+(x_2+y_2)) &= T((x_1+x_2)+(y_1+y_2)) \\ &= G(x_1+x_2) + H(y_1+y_2) \\ &\supseteq (G(x_1) \cap G(x_2)) + (H(y_1) \cap H(y_2)) \\ &= (G(x_1)+H(y_1)) \cap (G(x_2)+H(y_2)) \\ &= T(x_1+y_1) \cap T(x_2+y_2), \end{aligned}$$

$$\begin{aligned} T(\alpha(x_1+y_1)) &= T(\alpha x_1 + \alpha y_1) \\ &= G(\alpha x_1) + H(\alpha y_1) \\ &\supseteq G(x_1) + H(y_1) \\ &= T(x_1+y_1). \end{aligned}$$

Thus, $G_{U_1} + H_{U_2} \widetilde{<}_i V$.

Definition 11. *Let G_U be an IS-subspace of V. Then,*

a) G_U is said to be trivial if $G(x) = \{0_V\}$ for all $x \in U$.
b) G_U is said to be whole if $G(x) = V$ for all $x \in U$.

Proposition 3.3 *Let G_{U_1} and H_{U_2} be IS-subspaces of V. Then,*

i) If G_{U_1} and H_{U_2} are trivial IS-subspaces of V, then $G_{U_1} \Cap H_{U_2}$ is a trivial IS-subspace of V.
ii) If G_{U_1} and H_{U_2} are whole IS-subspaces of V, then $G_{U_1} \Cap H_{U_2}$ is a whole IS-subspace of V.
iii) If G_{U_1} is a trivial IS-subspace of V and H_{U_2} is a whole IS-subspace of V, then $G_{U_1} \Cap H_{U_2}$ is a trivial IS-subspace of V.
iv) If G_{U_1} and H_{U_2} are trivial IS-subspaces of V where $U_1 \cap U_2 = \{0_V\}$, then $G_{U_1} + H_{U_2}$ is a trivial IS-subspace of V.
v) If G_{U_1} and H_{U_2} are whole IS-subspaces of V where $U_1 \cap U_2 = \{0_V\}$, then $G_{U_1} + H_{U_2}$ is a whole IS-subspace of V.
vi) If G_{U_1} is a trivial IS-subspace of V and H_{U_2} is a whole IS-subspace of V where $U_1 \cap U_2 = \{0_V\}$, then $G_{U_1} + H_{U_2}$ is a whole IS-subspace of V.

Proof. The proof is easily seen by Definition 3, Definition 10, Definition 11, Theorem 31 and Theorem 33.

Proposition 3.4 *Let G_{U_1} and H_{U_2} be two IS-subspaces of V_1 and V_2, respectively. Then,*

i) If G_{U_1} and H_{U_2} are trivial IS-subspaces of V_1 and V_2, respectively, then $G_{U_1} \times H_{U_2}$ is a trivial IS-subspace of $V_1 \times V_2$.
ii) If G_{U_1} and H_{U_2} are whole IS-subspaces of V_1 and V_2, respectively, then $G_{U_1} \times H_{U_2}$ is a whole IS-subspace of $V_1 \times V_2$.

Proof. The proof is easily seen by Definition 9 , Definition 11 and Theorem 3 .

4 Applications of IS-subspaces

In this section, we give the applications of soft image, soft preimage, upper α-inclusion of soft sets and linear transformation of vector spaces on vector space with respect to IS-subspaces.

Theorem 41 *If $G_U \widetilde{<}_i V$, then $U_G = \{x \in U \mid G(x) = G(0_V)\}$ is a subspace of U.*

Proof. It is obvious that $0_V \in U_G$ and $\emptyset \neq U_G \subseteq U$. We need to show that $x+y \in U_G$ and $\alpha x \in U_G$ for all $x,y \in U_G$ and $\alpha \in F$, which means that $G(x+y) = G(0_V)$ and $G(\alpha x) = G(0_V)$ have to be satisfied. Since $x,y \in U_G$ and G_U is an IS-subspace of V, then $G(x) = G(y) = G(0_V)$,

$$G(x+y) \supseteq G(x) \cap G(y) = G(0_V), G(\alpha x) \supseteq G(x) = G(0_V)$$

for all $x,y \in U_G$ and $\alpha \in F$. Moreover, by Proposition 3.2,

$$G(0_V) \supseteq G(x+y) \text{ and } G(0_V) \supseteq G(\alpha x)$$

which completes the proof.

Theorem 42 *Let G_U be a soft set over V and α be a subset of V such that $G(0_V) \supseteq \alpha$. If G_U is an IS-subspace of V, then $G_U^{\supseteq \alpha}$ is a subspace of V.*

Proof. Since $G(0_V) \supseteq \alpha$, then $0_V \in G_U^{\supseteq \alpha}$ and $\emptyset \neq G_U^{\supseteq \alpha} \subseteq V$. Let $x,y \in G_U^{\supseteq \alpha}$, then

$$G(x) \supseteq \alpha \text{ and } G(y) \supseteq \alpha.$$

We need to show that $x+y \in G_U^{\supseteq \alpha}$ an $mx \in G_U^{\supseteq \alpha}$ for all $x,y \in G_U^{\supseteq \alpha}$ and $m \in F$. Since G_U is an IS-subspace of V, it follows that

$$G(x+y) \supseteq G(x) \cap G(y) \supseteq \alpha \cap \alpha = \alpha.$$

Furthermore, $G(mx) \supseteq G(x) \supseteq \alpha$, which completes the proof.

Theorem 43 *Let G_U and T_W be soft sets over V, where U and W are subspaces of V and Ψ be a linear isomorphism from U to W. If G_U is an IS-subspace of V, then so is $\Psi(G_U)$.*

Proof. Let $w_1, w_2 \in W$. Since Ψ is a surjective linear transformation, then there exists $u_1, u_2 \in U$ such that $\Psi(u_1) = w_1$ and $\Psi(u_2) = w_2$. Then,

$$\begin{aligned}
(\Psi(G_U))(w_1+w_2) &= \bigcup\{G(u) : u \in U, \Psi(u) = w_1+w_2\} \\
&= \bigcup\{G(u) : u \in U, u = \Psi^{-1}(w_1+w_2)\} \\
&= \bigcup\{G(u) : u \in U, u = \Psi^{-1}(\Psi(u_1+u_2)) = u_1+u_2\} \\
&= \bigcup\{G(u_1+u_2) : u_i \in U, \Psi(u_i) = w_i, i = 1,2\} \\
&\supseteq \bigcup\{G(u_1) \cap G(u_2) : u_i \in U, \Psi(u_i) = w_i, i = 1,2\} \\
&= (\bigcup\{G(u_1) : u_1 \in U, \Psi(u_1) = w_1\}) \\
&\cap (\bigcup\{G(u_2) : u_2 \in U, \Psi(u_2) = w_2\}) \\
&= (\Psi(G_U))(w_1) \cap (\Psi(G_U))(w_2)
\end{aligned}$$

Now, let $\alpha \in F$ and $w \in W$. Since Ψ is a surjective linear transformation, there exists $\tilde{u} \in U$ such that $\Psi(\tilde{u}) = w$. Then,

$$\begin{aligned}
(\Psi(G_U))(\alpha.w) &= \bigcup\{G(u) : u \in U, \Psi(u) = \alpha.w\} \\
&= \bigcup\{G(u) : u \in U, u = \Psi^{-1}(\alpha.w)\} \\
&= \bigcup\{G(u) : u \in U, u = \Psi^{-1}(\Psi(\alpha.\tilde{u})) = \alpha.\tilde{u}\} \\
&= \bigcup\{G(\alpha.\tilde{u}) : \alpha.\tilde{u} \in U, \Psi(\tilde{u}) = w\} \\
&\supseteq \bigcup\{G(\tilde{u}) : \tilde{u} \in U, \Psi(\tilde{u}) = w\} \\
&= (\Psi(G_U))(w)
\end{aligned}$$

Hence, $\Psi(G_U)$ is an IS-subspace of V.

Theorem 44 *Let G_U and T_W be soft sets over V, where U and W are subspaces of V and Ψ be a linear transformation from U to W. If T_W is an IS-subspace of V, then so is $\Psi^{-1}(T_W)$.*

Proof. Let $u_1, u_2 \in U$. Then,

$$\begin{aligned}
(\Psi^{-1}(T_W))(u_1+u_2) &= T(\Psi(u_1+u_2)) \\
&= T(\Psi(u_1)+\Psi(u_2)) \\
&\supseteq T(\Psi(u_1)) \cap T(\Psi(u_2)) \\
&= (\Psi^{-1}(T_W))(u_1) \cap (\Psi^{-1}(T_W))(u_2)
\end{aligned}$$

Now let $\alpha \in F$ and $u \in U$. Then,

$$\begin{aligned}
(\Psi^{-1}(T_W))(\alpha.u) &= T(\Psi(\alpha.u)) \\
&= T(\alpha.\Psi(u)) \\
&\supseteq T(\Psi(u)) \\
&= (\Psi^{-1}(T_W))(u)
\end{aligned}$$

Hence $\Psi^{-1}(T_W)$ is an IS-subspace of V.

Theorem 45 *Let V_1 and V_2 be two vector spaces and $(G_1, U_1) \widetilde{<}_i V_1$, $(G_2, U_2) \widetilde{<}_i V_2$. If $f : U_1 \to U_2$ is a linear transformation of vector spaces, then*

i) If f is surjective, then $(G_1, f^{-1}(U_2)) \widetilde{<}_i V_1$,
ii) $(G_2, f(U_1)) \widetilde{<}_i V_2$,
iii) $(G_1, Ker f) \widetilde{<}_i V_1$.

Proof.i) Since $U_1 < V_1$, $U_2 < V_2$ and $f : U_1 \to U_2$ is a surjective linear transformation, then it is clear that $f^{-1}(U_2) < V_1$. Since $(G_1, U_1) \widetilde{<}_i V_1$ and $f^{-1}(U_2) \subseteq U_1$, $G_1(x+y) \supseteq G_1(x) \cap G_1(y)$ and $G_1(\alpha x) \supseteq G_1(x)$ for all $x, y \in f^{-1}(U_2)$ and $\alpha \in F$. Hence $(G_1, f^{-1}(U_2)) \widetilde{<}_i V_1$.

ii) Since $U_1 < V_1$, $U_2 < V_2$ and $f : U_1 \to U_2$ is a vector space transformation, then $f(U_1) < V_2$. Since $f(U_1) \subseteq U_2$, the result is obvious by Definition 8.

iii) Since $Ker f < V_1$ and $Ker f \subseteq U_1$, the rest of the proof is clear by Definition 8.

Corollary 4.1 *Let $(G_1, U_1) \widetilde{<}_i V_1$, $(G_2, U_2) \widetilde{<}_i V_2$ and $f : U_1 \to U_2$ is a linear transformation, then $(G_2, \{0_{U_2}\}) \widetilde{<}_i V_2$.*

Proof. By Theorem 45 (iii), $(G_1, Ker f)\widetilde{<}_i V_1$. Then $(G_2, f(Ker f)) =$
$(G_2, \{0_{U_2}\})\widetilde{<}_i V_2$ by Theorem 45 (ii).

5 US-subspaces

In this section, we first define union soft subspace of a vector space, abbreviated by US-subspaces. We then investigate its related properties with respect to soft set operations.

Definition 12. *Let U be a subspace of V and T_U be a soft set over V. Then, the soft set T_U is called a US-subspace of V, denoted by $T_U\widetilde{<}_u V$, if the following properties are satisfied:*

s1) $T(x+y) \subseteq T(x)\cup T(y)$ *and*
s2) $T(\alpha x) \subseteq T(x)$,

for all $x, y \in U$ and $\alpha \in F$.

Example 5.1 *Consider the vector space V and the subspace U of V in Example 3.1. Let the soft set T_U over V, where $T : U \to P(V)$ is a set-valued function defined by*

$$
\begin{aligned}
T\left(\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix}\right\},\\
T\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right\},\\
T\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix}\right\},\\
T\left(\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right\}.
\end{aligned}
$$

Then, one can show that $T_U\widetilde{<}_u V$. However, if we define the soft set K_U over V such that

$$
\begin{aligned}
K\left(\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix}\right\},\\
K\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right\},\\
K\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix}\right\},\\
K\left(\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right) &= \left\{\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix},\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right\}
\end{aligned}
$$

then,
$K\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix}+\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix}\right) = K\left(\begin{bmatrix}\bar{0}&\bar{0}\\ \bar{1}&\bar{0}\end{bmatrix}\right) \nsubseteq K\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{1}&\bar{0}\end{bmatrix}\right) \cup K\left(\begin{bmatrix}\bar{0}&\bar{1}\\ \bar{0}&\bar{0}\end{bmatrix}\right)$. *Thus, K_U is not a US-subspace of V.*

It is easy to see that if we take the subspace of V as $U = \{0_V\}$, then it is obvious that G_U is a US-subspace of V no matter how G is defined. Thus, every vector space has at least one US-subspace as in the case of IS-subspace.

Proposition 5.2 *If $T_U\widetilde{<}_u V$, then $T(0_V) \subseteq T(x)$ for all $x \in U$.*

Proof. Since T_U is a US-subspace of V, then for all $x, y \in U$, $T(x+y) \subseteq T(x)\cup T(y)$. Since $(U,+)$ is a group, if we take $y = -x$ then,

$$T(x-x) = T(0_V) \subseteq T(x)\cup T(x) = T(x)$$

for all $x \in U$.

In Section 3, we showed that restricted intersection, the sum and the product of two IS-subspaces of V is an IS-subspace of V. Now, we show that the restricted union of two US-subspaces of V is a US-subspace of V with the following theorem:

Theorem 51 *If $G_{U_1}\widetilde{<}_u V$ and $T_{U_2}\widetilde{<}_u V$, then $G_{U_1}\cup_{\mathscr{R}} T_{U_2}\widetilde{<}_u V$.*

Proof. By Definition 4, let $G_{U_1}\cup_{\mathscr{R}} T_{U_2} = (G, U_1)\cup_{\mathscr{R}} (T, U_2) = (Q, U_1\cap U_2)$, where $Q(x) = G(x)\cup T(x)$ for all $x \in U_1\cap U_2 \neq \emptyset$. Since U_1 and U_2 are subspaces of V, then $U_1\cap U_2$ is a subspace of V. Let $x, y \in U_1\cap U_2$ and $\alpha \in F$, then

$$
\begin{aligned}
Q(x+y) &= G(x+y)\cup T(x+y)\\
&\subseteq (G(x)\cup G(y))\cup (T(x)\cup T(y))\\
&= (G(x)\cup T(x))\cup (G(y)\cup T(y))\\
&= Q(x)\cup Q(y)
\end{aligned}
$$

$$
\begin{aligned}
Q(\alpha x) &= G(\alpha x)\cup T(\alpha x)\\
&\subseteq G(x)\cup T(x)\\
&= Q(x)
\end{aligned}
$$

Therefore, $G_{U_1}\cup_{\mathscr{R}} T_{U_2} = Q_{U_1\cap U_2}\widetilde{<}_u V$.

Definition 13. *Let T_U be a US-subspace of V. Then,*

a) T_U is said to be trivial if $T(x) = \{0_V\}$ for all $x \in U$.
b) T_U is said to be whole if $T(x) = V$ for all $x \in U$.

Proposition 5.3 *Let G_{U_1} and T_{U_2} be US-subspaces of V, then*

i) If G_{U_1} and T_{U_2} are trivial US-subspaces of V, then $G_{U_1}\cup_{\mathscr{R}} T_{U_2}$ is a trivial US-subspace of V.
ii) If G_{U_1} and T_{U_2} are whole US-subspaces of V, then $G_{U_1}\cup_{\mathscr{R}} T_{U_2}$ is a whole US-subspace of V.
iii) If G_{U_1} is a trivial US-subspace of V and T_{U_2} is a whole US-subspace of V, then $G_{U_1}\cup_{\mathscr{R}} T_{U_2}$ is a whole US-subspace of V.

Proof. The proof is easily seen by Definition 4, Definition 13, Theorem 51.

6 Applications of US-subspaces

In this section, first we obtain the relation between IS-subspaces and US-subspaces and then give the applications of soft pre-image, soft anti image, lower α-inclusion of soft sets and linear transformation of vector spaces on vector spaces with respect to US-subspaces.

Theorem 61*Let T_U be a soft set over V. Then, T_U is a US-subspace of V if and only if T_U^r is an IS-subspace of V.*

*Proof.*Let T_U be a US-subspace of V. Then, for all $x,y \in U$ and $\alpha \in F$,

$$\begin{aligned} T^r(x+y) &= V \setminus T(x+y) \\ &\supseteq V \setminus ((T(x) \cup T(y)) \\ &= (V \setminus T(x)) \cap (V \setminus T(y)) \\ &= T^r(x) \cap T^r(y). \end{aligned}$$

$$\begin{aligned} T^r(\alpha x) &= V \setminus T(\alpha x) \\ &\supseteq V \setminus T(x) \\ &= T^r(x) \end{aligned}$$

Thus, T_U^r is an IS-subspace of V. Conversely, let T_U^r be an IS-subspace of V. Then, for all $x,y \in U$ and $\alpha \in F$,

$$\begin{aligned} T(x+y) &= V \setminus T^r(x+y) \\ &\subseteq V \setminus (T^r(x) \cap T^r(y)) \\ &= (V \setminus T^r(x)) \cup (V \setminus T^r(y)) \\ &= T(x) \cup T(y). \end{aligned}$$

$$\begin{aligned} T(\alpha x) &= V \setminus T^r(\alpha x) \\ &\subseteq V \setminus T^r(x) \\ &= T(x) \end{aligned}$$

Thus, T_U is a US-subspace of V.

Theorem 61 shows that if a soft set is a US-subspace of V, then its relative complement is an IS-subspace of V and vice versa.

Theorem 62*If $T_U \widetilde{<}_u V$, then $U_T = \{x \in U \mid T(x) = T(0_V)\}$ is a subspace of U.*

*Proof.*It is seen that $0_V \in U_T$ and $\emptyset \neq U_T \subseteq U$. We need to show that $x+y \in U_T$ and $\alpha x \in U_T$ for all $x,y \in U_T$ and $\alpha \in F$. Since $x,y \in U_T$ and T_U is a US-subspace of V, then $T(x) = T(y) = T(0_V)$,

$$T(x+y) \subseteq T(x) \cup T(y) = T(0_V), T(\alpha x) \subseteq T(x) = T(0_V)$$

for all $x,y \in U_T$ and $\alpha \in F$. Furthermore, by Proposition 5.2,

$$T(0_V) \subseteq T(x+y) \text{ and } T(0_V) \subseteq T(\alpha x).$$

Thus, the proof is completed.

Theorem 63*Let G_U be a soft set over V and α be a subset of V such that $\alpha \supseteq G(0_V)$. If G_U is a US-subspace of V, then $G_U^{\subseteq \alpha}$ is a subspace of V.*

*Proof.*Since $\alpha \supseteq G(0_V)$, then $0_V \in G_U^{\subseteq \alpha}$ and $\emptyset \neq G_U^{\subseteq \alpha} \subseteq V$. Let $x,y \in G_U^{\subseteq \alpha}$, then

$$G(x) \subseteq \alpha \text{ and } G(y) \subseteq \alpha.$$

We need to show that $x+y \in G_U^{\subseteq \alpha}$ an $mx \in G_U^{\subseteq \alpha}$ for all $x,y \in G_U^{\subseteq \alpha}$ and $m \in F$. Since G_U is a US-subspace of V, it follows that

$$G(x+y) \subseteq G(x) \cup G(y) \subseteq \alpha \cup \alpha = \alpha.$$

Moreover, $G(mx) \subseteq G(x) \subseteq \alpha$, which completes the proof.

Theorem 64*Let G_U and T_W be soft sets over V, where U and W are subspaces of V and Ψ be a linear transformation from U to W. If T_W is a US-subspace of V, then so is $\Psi^{-1}(T_W)$.*

*Proof.*Let T_W be a US-subspace of V. Then, T_W^r is an IS-subspace of V by Theorem 61 and $\Psi^{-1}(T_W^r)$ is an IS-subspace of V by Theorem 44. Thus, $\Psi^{-1}(T_W^r) = (\Psi^{-1}(T_W))^r$ is an IS-subspace of V by Theorem 21 (i). Therefore, $\Psi^{-1}(T_W)$ is a US-subspace of V by Theorem 61.

Theorem 65*Let G_U and T_W be soft sets over V, where U and W are subspaces of V and Ψ be a linear isomorphism from U to W. If G_U is a US-subspace of V, then so is $\Psi^\star(G_U)$.*

*Proof.*Let G_U be a US-subspace of V. Then, G_U^r is an IS-subspace of V by Theorem 61 and $\Psi(G_U^r)$ is an IS-subspace of V by Theorem 43. Thus, $\Psi(G_U^r) = (\Psi^\star(G_U))^r$ is an IS-subspace of V by Theorem 21 (ii). So, $\Psi^\star(G_U)$ is a US-subspace of V by Theorem 61.

Theorem 66*Let V_1 and V_2 be two vector spaces and $(T_1,U_1)\widetilde{<}_u V_1$, $(T_2,U_2)\widetilde{<}_u V_2$. If $f: U_1 \to U_2$ is a linear transformation, then*

i)If f is surjective, then $(T_1, f^{-1}(U_2))\widetilde{<}_u V_1$,
ii)$(T_2, f(U_1))\widetilde{<}_u V_2$,
iii)$(T_1, Ker f)\widetilde{<}_u V_1$.

*Proof.*Follows from Definition 12 and Theorem 45, therefore omitted.

Corollary 6.1*Let $(T_1,U_1)\widetilde{<}_u V_1$, $(T_2,U_2)\widetilde{<}_u V_2$ and $f: U_1 \to U_2$ is a linear transformation, then $(T_2, \{0_{V_2}\})\widetilde{<}_u V_2$.*

*Proof.*Follows from Theorem 66 (ii) by Theorem 66 (iii).

7 Conclusion

Throughout this paper, we have dealt with the IS-subspaces and US-subspaces of a vector space. We have investigated their related properties with respect to soft set operations and obtained the relations between IS-subspaces and US-subspaces. Furthermore, we have derived some applications of IS-subspaces and US-subspaces with respect to soft image, soft preimage, soft anti image, α-inclusion of soft sets and linear transformations of vector spaces. Further study could be done for soft substructures of different algebras.

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