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Common Fixed Point for Weakly Contractive Maps

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Abstract: In this paper, first we prove a common fixed point theorem for a pair of weakly compatible maps under weak contractive condition. Secondly, we prove common fixed point theorems for weakly compatible mappings along with E.A. and (CLR_f) properties. At the end, we prove a common fixed point theorem for variants of R-weakly commutative maps.

Keywords: Weakly compatible maps, weak contraction, E.A. property, (CLR_f) property, R-weakly commuting mapping of type (A_f), R-weakly commuting mapping of type (A_g), R-weakly commuting mapping of type (P)

1 Introduction

In 1922, the Polish mathematician, Banach proved a common fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banachs fixed point theorem or Banach contraction principle, which states that "Let (X,d) be a complete metric space. If T satisfies

$$d(Tx,Ty) \leq kd(x,y) \tag{1.1}$$

for each x,y in X, where $0 < k < 1$, then T has a unique fixed point in X". This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. This principle is basic tool in fixed point theory.

Many authors extended, generalized and improved Banach fixed point theorem in different ways. For the last quarter of the 20th century, there has been a considerable interest in the study of common fixed point of pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. The generalization of Banachs fixed point theorem by Jungck [9] gave a new direction to the "Fixed point theory Literature". This theorem has had many applications, but suffers from the drawback that the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand, Sessa [22] coined the notion of weak commutativity and proved common fixed point theorem for a pair of mappings.

Definition 1. *Two self-mappings f and g of a metric space (X,d) are said to be weakly commuting if $d(fgx,gfx) \leq d(gx,fx)$ for all x in X.*

Further, Jungck [10] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity.

Definition 2. *Two self-mappings f and g of a metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.*

This concept has been useful for obtaining fixed point theorems for compatible mappings satisfying contractive conditions and assuming continuity of atleast one of the mappings. It has been known from the paper of Kannan [12] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. This paper was a genesis for a multitude of fixed point papers over the next two decades.

In 1994, Pant [17] introduced the notion of R-weakly commuting mappings in metric spaces, firstly to widen

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the scope of the study of common fixed point theorems from the class of compatible to the wider class of R-weakly commuting mappings. Secondly, maps are not necessarily continuous at the fixed point.

Definition 3. *A pair of self-mappings (f,g) of a metric space (X,d) is said to be R-weakly commuting if there exists some $R \geq 0$ such that $d(fgx, gfx) \leq Rd(fx,gx)$ for all x in X.*

In 1997, Pathak et al. [18] introduced the improved notions of R-weakly commuting mappings and called these maps as R-weakly commuting mappings of type (A_f) and R-weakly commuting mappings of type (A_f).

Definition 4. *A pair of self-mappings (f,g) of a metric space (X,d) is said to be*

(i) *R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that*

$$d(fgx, ggx) \leq Rd(fx,gx) \quad \text{for all } x \text{ in } X.$$

(ii) *R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that*

$$d(gfx, ffx) \leq Rd(fx,gx) \quad \text{for all } x \text{ in } X.$$

In 1996, Jungck [11] introduced the concept of weakly compatible maps as follows:

Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Example 1. Let $X = \mathbb{R}$. Define $f,g : \mathbb{R} \to \mathbb{R}$ by $fx = x/3$, $x \in \mathbb{R}$ and $gx = x^2$, $x \in R$. Here 0 and $1/3$ are two coincidence points for the maps f and g. Note that f and g commute at 0, i.e., $fg(0) = gf(0) = 0$, but $fg(1/3) = f(1/9) = 1/27$ and $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible on $\mathbb{R}$.

Example 2. Weakly compatible maps need not be compatible. Let $X = [2,20]$ and d be the usual metric on X. Define mappings B, $T : X \to X$ by $Bx = x$ if $x = 2$ or > 5, $Bx = 6$ if $2 < x \leq 5$, $Tx = x$ if $x = 2$, $Tx = 12$ if $2 < x \leq 5$, $Tx = x - 3$ if $x > 5$. The mappings B and T are non-compatible, since sequence $\{x_n\}$ defined by $x_n = 5 + (1/n)$, $n \geq 1$. Then $Tx_n \to 2$, $Bx_n \to 2$, $TBx_n \to 2$ and $BTx_n \to 6$. But they are weakly compatible, since they commute at coincidence point at $x = 2$.

In 2009, Kumar et al. [14] introduced the notion of R-weakly commuting mapping of type (P) as follows:

Definition 5. *A pair of self-mappings (f,g) of a metric space (X,d) is said to be R-weakly commuting mapping of type (P) if there exists some $R > 0$ such that*

$$d(ffx, ggx) \leq Rd(fx,gx) \quad \text{for all } x \text{ in } X.$$

Remark. We have suitable examples to show that R-weakly commuting mappings, R-weakly commuting of type (A_f), R-weakly commuting of type (A_f) and R-weakly commuting of type (P) are distinct.

Example 3. Consider $X = [-1,1]$ with usual metric d defined by $d(x,y) = |x-y|$ for all x,y in X. Define $fx = |x|$ and $gx = |x| - 1$. Then by a straightforward calculation, one can show that $d(fx,gx) = 1$, $d(fgx,gfx) = 2(1-|x|)$, $d(fgx,ggx) = 1$, $d(gfx,ffx) = 1$, $d(ffx,ggx) = 2|x|$ for all x in X.

Now, we conclude the following:

(i) pair (f,g) is not weakly commuting.

(ii) for $R = 2$, pair (f,g) is R-weakly commuting, R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

(iii) for $R = \frac{3}{2}$, pair (f,g) is R-weakly commuting of type (A_f) but not R-weakly commuting of type (P) and R-weakly commuting.

Example 4. Consider $X = [0,1]$ with usual metric d defined by $d(x,y) = |x-y|$ for all x,y in X. Define $fx = x$ and $gx = x^2$. Then by a straightforward calculation, one can show that $ffx = x$, $gfx = x^2$, $fgx = x^2$, $ggx = x^4$ and $d(fgx,gfx) = 0$, $d(fgx,ggx) = |x^2(x-1)(x+1)|$, $d(gfx,ffx) = |x(x-1)|$, $d(ffx,ggx) = |(x^2+x+1)x(x-1)|$ and $d(fx,gx) = |x(x-1)|$ for all x in X.

Therefore, we conclude that

(i) pair (f,g) is R-weakly commuting for all positive real values of R.

(ii) for $R = 3$, pair (f,g) is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

(iii) for $R = 2$, pair (f,g) is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and not R-weakly commuting of type (P) (for this take $x = \frac{3}{4}$).

Example 5. Consider $X = [\frac{1}{2}, 2]$. Let us define self maps f and g by $fx = \frac{x+1}{3}$, $gx = \frac{x+2}{5}$.

We calculate the following:

$$d(fx,gx) = \frac{2x-1}{15},\ d(fgx,gfx) = 0,$$
$$d(fgx,ggx) = \frac{2x-1}{75},\ d(gfx,ffx) = \frac{2x-1}{45}$$
$$\text{and } d(ffx,ggx) = \frac{8(2x-1)}{225}.$$

Now, we conclude the following:

The pair (f,g) is R-weakly commuting for all positive real numbers.

For $R \geq \frac{8}{15}$, it is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

For $\frac{1}{3} \leq R < \frac{8}{15}$, it is R-weakly commuting of type (A_f) and R-weakly commuting of type (A_g) but not R-weakly commuting of type (P).

For $\frac{1}{5} \leq R < \frac{1}{3}$, it is R-weakly commuting of type (A_f) but not R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

Moreover, such mappings commute at their coincidence points. It is also obvious that f and g can fail to be pointwise R-weakly commuting only if there exists some x in X such that $fx = gx$ but $fgx \neq gfx$, that is, only if they possess a coincidence point at which they do not commute. Therefore, the notion of pointwise R-weak commutativity type mapping is equivalent to commutativity at coincidence points.

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 6. *Two self-mappings f and g of a metric space (X,d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.*

Example 6. Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ and let $fx = \frac{1}{5}x$ and $gx = \frac{3}{5}x$ for each $x \in X$.

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$, where $0 \in X$. Hence the pair (f,g) satisfy the E.A. property.

In 2011, Sintunavarat et al. [23] introduced the notion of (CLR_f) property as follows:

Definition 7. *Two self-mappings f and g of a metric space (X,d) are said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some x in X.*

Example 7. Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ and let $fx = x$ and $gx = x^2$ for each $x \in X$.

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = f(0)$. Hence the pair (f,g) satisfy the (CLR_f) property.

2 Main results

In 1984, Khan et al. [13] addressed a new category of fixed point problems with the help of a control function and called it altering distance function.

Definition 8. *A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*

(i) $\psi(0) = 0$,
(ii) *ψ is continuous and monotonically non-decreasing.*

In 1984, Khan et al. [13] proved the following fixed point theorem using altering distance function as follows:

Theorem 1. *Let (X,d) be a complete metric space. Let ψ be an altering distance function and $f : X \to X$ be a self-mapping which satisfies the following inequality:*

$$\psi(d(fx,fy)) \leq c\psi(d(x,y)) \tag{2.1}$$

for all $x,y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point.

Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [3, 16, 20, 21].

In 2000 and 2005, Chaudhary et al. ([6] and [7]) extend the notion of altering distance to two variables and three variables.

An interesting generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [2] in complete metric spaces as follows:

Definition 9. *A mapping $T : X \to X$, where (X,d) is a metric space, is said to be weakly contractive if*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \tag{2.2}$$

where $x,y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

If one takes $\varphi(t) = kt$ where $0 < k < 1$, then (2.2) reduces to (1.1).

Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [3, 5, 19, 24].

In 2001, Rhoades [18] proved the following theorem:

Theorem 2. *Let $T : X \to X$ be a weakly contractive mapping on a complete metric space (X,d), then T has a unique fixed point.*

In fact, Alber and Guerre-Delabriere [2] assumed an additional condition on φ which is $\lim_{t\to\infty} \varphi(t) = \infty$. But Rhoades [18] obtained the result noted in Theorem 2 without using this particular assumption.

It may be observed that though the function φ has been defined in the same way as the altering distance function, the way it has been used in Theorem 2 is completely different from the use of altering distance function.

In 2008, Dutta et al. [8] proved the following theorem:

Theorem 3. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y))\varphi(d(x,y)), \tag{2.3}$$

where $\psi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.

Then T has a unique fixed point.

In 2006, Beg et al. [4] generalized Theorem 3 in the following form:

Theorem 4. *Let (X,d) be a metric space and let f be a weakly contractive mapping with respect to g, that is,*

$$\psi(d(fx,fy)) \le \psi(d(gx,gy))\varphi(d(gx,gy)), \tag{2.4}$$

for all $x,y \in X$, where $\varphi,\psi : [0,\infty) \to [0,\infty)$ are two mappings with $\varphi(0) = \psi(0) = 0$, ψ is continuous nondecreasing and φ is lower semi-continuous.

If $fX \subset gX$ and gX is a complete subspace of X, then f and g have coincidence point in X.

In 2012, Moradi et al. [15] proved the following theorem:

Theorem 5. *Let T be self mapping on a complete metric space (X,d) satisfying the following:*

$$\psi(d(Tx,Ty)) \le \psi(d(x,y))\varphi(d(x,y)),$$

for all $x,y \in X$ (known as $(\psi-\varphi)$ weakly contractive), where $\varphi,\psi : [0,\infty) \to [0,\infty)$ be two mappings with $\varphi(0) = \psi(0) = 0$, $\varphi(t) > 0$ and $\psi(t) > 0$ for all $t > 0$.

Also suppose that either

(i) *ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} \varphi(t_(n)) = 0$, or*

(ii) *ψ is monotone non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_(n)) = 0$.*

Then T has a unique fixed point.

Now, we prove our results relaxing the condition of completeness on metric space for a pair of weakly compatible mappings.

Theorem 6. *Let f and g be self mappings on a metric space (X,d) satisfying the followings:*

$$gX \subset fX, \tag{2.5}$$

$$gX \text{ or } fX \text{ is complete}, \tag{2.6}$$

$$\psi(d(gx,gy)) \le \psi(d(fx,fy))\varphi(d(fx,fy)), \tag{2.7}$$

for all $x,y \in X$ ($(\psi-\varphi)$ weakly contractive), where $\varphi,\psi : [0,\infty) \to [0,\infty)$ are two mappings with $\varphi(0) = \psi(0) = 0$, $\varphi(t) > 0$ and $\psi(t) > 0$ for all $t > 0$.

Suppose also that either

(i) *ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} \varphi(t_(n)) = 0$, or*

(ii) *ψ is monotone non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_(n)) = 0$.*

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (2.5), one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_{n+1} = gx_n$, $n = 0,1,2,\ldots$.

Moreover, we assume that if $y_n = y_{n+1}$ for some $n \in N$, then there is nothing to prove. Now, we assume that $y_n \ne y_{n+1}$ for all $n \in \mathbb{N}$.

From (2.7), we have

$$\begin{aligned} \psi(d(y_{n+1},y_n)) &= \psi(d(gx_{n+1},gx_n)) \\ &\le \psi(d(fx_{n+1},fx_n))\varphi(d(fx_{n+1},fx_n)) \\ &= \psi(d(y_n,y_{n-1}))\varphi(d(y_n,y_{n-1})), \end{aligned} \tag{2.8}$$

for all $n \in \mathbb{N}$ and hence the sequence $\{\psi(d(y_{n+1},y_n))\}$ is monotone decreasing and bounded below. Thus, there exists $r \ge 0$ such that $\lim_{n\to\infty} \psi(d(y_{n+1},y_n)) = r$.

From (2.8), we deduce that

$$0 \le \varphi(d(y_n,y_{n-1})) \le \psi(d(y_n,y_{n-1})) - \psi(d(y_{n+1},y_n)). \tag{2.9}$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} \varphi((d(y_n,y_{n-1})) = 0$.

If (a) holds, then by hypothesis $\lim_{n\to\infty} d(y_n,y_{n-1}) = 0$.

If (b) holds, then from (2.9), we have

$$d(y_{n+1},y_n) < d(y_n,y_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Hence $\{d(y_{n+1},y_n)\}$ is monotonically decreasing and bounded below.

By hypothesis, $\lim_{n\to\infty} d(y_n,y_{n-1}) = 0$.

Therefore, in every case, we conclude that

$$\lim_{n\to\infty} d(y_n,y_{n-1}) = 0. \tag{2.10}$$

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exists $\varepsilon > 0$ and the subsequences $\{y_{m(k)}\}$ and $\{y_n(k)\}$ of $\{y_n\}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(y_{m(k)},y_{n(k)}) \le \varepsilon$ and by using the triangular inequality, we obtain

$$\begin{aligned} \varepsilon &< d(y_{m(k)},y_{n(k)}) \\ &\le d(y_{m(k)},y_{m(k)-1}) + d(y_{m(k)-1},y_{n(k)-1}) \\ &\quad + d(y_{n(k)-1},y_{n(k)}) \\ &\le d(y_{m(k)},y_{m(k)-1}) + d(y_{m(k)-1},y_{m(k)}) \\ &\quad + d(y_{m(k)},y_{n(k)-1}) + d(y_{n(k)-1},y_{n(k)}) \\ &\le 2d(y_{m(k)},y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1},y_{n(k)}). \end{aligned} \tag{2.11}$$

Letting $k \to \infty$ in the above inequality and using (2.10), we get

$$\lim_{k\to\infty} (d(y_{m(k)},y_{n(k)}) = \lim_{k\to\infty} (d(y_{m(k)-1},y_{n(k)-1}) = \varepsilon. \tag{2.12}$$

For all $k \in \mathbb{N}$, from (2.7), we have

$$\begin{aligned} &\psi(d(y_{m(k)},y_{n(k)})) \\ &\quad \le \psi(d(y_{m(k)-1},y_{n(k)-1}))\varphi(d(y_{m(k)-1},y_{n(k)-1})). \end{aligned} \tag{2.13}$$

If (a) holds, then

$$\lim_{k\to\infty}\psi(d(y_{m(k)-1},y_{n(k)-1})) \\ =\lim_{k\to\infty}\psi(d(y_{m(k)},y_{n(k)}))=\psi(\varepsilon), \qquad (2.14)$$

Now, from , we conclude that

$$\lim_{k\to\infty}\varphi(d(y_{m(k)-1},y_{n(k)-1}))=0.$$

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1},y_{n(k)-1})=0$, a contradiction.

If (b) holds, then from (2), we have

$$\varepsilon<d(y_{m(k)},y_{n(k)})<d(y_{m(k)-1},y_{n(k)-1}),$$

and so

$$d(y_{m(k)},y_{n(k)})\to\varepsilon^+$$

and

$$d(y_{m(k)-1},y_{n(k)-1})\to\varepsilon+ \text{ as } k\to\infty.$$

Hence

$$\lim_{k\to\infty}\psi(d(y_{m(k)-1},y_{n(k)-1}))=\lim_{k\to\infty}\psi(d(y_{m(k)},y_{n(k)})) \\ =\psi(\varepsilon+),$$

where $\psi(\varepsilon^+)$ is the right limit of ψ at ε.

Therefore, from (2), we get

$$\lim_{k\to\infty}\varphi(d(y_{m(k)-1},y_{n(k)-1}))=0.$$

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1},y_{n(k)-1})=0$, a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence.

Since fX is complete, so there exists a point $z\in fX$ such that $\lim_{n\to\infty} y_n=\lim_{n\to\infty} fx_{n+1}=z$.

Now, we show that z is the common fixed point of f and g. Since $z\in fX$, so there exists a point $p\in X$ such that $fp=z$.

If (a) holds, then from (2.7), for all $n\in\mathbb{N}$, we have

$$\psi(d(fp,gp))=\lim_{n\to\infty}\psi(d(gp,gx_n)) \\ \leq\lim_{n\to\infty}\psi(d(fp,fx_n))\lim_{n\to\infty}\varphi(d(fp,fx_n)) \\ \leq\lim_{n\to\infty}\psi(d(fp,fx_n)). \qquad (2.15)$$

Using condition (a) and $\lim_{n\to\infty} y_n=z$, we get

$$\psi(d(fp,gp))\leq\psi(d(z,z))=\psi(0)=0$$

and so $d(gp,fp)=0$ (note that φ and ψ are non-negative with $\varphi(0)=\psi(0)=0$), which implies that $gp=fp=z$.

If (b) holds, then from (2.7), we have

$$\psi(d(fp,gp))=\lim_{n\to\infty}\psi(d(gp,gx_n)) \\ \leq\lim_{n\to\infty}\psi(d(fp,fx_n))\lim_{n\to\infty}\varphi(d(fp,fx_n)) \\ =\lim_{n\to\infty}\psi(d(fp,fx_n)). \qquad (2.16)$$

Using condition (b) and $\lim_{n\to\infty} y_n=z$, we get

$d(fp,gp)\leq d(z,z)=0$, which implies that $fp=gp=z$.

Now, we show that $z=fp=gp$ is a common fixed point of f and g.

Since $fp=gp$ and f, g are weakly compatible maps, we have $fz=fgp=gfp=gz$.

We claim that $fz=gz=z$.

Let, if possible, $gz\neq z$.

If (a) holds, then from (2.7), we have

$$\psi(d(gz,z))=\psi(d(gz,gp))\leq\psi(d(fz,fp))\varphi(d(fz,fp)) \\ =\psi(d(gz,z))\varphi(d(gz,z)) \\ <\psi(d(gz,z)), \text{ a contradiction.}$$

If (b) holds, then we have

$$d(gz,z)<d(gz,z), \text{ a contradiction.}$$

Hence $gz=z=fz$, so z is the common fixed point of f and g.

For the uniqueness, let u be another common fixed point of f and g, so that $fu=gu=u$.

We claim that $z=u$.

Let, if possible, $z\neq u$.

If (a) holds, then from (2.7), we have

$$\psi(d(z,u))=\psi(d(gz,gu))\leq\psi(d(fz,fu))\varphi(d(fz,fu)) \\ =\psi(d(z,u))\varphi(d(z,u)) \\ <\psi(d(z,u)), \text{ a contradiction.}$$

If (b) holds, then we have

$$d(z,u)<d(z,u), \text{ a contradiction.}$$

Thus, we get $z=u$. Hence z is the unique common fixed point of f and g.

*Example 8.*Let $X=[0,1]$ be endowed with the Euclidean metric $d(x,y)=|x-y|$ for all x,y in X and let $gx=\frac{1}{5}x$ and $fx=\frac{3}{5}x$ for each $x\in X$. Then

$$d(gx,gy)=\frac{1}{5}|x-y| \quad\text{and}\quad d(fx,fy)=\frac{3}{5}|x-y|.$$

Let $\psi(t)=5t$ and $\varphi(t)=t$. Then

$$\psi(d(gx,gy))=\psi\left(\frac{1}{5}|x-y|\right)=5\frac{1}{5}|x-y|=|x-y|.$$

$$\psi(d(fx,fy))=\psi\left(\frac{3}{5}|x-y|\right)=5\frac{3}{5}|x-y|=3|x-y|.$$

$$\varphi(d(fx,fy))=\varphi\left(\frac{3}{5}|x-y|\right)=\frac{3}{5}|x-y|.$$

Now

$$\psi(d(fx,fy))-\varphi(d(fx,fy))=\left(3-\frac{3}{5}\right)|x-y| \\ =\frac{12}{5}|x-y|.$$

So

$$\psi(d(gx,gy)) < \psi(d(fx,fy)) - \varphi(d(fx,fy)).$$

From here, we conclude that f, g satisfy the relation (2.7).

Also $gX = [0,\frac{1}{5}] \subseteq [0,\frac{3}{5}] = fX$, gX is complete and f, g are weakly compatible. Hence all the conditions of Theorem 6 are satisfied. Here 0 is the unique common fixed point of f and g.

3 E.A. and (CLR$_\mathrm{f}$) properties

Theorem 7. *Let (X,d) be a metric space and let f and g be weakly compatible self-maps of X satisfying (2.7), (a), (b) and the followings:*

$$f \text{ and } g \text{ satisfy the E.A. property,} \tag{3.1}$$

$$fX \text{ is closed subset of } X. \tag{3.2}$$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x_0$ for some $x_0 \in X$. Now, fX is closed subset of X, therefore, for $z \in X$, we have $\lim_{n\to\infty} fx_n = fz$.

We claim that $fz = gz$.

From (2.7), we have

$$\psi(d(gx_n,gz)) \le \psi(d(fx_n,fz))\varphi(d(fx_n,fz)).$$

Letting $n \to \infty$, we have

$$\begin{aligned}\psi(d(fz,gz)) &\le \lim_{n\to\infty} \psi(d(fx_n,fz)) \lim_{n\to\infty} \varphi(d(fx_n,fz)) \\ &= \psi(d(fz,fz))\varphi(d(fz,fz)) \\ &= \psi(0)\varphi(0).\end{aligned}$$

If (a) holds, then

$$\psi(d(fz,gz)) \le 0, \quad \text{implies that } d(fz,gz) = 0,$$

that is,

$$fz = gz.$$

If (b) holds, then

$$d(fz,gz) \le 0, \quad \text{implies that} \quad fz = gz.$$

Therefore, $fz = gz$.

Now, we show that gz is the common fixed point of f and g.

Suppose that $gz \ne ggz$. Since f and g are weakly compatible $gfz = fgz$ and therefore $ffz = ggz$.

From (2.7), we have

$$\begin{aligned}\psi(d(gz,ggz)) &\le \psi(d(fz,fgz))\varphi(d(fz,fgz)) \\ &= \psi(d(gz,gfz))\varphi(d(gz,gfz)) \\ &= \psi(d(gz,ggz))\varphi(d(gz,ggz)).\end{aligned}$$

If (a) holds, then

$$\psi(d(gz,ggz)) < \psi(d(gz,ggz)), \text{ a contradiction.}$$

If (b) holds, then

$$d(gz,ggz) < d(gz,ggz), \quad \text{a contradiction.}$$

Hence $ggz = gz$. Hence gz is the common fixed point of f and g.

Finally, we show that the fixed point is unique.

Let u and v be two common fixed points of f and g such that $u \ne v$.

From (2.7), we have

$$\begin{aligned}\psi(d(u,v)) &= \psi(d(gu,gv)) \\ &\le \psi(d(fu,fv))\varphi(d(fu,fv)) \\ &= \psi(d(u,v))\varphi(d(u,v)).\end{aligned}$$

If (a) holds, then we have

$$\psi(d(u,v)) < \psi(d(u,v)), \quad \text{a contradiction.}$$

If (b) holds, then we have

$$d(u,v) < d(u,v), \quad \text{a contradiction.}$$

Therefore, $u = v$, which proves the uniqueness.

Theorem 8. *Let (X,d) be a metric space and let f and g be weakly compatible self-mappings of X satisfying (2.7), (a), (b) and the following:*

$$f \text{ and } g \text{ satisfy (CLR}_f\text{) property.} \tag{3.3}$$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLR$_\mathrm{f}$) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx \quad \text{for some } x \in X.$$

Now, we claim that $fx = gx$.

From (2.7), we have

$$\psi(d(gx_n,gx)) \le \psi(d(fx_n,fx))\varphi(d(fx_n,fx)) \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we have

$$\begin{aligned}\psi(d(fx,gx)) &\le \lim_{n\to\infty} \psi(d(fx_n,fx)) \lim_{n\to\infty} \varphi(d(fx_n,fx)) \\ &= \psi(d(fx,fx)) - \varphi(d(fx,fx) \\ &= \psi(0) - \varphi(0).\end{aligned}$$

If (a) holds, then we have

$$\psi(d(fx,gx)) \le 0, \quad \text{implies that } d(fx,gx) = 0,$$

that is,

$$gx = fx.$$

If (b) holds, then we have

$$d(fx,gx) \le 0, \text{ that is, } gx = fx.$$

Let $w = fx = gx$.

Since f and g are weakly compatible $gfx = fgx$, implies that, $fw = fgx = gfx = gw$.

Now, we claim that $gw = w$. Let, if possible, $gw \ne w$.

If (a) holds, then from (2.7), we have

$$\begin{aligned}\psi(d(gw,w)) &= \psi(d(gw,gx))\\ &\le \psi(d(fw,fx))\varphi(d(fw,fx))\\ &< \psi(d(fw,fx))\\ &= \psi(d(gw,w)), \text{ a contradiction.}\end{aligned}$$

If (b) holds, then we have

$$d(gw,w) < d(gw,w), \text{ a contradiction.}$$

Thus, we get $gw = w = fw$.

Hence w is the common fixed point of f and g.

For the uniqueness, let u be another common fixed point of f and g such that $fu = u = gu$.

Now, we claim that $w = u$.

Let, if possible, $w \ne u$.

If (a) holds, then from (2.7), we have

$$\begin{aligned}\psi(d(w,u)) &= \psi(d(gw,gu))\\ &\le \psi(d(fw,fu))\varphi(d(fw,fu))\\ &= \psi(d(w,u))\varphi(d(w,u))\\ &< \psi(d(w,u)), \quad \text{a contradiction.}\end{aligned}$$

If (b) holds, then we have

$$d(w,u) < d(w,u), \quad \text{a contradiction.}$$

Thus, we get, $w = u$. Hence w is the unique common fixed point of f and g.

*Example 9.*Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ and let $gx = \frac{1}{5}x$ and $fx = \frac{3}{5}x$ for each $x \in X$. Then $d(gx,gy) = \frac{1}{5}|x-y|$ and $d(fx,fy) = \frac{3}{5}|x-y|$.

Let $\psi(t) = 5t$ and $\varphi(t) = t$. Then

$$\psi(d(gx,gy)) = \psi\left(\frac{1}{5}|x-y|\right) = 5\frac{1}{5}|x-y| = |x-y|.$$

$$\psi(d(fx,fy)) = \psi\left(\frac{3}{5}|x-y|\right) = 5\frac{3}{5}|x-y| = 3|x-y|.$$

$$\varphi(d(fx,fy)) = \varphi\left(\frac{3}{5}|x-y|\right) = \frac{3}{5}|x-y|.$$

Now

$$\psi(d(fx,fy)) - \varphi(d(fx,fy)) = \left(3 - \frac{3}{5}\right)|x-y|$$

$$= \frac{12}{5}|x-y|.$$

So

$$\psi(d(gx,gy)) < \psi(d(fx,fy)) - \varphi(d(fx,fy)).$$

Now, we conclude that f, g satisfy (2.7).

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = f(0)$, hence the pair (f,g) satisfy the (CLR_f) property. Also f and g are weakly compatible. From here, we also deduce that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$, where $0 \in X$, implies that f and g satisfy E.A. property. Hence all the conditions of Theorem 7 and 8 are satisfied. Here 0 is the unique common fixed point of f and g.

Theorem 9.*The Theorems 6, 7 and 8 remains true if a weakly compatible property is replaced by any one (retaining the rest of hypothesis) of the following:*

(i) *R-weakly commuting property,*
(ii) *R-weakly commuting property of type (A_f),*
(iii) *R-weakly commuting property of type (A_g),*
(iv) *R-weakly commuting property of type (P),*
(v) *weakly commuting property.*

*Proof.*Since all the conditions of all above theorem are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pairs (f,g), then using R-weak commutativity one gets

$$d(fgx,gfx) \le Rd(fx,gx) = 0,$$

which amounts to say that $fgx = gfx$. Thus the pair (f,g) is weakly compatible. Now applying above theorems one concludes that f and g have a unique common fixed point.

In case (f,g) is an R-weakly commuting pair of type (A_f), then

$$d(fgx,g^2x) \le d(fx,gx) = 0,$$

which amounts to say that $fgx = g^2x$.

Now $d(fgx,gfx) \le d(fgx,g^2x) + d(g^2x,gfx) \le 0 + 0 = 0$, yielding thereby $fgx = gfx$.

In case (f,g) is an R-weakly commuting pair of type (A_f), then

$$d(fgx,f^2x) \le d(fx,gx) = 0,$$

which amounts to say that $gfx = f^2x$.

Now $d(fgx,gfx) \le d(fgx,f^2x) + d(f^2x,gfx) \le 0 + 0 = 0$, yielding thereby $fgx = gfx$.

Similarly, if pair is R-weakly commuting mapping of type (P) or weakly commuting, then (f,g) also commutes at their points of coincidence. Now in view of above theorems, in all four cases f and g have a unique common fixed point.

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