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Multiple Criteria Secretary Problem: A New Approach

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Abstract: In multiple criteria secretary problem selection of a unit is based on two independent characteristics. The units, appeared before an observer are known (say N). N is considered as best rank of a unit. A unit is selected, if it is better with respect to either first or second or both the characteristics. In this paper, joint probability distributions along with marginal distributions (of real ranks of both the characteristics and the position at which the selection is made) are derived systematically using simple and explicable method. Further these marginal distributions are used to derive expected cost of inspection and expected value of real rank of selected unit. A new criterion for selecting the best unit based on expected real rank is developed.

Keywords: Secretary Problem, joint distribution, marginal distribution, real ranks, selection criterion

1 Introduction

The ‘Secretary Problem’ deals with the sequential decision procedure. According to the classical secretary problem, N randomly arranged units are to be observed one after another with the aim of stopping at a ‘suitable’ position and selecting a unit appearing at that position such that the probability of selecting the best unit is maximum. This is done with the condition that the units once observed and rejected are not allowed to be called back by the observer at any time in future, and more over if the observer reaches the last, the N^{th} one, then it must be accepted. Many solutions and versions of the problem are available in literature.

Multiple criteria optimal selection problems were introduced in more general form, with observations in a partially ordered set and with an arbitrary payoff utility by Berezovskii, Geninson and Rubchinskii (1980) and Stadje (1980). The multiple criteria problem of optimum stopping of the selection process was solved by Gnedin (1982). Such a problem may be considered as a generalization of the classical secretary problem (one criterion best choice problem) as found in Gilbert and Mosteller (1966). Problems in which the unit selected is said to be the best if it is optimal with respect to a social choice function, for example Pareto optimal, were treated by Berezovskii and Gnedin (1981), Gnedin (1983), Baryshnikov, Berezovskii and Gnedin (1984).

Baryshnikov and Gnedin (1986), Samuels and chotlos (1987) discussed the problem where the goal of the observer is to minimize the expectations of the sum of the ranks of the unit selected, rank one being the best. Ferguson (1992) generalized the problem presented by Gnedin by allowing dependencies between the attributes, and showed that the optimal policy has the same threshold form as the standard single attribute Classical Secretary Problem (CSP). Samuels and Chotlos (1987) extended the rank minimization problem of Chow et al. (1964). They sought an optimal policy for minimizing the sum of two ranks for independent attributes. Bearden et al. (2004) proposed a multi-attribute (or multi-dimensional) generalization of generalized secretary problem, presenting a method for computing its optimal policies, and testing it in two experiments with incentive-compatible payoffs.

In the present study of multiple criteria secretary problem random variables are real ranks of both the characteristics and the position at which the selection is made. The joint distribution and marginal distributions of these random variables are derived using simple algebra given in section 2. In section 3, expected values of these random variables are found to obtain expected real rank of the selected unit and cost incurred in the selection process. In section 4, optimality criterion based on probabilistic approach is discussed. A new optimality criterion based on expected real rank is developed and its usefulness over probabilistic approach is revealed in the last section.

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2 Multiple Criteria Secretary Problem

There are N known units. Assume that each unit can be ranked with respect to two observable characteristics which are common for all units. The best unit is given rank N, the second best rank (N-1), etc. and the worst rank 1. It is assumed that these two characteristics are independent.

The selection procedure is as follows:

1. Observe the first r units without selecting any.
2. Select i thunit if it is better than the best of the first r units with respect to the first or the second or both the characteristics then stop (r+1 $\leq$ i $\leq$ N-1).
3. If none of the (N-1) units is selected, then Nthunit is to be selected.

Let R_k(i) (k = 1,2 ; i= 1,2,. . . ,N) be the real rank of the ith unit with respect to the kthcharacteristic.

Let X_1 and X_2be the real ranks of Ythunit with respect to the first and second characteristics respectively.

Note that X_1, X_2 and Y are discrete random variables having the ranges X_1 = 1,2,. . . ,N ; X_2 = 1,2,. . . ,N and Y = r+1. . . N. The total number of permutations of [R_1(1), R_1(2),. . . , R_1(N)] and [R_2(1), R_2 (2),. . . , R_2(N)] are $(N!)^2$and each pair is assumed to be equally likely.

Notations:

1. α (x, y) = $\frac{\binom{x-1}{y-1}}{\binom{N-2}{y-2}}$
2. P(x_1, x_2, y| r, N) = P(X_1=x_1, X_2=x_2, Y= y | r, N) is the probability that the observer stops after examining y units and the selected unit has real ranks x_1,x_2 with respect to characteristic 1 and characteristic 2 respectively for fixed values of r and N.
3. P(x_1,x_2, |r, N) is the joint probability distribution of X_1 and X_2.
4. P_{X1}(x | r, N) is the marginal probability distribution of X_1.
5. Px_2 (x | r, N) is the marginal probability distribution of X_2.
6. P_y(y | r, N) is the marginal probability distribution of Y.

To analyze this problem we need a result derived by Kane S. P. (1988) for one characteristic, is as given below:

Joint distribution of X and Y:

Result: The probability of (X = x, Y = y) is given by

$$P(x,\ y|\ r,\ N) = \begin{cases} \frac{r}{N\ (N-1)}\alpha(x,y) & r+1\ \leq y\ \leq N-1;\quad y\leq x\ \leq N \\ \frac{r}{N\ (N-1)} & y\ =\ N;\quad 1\ \leq\ x\ \leq\ N \\ 0, & otherwise. \end{cases}$$

Proof : Proof is divided into two parts.

1) When Y $\leq$ N-1

Let j be the maximum real rank of the first r units. The real ranks of the units in y-1 positions must be less than j. The unit with real rank j can be at any of the r positions. Further, j can be at most y-1and it cannot exceed x-1. Therefore, the number of permutations qualifying such condition is equal to

r $[(j-1)^{(y-2)}]$ (N-y) !

Therefore,

$$P(x,y\ |r,N) = \frac{r\ (N-y)!}{N!}\sum_{j=y-1}^{x-1}(j-1)^{(y-2)}$$

Using the result from $A_{1.2}$ from Appendix and simplifying, we get

P (x, y| r, N) = $\frac{r\ (N-y)!}{(y-1)N!}\left[(x-1)^{(y-1)}-(y-2)^{(y-1)}\right]$

P (x, y| r, N) = $\frac{r\ (N-y)!}{(y-1)N!}(x-1)^{(y-1)}$

P (x, y| r, N) = $\frac{r}{N(N-1)}$ $\alpha(x,y)$, y $\leq$ x $\leq$ N, r+1 $\leq$ y $\leq$ N-1

2) When Y = N

i] If Y = N and X = 1, 2, . . . , N-1, then it is obvious that the best unit has already been appeared in the first r units. Hence excluding the best unit which is in first r units and the last inspected unit, the remaining (N-2) units can appear in (N-2) ways. The best unit can be at any one of the first r positions therefore the number of permutations qualifying this condition is r (N-2) .

Hence the probability of this event is

P (x, y| r, N) = $\frac{r\ (N-2)!}{N!}$

Thus,

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P (x, y| r, N) = $\frac{r}{N(N-1)}$, $1 \le x \le N-1$

ii] If Y = N and X = N, then it is obvious that the second best unit has already been appeared in the first r units. Hence, the probability of this event is $\frac{r\ (N-2)!}{N!}$.

P (N, N| r, N) = $\frac{r}{N(N-1)}$

For other pairs of X and Y, P (x, y| r, N) s are zero.

Thus we have,

$$P(x,\ y\,|r,\ N)\ = \begin{cases} \frac{r}{N\ (N-1)}\alpha\,(x,y)\,, & r+1\ \le y\ \le N-1; y\le x\ \le N \\ \frac{r}{N\ (N-1)}, & y\ =\ N\ ;1\ \le\ x\ \le\ N \\ 0, & Otherwise. \end{cases}$$

Lemma: The probability that the units in the positions from (r + 1) through y are not relatively better than the best of the first r units (x is real rank of the unit at the yth position), denoted by P (x, y) is given by

P (x, y) = $\frac{r}{N\ (N-1)}$ $[\alpha(N,y) -\ \alpha\ (x,\ y)]$, if $r+1 \le\ y \le\ N, 1 \le\ x \le\ N-1$,

Proof: Let j be the real rank of the first 'r' units. j must be at least x + 1and at the most N. Since j can be in any one of the r positions, the number of permutations are:

r $[(j-2)^{(y-2)}]$ (N-y) !

$$\therefore P(x,y) = \frac{r\ (N-y)!}{N!}\sum_{j=x+1}^{N}(j-2)^{(y-2)}$$

P (x, y) = $\frac{r\ (N-y)!}{N!}\left[\frac{(N-1)!}{(N-y)!} - \frac{(x-1)!}{(x-y)!}\right]$

= $\frac{r}{N\ (N-\ 1)}$ $[\alpha(N,y) -\ \alpha\ (x,\ y)]$, if $r+1 \le\ y \le\ N, 1 \le\ x \le\ N-1$,

Remark: Using the above lemma and the result by Kane S. P. for one characteristic, joint probability distribution in multiple criteria secretary problem is derived.

2.1 Joint Probability Distribution of $\boldsymbol{X_1}$*,* $\boldsymbol{X_2}$ ***and Y***

Theorem 1: The joint probability distribution of (X_1, X_2, Y) is given by,

$$P\,(x_1,x_2,y|\ r,N) = \begin{cases} \frac{r^2}{(N-1)^2N^2}\left[\alpha\ (x_1,\ y)\alpha\,(N,\ y) + \alpha\ (x_2,\ y)\alpha\,(N,\ y) - \alpha\ (x_1,\ y)\alpha\ (x_2,\ y)\right]\ , \\ r+1\ \le\ y\ \le\ N-1;\ \ y\ \le x_1,\ x_2 \le\ N \\ \frac{r^2}{(N-1)^2N^2}, & y = \ N;\ \ 1 \le x_1,\ x_2\ \le N \\ 0, & otherwise \end{cases} \tag{1}$$

Proof: Following are the four mutually exclusive and exhaustive events, defined by, A_1, A_2, A_3, A_4 associated with the Ythunit :

A_1: Ythunit is relatively better with respect to X_1 but it is not better with respect to X_2.
A_2: Ythunit is relatively better with respect to X_2 but it is not better with respect to X_1.
A_3: Yth unit is relatively better with respect to both characteristics.
A_4: Yth unit is not relatively better with respect to both characteristics.

The joint probabilities of (X_1, X_2, Y) with respect to above four events are as follows:

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$$\frac{r^2}{(N-1)^2N^2}\alpha(x_1, y)\left[\alpha(N, y)-\alpha(x_2, y)\right], \qquad if\ y \le x_1 \le N;\ 1 \le x_2 \le N-1;$$
$$r+1 \le y \le N-1, \quad under\ A_1$$

$$\frac{r^2}{(N-1)^2N^2}\alpha(x_2, y)\left[\alpha(N, y)-\alpha(x_1, y)\right], \qquad if\ y \le x_2 \le N;\ 1 \le x_1 \le N-1;$$
$$r+1 \le y \le N-1, \quad under\ A_2$$

$$\frac{r^2}{(N-1)^2N^2}\alpha(x_1, y)\ \alpha(x_2, y), \qquad if\ y \le x_1 \le N;\ y \le x_2 \le N;$$
$$r+1 \le y \le N-1, \quad under\ A_3$$

$$\frac{r^2}{(N-1)^2N^2}, \qquad if\ y = N;\quad 1 \le x_1, x_2 \le N-1, \quad under\ A_4$$

Combining the above four probabilities, joint probability distribution of X_1, X_2 and Y as given in (1) is obtained.

2.2 Joint Distribution of X_1 and X_2

Corollary 1: The joint distribution of (X_1, X_2) is

$$P(x_1,x_2,|r,N) = \begin{cases} \frac{r^2}{(N-1)^2N^2} + \frac{r^2}{(N-1)^2N^2}\sum_{y=r+1}^{N-1}\left[\alpha(x_1, y)\alpha(N, y)+\alpha(x_2, y)\alpha(N, y)-\alpha(x_1, y)\alpha(x_2, y)\right], & 1 \le x_1, x_2 \le N \\ 0, & otherwise \end{cases} \tag{2}$$

Proof:

$$P(x_1,x_2,|r,N) = \sum_{y=r+1}^{N-1} P(x_1,x_2,y|r,N)$$
$$= \sum_{y=r+1}^{N-1} P(x_1,x_2,y|r,N) + P(x_1,x_2,N|r,N)$$
$$= \frac{r^2}{(N-1)^2N^2} + \frac{r^2}{(N-1)^2N^2}\sum_{y=r+1}^{N-1}\left[\alpha(x_1, y)\alpha(N, y)+\alpha(x_2, y)\alpha(N, y)-\alpha(x_1, y)\alpha(x_2, y)\right]$$

2.3 Marginal Distributions of X_1, X_2 and Y

Corollary2: The marginal probability distributions of X_1 and $X_2$are respectively given by,

$$P_{x_1}(x|r,N) = \begin{cases} \frac{r^2}{N(N-1)^2} + \frac{r^2}{N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2} + \frac{r^2}{N(N-1)^2}\sum_{y=r+1}^{N-1}\frac{1}{y}\ \alpha(x_1, y) & if\ 1 \le x_1 \le N \\ 0, & otherwise \end{cases} \tag{3}$$

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$$P_{x_2}(x|r,N) = \begin{cases} \frac{r^2}{N(N-1)^2} + \frac{r^2}{N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2} + \frac{r^2}{N(N-1)^2}\sum_{y=r+1}^{N-1}\frac{1}{y}\,\alpha(x_2,\,y) \\ \qquad\qquad if\ \ 1 \le x_2 \le N \\ 0, \qquad otherwise \end{cases} \tag{4}$$

Proof:

$$P_{x_1}(x|r,N) = \sum_{x_2=1}^{N} P(x_1,x_2,|r,N)$$

$$= \frac{r^2}{N(N-1)^2} + \frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1} N\alpha(x_1,\,y)\,\alpha(N,\,y) +$$

$$\frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1}\alpha(N,\,y)\sum_{x_2=y}^{N}\alpha(x_2,\,y) - \frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1}\alpha(x_1,\,y)\sum_{x_2=y}^{N}\alpha(x_2,\,y)$$

$$= \frac{r^2}{N(N-1)^2} + \frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1} N\alpha(x_1,\,y)\,\alpha(N,\,y) + \frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1}\frac{\binom{N-1}{y-1}}{\binom{N-2}{y-2}}\frac{\binom{N}{y}}{\binom{N-2}{y-2}} -$$

$$\frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1}\alpha(x_1,\,y)\frac{\binom{N}{y}}{\binom{N-2}{y-2}}$$

(by using $A_{1.3}$ from Appendix)

$$P_{x_1}(x|r,N) = \frac{r^2}{N(N-1)^2} + \frac{r^2}{N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2} + \frac{r^2}{N\,(N-1)}\sum_{y=r+1}^{N-1}\frac{1}{y}\alpha(x_1,\,y),$$

$$if\ \ 1 \le x_1 \le N$$

Since $P(x_1,x_2,\ r,N)$ is symmetric in $X_1$and X_2, the marginal probability distribution of X_2 is given by (4).
Corollary 3: Marginal probability distribution of Y is,

$$P_y(y\mid r,\,N) = \begin{cases} \frac{r^2(2y-1)}{y^2(y-1)^2}, & if\quad r+1 \le y \le N-1 \\ \frac{r^2}{(N-1)^2}, & if\quad y = N. \\ 0, & otherwise \end{cases} \tag{5}$$

Proof: The proof is given in two parts:
Case i) r + 1 ≤ y ≤ N -1

$$P_y(y\ |r,\,N) = \sum_{x_1=1}^{N}\sum_{x_2=1}^{N} P(x_1,x_2,y|r,N)$$

$$= \frac{r^2}{N^2(N-1)^2}\sum_{x_1=1}^{N}\sum_{x_2=1}^{N}\left[\alpha(x_1,\, y)\,\alpha(N,\, y)+\alpha(x_2,\, y)\,\alpha(N,\, y)-\alpha(x_1,\, y)\alpha(x_2,\, y)\right] \quad \text{from (1)}$$

$$= \frac{r^2}{N^2(N-1)^2}\left[N\frac{(N-1)}{(y-1)}\frac{\binom{N}{y}}{\binom{N-2}{y-2}}+N\frac{(N-1)}{(y-1)}\frac{\binom{N}{y}}{\binom{N-2}{y-2}}-\frac{\binom{N}{y}}{\binom{N-2}{y-2}}\frac{\binom{N}{y}}{\binom{N-2}{y-2}}\right]$$

$$= \text{r}\ \frac{r^2(2y-1)}{y^2(y-1)^2}$$

Case ii) y = N

$$P_y\ (y\mid r,\ N)\ =\sum_{x_1=1}^{N}\sum_{x_2=1}^{N}P(x_1,x_2,N|r,N)$$

$$= \frac{r^2}{(N-1)^2} \quad \text{from (1)}$$

3 Expected Values of X_1, X_2 and Y

Corollary 4: The mathematical expectation of X_1 is given by

$$E\ \ (X_1|\ r,\ N)=\frac{r^2\,(N+1)}{2}\left\{\frac{1}{(N-1)^2}+\sum_{y=r+1}^{N-1}\left[\frac{1}{y(y-1)^2}\ +\ \frac{2}{(y+1)y(y-1)}\right]\right\} \tag{6}$$

Proof:

$$E\,(X_1|\ r,\ N)=\sum_{x_1=1}^{N}x_1P_{x_1}\,(x_1|\ r,N)$$

$$=\sum_{x_1=1}^{N}x_1\left[\frac{r^2}{N(N-1)^2}+\frac{r^2}{N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2}+\frac{r^2}{N(N-1)}\sum_{y=r+1}^{N-1}\frac{1}{y}\alpha(x_1,\, y)\right]\quad from(2)$$

$$=\frac{N(N+1)}{2}\frac{r^2}{N(N-1)^2}+\frac{r^2N(N+1)}{2N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2}+\frac{r^2}{N(N-1)}\sum_{x_1=1}^{N}x_1\sum_{y=r+1}^{N-1}\frac{1}{y}\frac{\binom{x_1-1}{y-1}}{\binom{N-2}{y-2}}$$

$$=\frac{r^2}{(N-1)^2}\frac{(N+1)}{2}+\frac{r^2\,(N+1)}{2}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2}+\frac{r^2}{N(N-1)}\sum_{y=r+1}^{N-1}\frac{1}{\binom{N-2}{y-2}}\sum_{x_1=1}^{N}\binom{x_1}{y}$$

$$=\frac{r^2}{(N-1)^2}\frac{(N+1)}{2}+\frac{r^2\,(N+1)}{2}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2}+\frac{r^2}{N(N-1)}\sum_{y=r+1}^{N-1}\frac{\binom{N+1}{y+1}}{\binom{N-2}{y-2}}$$

$$\therefore E\ \ (X_1|\ r,\ N)=\frac{r^2\,(N+1)}{2}\left\{\frac{1}{(N-1)^2}+\sum_{y=r+1}^{N-1}\left[\frac{1}{y(y-1)^2}\ +\ \frac{2}{(y+1)y(y-1)}\right]\right\}$$

Remark: From (3) and (4), it may be seen that, E (X_2| r, N) = E (X_1| r, N).
E(X | r, N) is computed for various combinations of r (from 1 to N-1) and N (=10, 15, 20) as listed in Table 1.

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TABLE1: Computation of E (X| r, N) showing the values of E(X| r, N) for some N

r	E(X | r,10)	E(X| r, 15)	E(X | r, 20)	E(X| r, 25)	E(X | r, 30)
1	6.268723	9.141301	12.008465	14.873605	17.737772
2	6.741557	9.898534	13.033855	16.161083	19.284418
3	6.918503	10.271701	13.576178	16.862438	20.139942
4	**6.921783**	10.438579	13.868758	17.266558	20.648785
5	6.804869	**10.476946**	14.013685	17.499830	20.961643
6	6.593297	10.423944	**14.059707**	17.622610	21.150482
7	6.300598	10.299257	14.032656	**17.666885**	21.253431
8	5.934567	10.114223	13.947415	17.651081	**21.292463**
9	5.5	9.875814	13.813134	17.586523	21.281086
10		9.588547	13.635742	17.480555	21.228048
11		9.255475	13.419249	17.338139	21.139227
12		8.878735	13.166455	17.1627	21.018898
13		8.459864	12.879381	16.956873	20.869898
14		8.00	12.559519	16.722368	20.694477
15			12.207994	16.460627	20.494297
16			11.825671	16.172718	20.270636
17			11.413224	15.859480	20.024490
18			10.971190	15.521572	19.756647
19			10.50	15.159530	19.467747
20				14.773783	19.158300
21				14.364686	18.828730
22				13.932533	18.479387
23				13.477569	18.110561
24				13.00	17.722502
25					17.315416
26					16.889481
27					16.444847
28					15.981648
29					15.50

From the above table, it is observed that E(X | r, N) attains maximum at some r for given N.

Expected value of Y:

Corollary 5: Expectation of Y is given by

E(Y | r, N) = r + r^2 $\left(\frac{1}{r^2}+\frac{1}{(r+1)^2}++\frac{1}{(N-1)^2}\right)\ldots$ (7)

Proof:

$$E\left(Y \mid r,\ N\right)=\sum_{y=r+1}^{N} yP_y\left(y\mid r,\ N\right)$$

$$=\sum_{y=r+1}^{N-1} y\,P_y\left(y\mid r,\ N\right)+NP_y\left(y\mid r,\ N\right)$$

$$=r^2\sum_{y=r+1}^{N-1}\frac{(2y-1)}{y(y-1)^2}+\frac{Nr^2}{(N-1)^2}$$

$$=r+r^2\left(\frac{1}{r^2}+\frac{1}{(r+1)^2}++\frac{1}{(N-1)^2}\right)$$

4 Optimality Criterion for Selection 'r'

It may be recalled that r denotes the number of units that are passed without selection. In original Secretary Problem, the usual criterion for the choice of optimum r is to maximize the probability that the best unit is selected. On the same lines, we suggest the following optimality criterion for selection of r.

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Select r such that P[X_1 = N or X_2 = N | r, N] is maximum.
We know that, P_{x_1}[N | r, N] = P_{x_2}[N | r, N].

$$= \frac{r^2}{N(N-1)^2} + \frac{r^2}{N}\sum_{y=r+1}^{N-1}\frac{1}{y(y-1)^2} + \frac{r^2}{N(N-1)}\sum_{y=r+1}^{N-1}\frac{(N-1)}{y(y-1)} \qquad from(3)$$

$$= \frac{r^2}{N}\left[\frac{1}{r^2} + \frac{1}{(r+1)^2} + + \frac{1}{(N-1)^2}\right]$$

$$P(N,N \mid r,\ N) = \frac{r^2}{N^2(N-1)^2} + \frac{r^2}{N^2(N-1)^2}\sum_{y=r+1}^{N-1}\frac{(N-1)^2}{(y-1)^2} \qquad from\ (2)$$

$$= \frac{r^2}{N^2}\left[\frac{1}{r^2} + \frac{1}{(r+1)^2} + + \frac{1}{(N-1)^2}\right]$$

Therefore, P[X_1 = N or X_2 = N | r, N] = P_{x_1}[N | r, N] + P_{x_2} [N | r, N] –P [N, N | r, N]

$$= \frac{r^2(2N-1)}{N^2}\left[\frac{1}{r^2} + \frac{1}{(r+1)^2} + + \frac{1}{(N-1)^2}\right]$$

P[X_1 = N or X_2 = N | r, N] is computed for various combinations of r (from 1 to N-1) and N (=10, 15, 20) as listed in Table 2.

TABLE 2: Computation of P[X_1 = N or X_2 = N | r, N]

R	P(X_1 = N or X_2 = N | r,10)	P(X_1 = N or X_2 =N | r,15)	P(X_1 = N or X_2=N | r,20)
1	0.292556	0.203128	0.155382
2	0.410223	0.296958	0.231529
3	0.495503	0.378155	0.301564
4	0.543116	0.443140	0.362781
5	**0.551744**	0.491017	0.414502
6	0.520911	0.521465	0.456483
7	0.450407	**0.534340**	0.488616
8	0.340123	0.529569	0.510845
9	0.190000	0.507110	0.523140
10		0.466940	**0.525481**
11		0.409042	0.517857
12		0.333405	0.500260
13		0.240023	0.472684
14		0.128889	0.435124
15			0.387579
16			0.330045
17			0.262522
18			0.185007
19			0.097500

From the above table it may be noted that P[X_1 = N or X_2 = N | r, N] attains maximum at a value of r (say r_0) for given N. This r_0 is the optimum value of r under said optimality criterion.
In this connection we have the following corollary.
Corollary 6: For given N, r_0, satisfies the following inequalities:

$$(2r_0+1)\sum_{k=r_0+1}^{N-1}\frac{1}{k^2} \leq 1 \leq (2r_0-1)\sum_{k=r_0}^{N-1}\frac{1}{k^2} \qquad (8)$$

Proof: We have noticed earlier that P[X_1 = N or X_2 = N | r, N] is maximum at r = r_0 .
So for this r_0, we must have
P[X_1 = N or X_2 = N | r_0-1, N] ≤ P[X_1 = N or X_2 = N | r_0, N] ≥ P[X_1 = N or X_2 = N | r_0+1, N]
The first half of the above inequality leads to

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$$\frac{(r_0-1)^2(2N-1)}{N^2}\left[\frac{1}{(r_0-1)^2}+\frac{1}{r_0{}^2}+ + \frac{1}{(N-1)^2}\right]$$

$\leq \frac{r_0{}^2(2N-1)}{N^2}\left[\frac{1}{r_0{}^2}+\frac{1}{(r_0+1)^2}+ + \frac{1}{(N-1)^2}\right]$

On simplifying, we get

$$1 \leq (2r_0-1)\sum_{k=r_0}^{N-1}\frac{1}{k^2} \quad ...(*)$$

$P[X_1 = N \text{ or } X_2 = N \mid r_0, N] \geq P[X_1 = N \text{ or } X_2 = N \mid r_0+1, N]$ leads to

$$\frac{r_0{}^2\,(2N-1)}{N^2}\left[\frac{1}{r_0{}^2}+\frac{1}{(r_0+1)^2}+ + \frac{1}{(N-1)^2}\right]$$

$\geq \frac{(r_0+1)^2(2N-1)}{N^2}\left[\frac{1}{(r_0+1)^2}+\frac{1}{(r_0+2)^2}+ + \frac{1}{(N-1)^2}\right]$

On simplifying, we get

$$(2r_0+1)\sum_{k=r_0+1}^{N-1}\frac{1}{k^2} \leq 1 \qquad ...(**)$$

From (*) and (**) we get,

$$(2r_0+1)\sum_{k=r_0+1}^{N-1}\frac{1}{k^2} \leq 1 \leq (2r_0-1)\sum_{k=r_0}^{N-1}\frac{1}{k^2}$$

These inequalities are too complex to give r_0 explicitly. However for large N, using Euler's summation formula in (8), we get a good approximate solution for r_0 which is approximately equal to the integral part of N/2. And the maximum value of $P[X_1 = N \text{ or } X_2 = N \mid r, N] = P[X_1 = N \text{ or } X_2 = N \mid r_0, N]$ is approximately equal to 1/2.

This fact was also noticed by Gnedin (1982) using different approach. If r_0 is chosen to be integral part of N/2, then $E(Y/r_0, N)$ is approximately equal to 3N/4.That is 75% of total number units are expected to be observed which is quite large.

5 New Approach for Optimum 'r'

The aim is to select a unit such that it should be sufficiently good as per either criterion. This means that select a unit such that $E(X_1 \mid r, N)$ or $E(X_2 \mid r, N)$ is maximum. Therefore, we propose the following criterion.
Choose r such that $E(X_1 \mid r, N)$ is maximum.
Since $E(X_1\mid r, N) = E(X_2\mid r, N)$, the optimum r, denoted by r^* should satisfy
$E(X_1\mid r^*, N) \geq E(X_1 \mid r, N)$, for all r
Such r* exists in view of the discussion in section 3.
Therefore, for optimum r*, we have the following inequalities:
$E(X_1\mid r^*-1, N) \leq E(X_1\mid r^*, N) \geq E(X_1\mid r^*+1, N)$ … (9)
Putting the expressions in (9) and simplifying, we get

$$\frac{(r^*-1)(3r^*-1)}{(r^*+1)r^{*2}(2r^*-1)} - B \leq A \leq \frac{(r^*+1)(3r^*+2)}{(r^*+2)r^{*2}(2r^*+1)} - B \qquad (10)$$

Where $A = \frac{1}{(N-1)^2}$ and $B = \sum_{y=r+1}^{N-1}\left[\frac{1}{y(y-1)^2}+\frac{2}{(y+1)y(y-1)}\right]$

These inequalities are also too complex to provide explicit expression for r^*.
In the following table, Expected values of X and Expected values of Y are given for r_0 and r^*, for some selected values of N.

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TABLE 3. VALUES OF E (Y | r^*, N), E(Y | r_0, N), E (X | r^*, N) and E(X | r_0, N) FOR SOME SET OF (r^*, r_0, N)

N	r_0	r^*	E(Y | r_0, N)	E(Y | r^*, N)	E(X | r_0, N)	E(X | r^*, N)
10	5	4	7.903916	6.858506	6.804869	6.921783
15	7	5	11.14574	8.809618	10.299257	10.476946
20	10	6	15.38955	10.68188	13.635742	14.059707

6 Conclusion

From the above table, it may be noted that, in the criterion, which considers the maximization of probability of selecting the best unit, attention is not paid to the real ranks of the selected units if the procedure fails to select the best unit. However, if we select the new approach of optimality criterion based on ranks, we can see that E(X | r^*, N) of the selected unit is large. Further it can be observed that $r^*$is always less than r_0.
Hence, the expected cost of inspection corresponding to the scheme which allows to inspect first r^*units only, without selecting any from them is less than the expected cost of inspection corresponding to the scheme which allows to inspect r_0 units without selecting any from them.
This, therefore, suggests that it is more appropriate to choose optimum value of
r = r^* as it is going to reduce the observation cost, and at the same time rank of the selected unit is approximately 0.7N.

APPENDIX

$A_{1.1}: n^{(r)} = n(n-1)(n-2)\ldots(n-r+1)$. $A_{1.2}: \sum_{n=a}^{b} n^{(x)} = \frac{1}{x+1}\left[(b+1)^{(x+1)} - a^{(x+1)}\right]$ $A_{1.3}: \sum_{n=a}^{b} \binom{n}{x} = \binom{b+1}{x+1} - \binom{a}{x+1}$ $A_{1.4}: \binom{x-1}{y-1} = \frac{x-y+1}{y-1}\binom{x-1}{y-2}$

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