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On Paranorm I-Convergent Sequence Spaces of Interval Numbers Defined by Modulus Function

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Abstract: In this article we introduce and study the I-convergent sequence spaces $\mathscr{C}^I(\bar{\mathscr{A}}, f, p)$, $\mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$ and $\ell^I_\infty(\bar{\mathscr{A}}, f, p)$ on interval numbers with the help of a modulus function f and a bounded sequence $p = (p_k)$ of strictly positive real numbers. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Keywords: Interval numbers, Ideal, filter, I-convergent sequence, solid and monotone space, modulus function.

1 Introduction and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.

Let ℓ_∞, c and c_0 be denote the Banach spaces of bounded, convergent and null sequences respectively with norm

$$\|x\| = \sup_k | x_k |$$

We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Any subspace λ of the linear space ω of sequences is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous, for all $i \in \mathbb{N}$. A space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space.

It is an admitted fact that the real and complex numbers are playing a vital role in the world of mathematics. Many mathematical structures have been constructed with the help of these numbers. In recent years, since 1965 fuzzy numbers and interval numbers also managed their place in the world of mathematics and credited into account some alike structures . Interval arithmetic was first suggested by P.S.Dwyer[13] in 1951. Further development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R.E.Moore[24] in 1959 and Moore and Yang[25] and others and have developed applications to differential equations.

Recently, Chiao [12] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryılmaz[34] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

Hereafter, we give the definitions that will be used in the paper.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $I\mathbb{R}$. Any element of $I\mathbb{R}$ is called an interval number and it is denoted by $\bar{A} = [x_l, x_r]$ where x_l and x_r are the smallest and the greatest point of an interval number $\bar{A}$. An interval number is closed subset of real numbers [12]. The algebraic operations for interval numbers can be found in

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[34] and it is a quasilinear space(see[37]).

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{A}_1,\bar{A}_2) = \max\{| x_{1_l} - x_{2_l} |, | x_{1_r} - x_{2_r} |\}, (\text{see}[25,34]). \quad (1)$$

where x_l and x_r be first and last points of $\bar{A}$, respectively.

In a special case, $\bar{A}_1 = [a,a]$, $\bar{A}_2 = [b,b]$, we obtain the usual metric of $\mathbb{R}$ with

$$d(\bar{A}_1,\bar{A}_2) = | a-b | .$$

Let us define transformation f from $\mathbb{N}$ to $I\mathbb{R}$ by $k \to f(k) = \bar{\mathscr{A}}$, $\bar{\mathscr{A}} = (\bar{A}_k)$. Then, the sequence $(\bar{A}_k)$ is called sequence of interval numbers, where $\bar{A}_k$ is the k^{th} term of the sequence $(\bar{A}_k)$.

Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{\mathscr{A}}) = \{\bar{\mathscr{A}} = (\bar{A}_k) : \bar{A}_k \in I\mathbb{R}\}. \quad (2)$$

The algebraic properties of $\omega(\bar{\mathscr{A}})$ can be found in [12,34].

The following definitions were given by Şengönül and Eryılmaz in [34].
A sequence $\bar{\mathscr{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ of interval numbers is said to be convergent to an interval number $\bar{A}_0 = [x_{0_l}, x_{0_r}]$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that $d(\bar{A}_k, \bar{A}_0) < \varepsilon$, for all $k \geq n_0$ and we denote it as $\lim_k \bar{A}_k = \bar{A}_0$.

Thus, $\lim_k \bar{A}_k = \bar{A}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$ and it is said to be Cauchy sequence of interval numbers if for each $\varepsilon > 0$, there exists a positive integer k_0 such that $d(\bar{A}_k, \bar{A}_m) < \varepsilon$, whenever $k,\ m \geq k_0$.

Let us denote the space of all convergent, null and bounded sequences of interval numbers by $\mathscr{C}(\bar{\mathscr{A}})$, $\mathscr{C}_\circ(\bar{\mathscr{A}})$ and $\ell_\infty(\bar{\mathscr{A}})$, respectively. The sets $\mathscr{C}(\bar{\mathscr{A}})$, $\mathscr{C}_\circ(\bar{\mathscr{A}})$ and $\ell_\infty(\bar{\mathscr{A}})$ are complete metric spaces with the metric

$$\hat{d}(\bar{A}_k, \bar{B}_k) = \sup_k \max\{|x_{k_l} - y_{k_l}|, |x_{k_r} - y_{k_r}|\} (see[34]). \quad (3)$$

If we take $\bar{B}_k = \bar{O}$ in (3) then the metric $\hat{d}$ reduces to

$$\hat{d}(\bar{A}_k, \bar{O}) = \sup_k \max\{|x_{k_l}|, |x_{k_r}|\}. \quad (4)$$

In this paper, we assume that a norm $||\bar{A}_k||$ of the sequence of interval numbers $(\bar{A}_k)$ is the distance from $(\bar{A}_k)$ to $\bar{O}$ and satisfies the following properties:
$\forall \bar{A}_k, \bar{B}_k \in \lambda(\bar{\mathscr{A}})$ and $\forall \alpha \in \mathbb{R}$
(N_1) $\forall \bar{A}_k \in \lambda(\bar{\mathscr{A}}) - \{\bar{O}\}, ||\bar{A}_k||_{\lambda(\bar{\mathscr{A}})} > 0;$
(N_2) $||\bar{A}_k||_{\lambda(\bar{\mathscr{A}})} = 0 \Leftrightarrow \bar{A}_k = \bar{O};$
(N_3) $||\bar{A}_k + \bar{B}_k||_{\lambda(\bar{\mathscr{A}})} \leq ||\bar{A}_k||_{\lambda(\bar{\mathscr{A}})} + ||\bar{B}_k||_{\lambda(\bar{\mathscr{A}})}$
(N_4) $||\alpha \bar{A}_k||_{\lambda(\bar{\mathscr{A}})} = |\alpha|||\bar{A}_k||_{\lambda(\bar{\mathscr{A}})}$, where $\lambda(\bar{\mathscr{A}})$ is a subset of $\omega(\bar{\mathscr{A}})$.

Let $\bar{\mathscr{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ be the element of $\mathscr{C}(\bar{\mathscr{A}})$, $\mathscr{C}_\circ(\bar{\mathscr{A}})$ or $\ell_\infty(\bar{\mathscr{A}})$. Then, in the light of above discussion, the classes of sequences $\mathscr{C}(\bar{\mathscr{A}})$, $\mathscr{C}_\circ(\bar{\mathscr{A}})$ and $\ell_\infty(\bar{\mathscr{A}})$ are normed interval spaces normed by

$$\| \bar{\mathscr{A}} \| = \sup_k \max\{| x_{k_l} |, | x_{k_r} |\}\ (\text{see}[34]). \quad (5).$$

Throughout, $\bar{O} = [0,0]$ and $\bar{I} = [1,1]$ represent zero and identity interval numbers according to addition and multiplication, respectively.

As a generalisation of usual convergence for the sequences of real or complex numbers, the concept of statistical convergence was first introduced by Fast [14] and also independently by Buck [11] and Schoenberg [33]. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy[15], Šalát [30], Tripathy [35] and many others. The notion of statistical convergence has been extended to interval numbers by Esi as follows in [1],[2],[3],[4],[5],[6],[7],[8],[9],[10].

Let us suppose that $\bar{\mathscr{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathscr{A}})$. If, for every $\varepsilon > 0$,

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \varepsilon,\ n \leq k\}| = 0. \quad (6)$$

then the sequence $\bar{\mathscr{A}} = (\bar{A}_k)$ is said to be statistically convergent to an interval number $\bar{A}_0$, where vertical lines denote the cardinality of the enclosed set. That is, if $\delta(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{k \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \varepsilon\}$.

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salǎt and Wilczyński [21,22]. Later on, it was studied by Šalát, Tripathy and Ziman [31,32], Esi and Hazarika[1], Tripathy and Hazarika [36], Khan *et al* [18,19,20] and many others.

Let us denote the classes of I-convergent, I-null, bounded I-convergent and bounded I-null sequences of interval numbers with $\mathscr{C}^I(\bar{\mathscr{A}})$, $\mathscr{C}^I_\circ(\bar{\mathscr{A}})$, $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}})$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}})$, respectively.

Definition 1.1. *Let $\mathbb{N}$ be a non empty set. Then, a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if*
(i) I is additive i.e $\forall A,B \in I \Rightarrow A \cup B \in I$
(ii) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on N if and only if

(i) $\Phi \notin \pounds(I)$,
(ii) $\forall A,\ B \in \pounds(I)$ *we have* $A \cap B \in \pounds(I)$,
(iii) $\forall A \in \pounds(I)$ *and* $A \subseteq B \Rightarrow B \in \pounds(I)$.
An Ideal $I \subseteq 2^N$ *is called non-trivial if* $I \neq 2^{\mathbb{N}}$.
A non-trivial ideal $I \subseteq 2^N$ *is called admissible if*

$$\{\{x\} : x \in N\} \subseteq I.$$

Let us suppose that I *be an ideal. Then a sequence* $\bar{\mathscr{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathscr{A}}) \subset \omega(\bar{\mathscr{A}})$

(i) is said to be I-convergent to an interval number $\bar{A}_0$ *if for every* $\varepsilon > 0$, *the set*

$$\{k \in N : \| \bar{A}_k - \bar{A}_0 \| \geq \varepsilon\} \in I.$$

In this case, we write $I - \lim \bar{A}_k = \bar{A}_0$. *If* $\bar{A}_0 = \bar{O}$ *then the sequence* $\bar{\mathscr{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathscr{A}})$ *is said to be I-null. In this case, we write* $I - \lim \bar{A}_k = \bar{O}$.

(ii) is said to be I-cauchy if for every $\varepsilon > 0$ *there exists a number* $m = m(\varepsilon)$ *such that*

$$\{k \in N : \| \bar{A}_k - \bar{A}_m \| \geq \varepsilon\} \in I.$$

(iii) is said to be I-bounded if there exists some $M > 0$ *such that*

$$\{k \in N : \| \bar{A}_k \| \geq M\} \in I.$$

We know that for each ideal I, there is a filter $\pounds(I)$ corresponding to I,
i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.2. *A sequence space* $\lambda(\bar{\mathscr{A}})$ *of interval numbers is*

(iv) said to be solid(normal) if $(\alpha_k \bar{A}_k) \in \lambda(\bar{\mathscr{A}})$ *whenever* $(\bar{A}_k) \in \lambda(\bar{\mathscr{A}})$ *and for any sequence* (α_k) *of scalars with* $|\alpha_k| \leq 1$, *for all* $k \in \mathbb{N}$,

(v) said to be symmetric if $(\bar{A}_{\pi(k)}) \in \lambda(\bar{\mathscr{A}})$ *whenever*

$$\bar{A}_k \in \lambda(\bar{\mathscr{A}}),$$

where π *is a permutation on* $\mathbb{N}$,

(vi) said to be sequence algebra if $(\bar{A}_k) * (\bar{B}_k) = (\bar{A}_k.\bar{B}_k) \in \lambda(\bar{\mathscr{A}})$ *whenever* $(\bar{A}_k),\ (\bar{B}_k) \in \lambda(\bar{\mathscr{A}})$,

(vii) said to be convergence free if $(\bar{B}_k) \in \lambda(\bar{\mathscr{A}})$ *whenever* $(\bar{A}_k) \in \lambda(\bar{\mathscr{A}})$ *and* $\bar{A}_k = \bar{O}$ *implies* $\bar{B}_k = \bar{O}$, *for all k.*

Definition 1.3. *Let* $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$. *The K-step space of the* $\lambda(\bar{\mathscr{A}})$ *is a sequence space* $\mu_K^{\lambda(\bar{\mathscr{A}})} = \{(\bar{A}_{k_n}) \in \omega(\bar{\mathscr{A}}) : (\bar{A}_k) \in \lambda(\bar{\mathscr{A}})\}$.

Definition 1.4. *A canonical pre-image of a sequence* $(\bar{A}_{k_n}) \in \mu_K^{\lambda(\bar{\mathscr{A}})}$ *is a sequence*
$(\bar{B}_k) \in \omega(\bar{\mathscr{A}})$ *defined by*

$$\bar{B}_k = \begin{cases} \bar{A}_k, & \text{if } k \in K, \\ \bar{O}, & \text{otherwise}. \end{cases}$$

A canonical preimage of a step space $\mu_K^{\lambda(\bar{\mathscr{A}})}$ *is a set of canonical preimages of all elements in* $\mu_K^{\lambda(\bar{\mathscr{A}})}$, *i.e.* $\bar{\mathscr{B}}$ *is in the canonical preimage of* $\mu_K^{\lambda(\bar{\mathscr{A}})}$ *iff* $\bar{\mathscr{B}}$ *is the canonical preimage of some* $\bar{\mathscr{A}} \in \mu_K^{\lambda(\bar{\mathscr{A}})}$.

Definition 1.5. *A sequence space* $\lambda(\bar{\mathscr{A}})$ *is said to be monotone if it contains the canonical preimages of its step space.*

Definition 1.6. *A function* $f : [0,\infty) \longrightarrow [0,\infty)$ *is called a modulus function if*
(1) $f(t) = 0$ *if and only if* $t = 0$,
(2) $f(t+u) \leq f(t) + f(u)$ *for all* $t,\ u \geq 0$,
(3) f *is increasing, and*
(4) f *is continuous from the right at zero.*

For any modulus function f, *we have the inequalities*

$$|f(x) - f(y)| \leq f(x-y)$$

and

$$f(nx) \leq nf(x), \quad \text{for all } x,\ y \in [0,\infty]. \tag{7}$$

A modulus function f *is said to satisfy* $\Delta_2 - Condition$ *for all values of* u *if there exists a constant* $K > 0$ *such that* $f(Lu) \leq KLf(u)$ *for all values of* $L > 1$.

The idea of modulus was introduced by Nakano in $1953.(See[26],\ Nakano, 1953)$.

Ruckle [27,28,29] used the idea of a modulus function f *to construct the sequence space*

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}. \tag{8}$$

This space is an FK-space and Ruckle[27,28,29] proved that the intersection of all such $X(f)$ *spaces is* ϕ, *the space of all finite sequences. The space* $X(f)$ *is closely related to the space* ℓ_1 *which is an* $X(f)$ *space with* $f(x) = x$ *for all real* $x \geq 0$. *Thus Ruckle[27,28,29] proved that, for any modulus* f.

$$X(f) \subset \ell_1 \text{ and } X(f)^\alpha = \ell_\infty$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}. \tag{9}$$

Spaces of the type $X(f)$ *are a special case of the spaces structured by B.Gramsch [17]. From the point of view of local convexity, spaces of the type* $X(f)$ *are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[16], G.Köthe[23] and W.H.Ruckle[27,28,29].*

We need the following popular inequalities throughout the paper.

Let $p=(p_k)$ be the bounded sequence of positive reals numbers. For any complex λ, whenever $H=\sup_k(p_k)<\infty$, we have

$$|\lambda|^{p_k}\le \max(1,|\lambda|^H)$$

Also, whenever $H=\sup_k(p_k)$, we have

$$|a_k+b_k|^{p_k}\le C(|a_k|^{p_k}+|b_k|^{p_k}),$$

where $C=\max(1,2^{H-1})$.

Now, we give some important Lemmas.

Lemma.1.7. Every solid space is monotone.

Lemma.1.8. Let $K\in \pounds(I)$ and $M\subseteq N$. If $M\notin I$, then $M\cap K\notin I$, where $\pounds(I)\subseteq 2^N$ filter on N.

Lemma.1.9. If $I\subseteq 2^N$ and $M\subseteq N$. If $M\notin I$, then $M\cap N\notin I$.

2 Main Results

Let us give a most important definition for this paper:

Definition 2.1(see [37]). Let $\bar{X}$ be a space of interval numbers. A function $g:\bar{X}\longrightarrow \mathbb{R}$ is called paranorm on $\bar{X}$, if for all $A,\ B\in\bar{X}$,
(P_1) $g(A)=0$ if $A=\bar{0}$,
(P_2) $g(A)\ge 0$
(P_3) $g(-A)=g(A)$,
(P_4) $g(A+B)\le g(A)+g(B)$,
(P_5) If (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ $(n\to\infty)$ and $(A_n),A_0\in\bar{X}$ with $g(A_n)\to g(A_0)$ $(n\to\infty)$, then $g(\lambda_nA_n)\to g(\lambda A_0)\to 0$ $(n\to\infty)$.
(P_6) If $A\le B$, then $g(A)\le g(B)$.

In this article, we introduce and study the following classes of sequences;

$$\mathscr{C}^I(\bar{\mathscr{A}},f,p)=\left\{\bar{\mathscr{A}}=(\bar{A}_k)\in\ell_\infty(\bar{\mathscr{A}}):\{k\in\mathbb{N}:f(\|\bar{A}_k-\bar{A}\|)^{p_k}\ge\varepsilon\}\in I,\right\};\quad(10)$$

$$\mathscr{C}_0^I(\bar{\mathscr{A}},f,p)=\left\{\bar{\mathscr{A}}=(\bar{A}_k)\in\ell_\infty(\bar{\mathscr{A}}):\{k\in\mathbb{N}:f(\|\bar{A}_k\|)^{p_k}\ge\varepsilon\}\in I\right\};\quad(11)$$

$$\ell_\infty(\bar{\mathscr{A}},f,p)=\left\{\bar{\mathscr{A}}=(\bar{A}_k)\in\ell_\infty(\bar{\mathscr{A}}):\sup_k f(\|\bar{A}_k\|)^{p_k}<\infty\right\}.\quad(12)$$

We also denote $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)=\ell_\infty(\bar{\mathscr{A}},f,p)\cap\mathscr{C}^I(\bar{\mathscr{A}},f,p)$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}},f,p)=\ell_\infty(\bar{\mathscr{A}},f,p)\cap\mathscr{C}^I_0(\bar{\mathscr{A}},f,p)$, where $p=(p_k)$ is a bounded sequence of positive real numbers and f is a modulus function.

Theorem 2.2. The classes of sequences $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}},f,p)$ are paranormed spaces, paranormed by

$g(\bar{\mathscr{A}})=g((\bar{A}_k))=\sup_k f(\|\bar{A}_k\|)^{\frac{p_k}{M}}$, where $M=\max\{1,\sup_k p_k\}$.

Proof. Let $\bar{\mathscr{A}}=(\bar{A}_k),\ \bar{\mathscr{B}}=(\bar{B}_k)\in\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$.
(P_1). It is Clear that $g(\bar{\mathscr{A}})=0$ if $\bar{\mathscr{A}}=\bar{\theta}$.
(P_2). It is obvious that $g(\bar{\mathscr{A}})\ge 0$.
(P_3). $g(\bar{\mathscr{A}})=g(-\bar{\mathscr{A}})$ is obvious.
(P_4). Since $\frac{p_k}{M}\le 1$ and $M>1$, using Minkowski's inequality, we have]

$$g(\bar{\mathscr{A}}+\bar{\mathscr{B}})=g(\bar{A}_k+\bar{B}_k)=\sup_k f(\|\bar{A}_k+\bar{B}_k)\|)^{\frac{p_k}{M}}\le\sup_k f(\|\bar{A}_k\|)^{\frac{p_k}{M}}$$

$$+\sup_k f(\|\bar{B}_k)\|^{\frac{p_k}{M}}=g(\bar{A}_k)+g(\bar{B}_k)=g(\bar{\mathscr{A}})+g(\bar{\mathscr{B}})$$

Therefore, $g(\bar{\mathscr{A}}+\bar{\mathscr{B}})\le g(\bar{\mathscr{A}})+g(\bar{\mathscr{B}})$.
(P_5). Let (λ_k) be a sequence of scalars with $(\lambda_k)\to\lambda$ $(k\to\infty)$ and
$(\bar{A}_k),\ \bar{A}_0\in\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$ such that

$$g(\bar{A}_k)\to g(\bar{A}_0)\ (k\to\infty),$$

Then, since the inequality

$$g(\bar{A}_k)\le g(\bar{A}_k-\bar{A}_0)+g(\bar{A}_0),\qquad(14)$$

holds by subadditivity of g, the sequence $\{g(\bar{A}_k)\}$ is bounded.
Therefore,

$$|g(\lambda_k\bar{A}_k)-g(\lambda\bar{A}_0)|=|g(\lambda_k\bar{A}_k)-g(\lambda\bar{A}_k)+g(\lambda\bar{A}_k)-g(\lambda\bar{A}_0)|$$

$$\le|\lambda_k-\lambda|^{\frac{p_k}{M}}|g(\lambda_k\bar{A}_k)|+|\lambda|^{\frac{p_k}{M}}|g(\bar{A}_k)-g(\bar{A}_0)|\to 0$$

as $(k\to\infty)$. That is to say that scalar multiplication is continuous.
(P_6). Since f is increasing, it is clear that $g(\bar{\mathscr{A}})\le g(\bar{\mathscr{B}}$, if $\mathscr{A}\le\mathscr{B}$.
Hence $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$ is a paranormed space.
For $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}},f,p)$, the result is similar.

Theorem 2.3. The set $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$ is closed subspace of $\ell_\infty(\bar{\mathscr{A}},f,p)$.

Proof. Let $(\bar{A}^{(n)}_k)$ be a Cauchy sequence in $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$ such that $\bar{A}^{(n)}_k\to\bar{A}$.
We show that $\bar{A}\in\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$
Since $(\bar{A}^{(n)}_k)\in\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)$. Then, there exists $\bar{A}_n$ such that

$$\{k\in\mathbb{N}:f(\|\bar{A}^{(n)}_k-\bar{A}_n\|)^{p_k}\ge\varepsilon\}\in I.$$

We need to show that
(1) $(\bar{A}_n)$ converges to $\bar{A}_0$.
(2) If $U=\{k\in\mathbb{N}:f(\|\bar{A}_k-\bar{A}_0\|)^{p_k}<\varepsilon\}$, then $U^c\in I$.
(1) Since $(\bar{A}^{(n)}_k)$ is Cauchy sequence in $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}},f,p)\Rightarrow$ for a given $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that

$\sup_k f(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \|)^{\frac{p_k}{M}} < \frac{\varepsilon}{3}$, for all $n,\ q \geq k_0$.
For $\varepsilon > 0$, we have

$$B_{nq} = \{k \in \mathbb{N} : f(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \|)^{p_k} < (\frac{\varepsilon}{3})^M\}$$

$$B_q = \{k \in \mathbb{N} : f(\| \bar{A}_k^{(q)} - \bar{A}_q \|)^{p_k} < (\frac{\varepsilon}{3})^M\}$$

$$B_n = \left\{k \in \mathbb{N} : f(\| \bar{A}_k^{(n)} - \bar{A}_n \|)^{p_k} < (\frac{\varepsilon}{3})^M\right\}$$

Then, $B_{nq}^c, B_q^c, B_n^c \in I$
Let $B^c = B_{nq}^c \cup B_q^c \cup B_n^c$, where $B = \{k \in \mathbb{N} : f(\| \bar{A}_q - \bar{A}_n \|)^{p_k} < \varepsilon\}$. Then $B^c \in I$.
We choose $k_0 \in B^c$. Then for each $n,\ q \geq k_0$, we have
$\{k \in \mathbb{N} : f(\| \bar{A}_q - \bar{A}_n \|)^{p_k} < \varepsilon\} \supseteq \Big[\{k \in \mathbb{N} : f(\| \bar{A}_q - \bar{A}_k^{(q)} \|)^{p_k} < (\frac{\varepsilon}{3})^M\}$

$$\cap\{k \in \mathbb{N} : f(\| \bar{A}_k^{(q)} - \bar{A}_k^{(n)} \|)^{p_k} < (\frac{\varepsilon}{3})^M\}$$

$$\cap\{k \in \mathbb{N} : f(\| \bar{A}_k^{(n)} - \bar{A}_n \|)^{p_k} < (\frac{\varepsilon}{3})^M\}\Big]$$

Then $(\bar{A}_n)$ is a Cauchy sequence of interval numbers, so there exists some interval number $\bar{A}_0$ such that $\bar{A}_n \to \bar{A}_0$ as $n \to \infty$.
(2) Let $0 < \delta < 1$ be given. Then, we show that if

$$U = \{k \in \mathbb{N} : f(\| \bar{A}_k - \bar{A}_0 \|)^{p_k} < \delta\},$$

then $U^c \in I$.
Since $(\bar{A}_k^{(n)}) \to \bar{A}$ then, there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : f(\| \bar{A}_k^{(q_0)} - \bar{A}_k \|)^{p_k} < (\frac{\delta}{3D})^M\} \tag{14}$$

implies $P^c \in I$.
where

$$D = \max\{1, 2^{H-1}\},\ \ H = \sup_k p_k \geq 0,$$

The number q_0 can be chosen that together with (14), we have

$$Q = \{k \in \mathbb{N} : f(\| \bar{A}_{q_0} - \bar{A}_0 \|)^{p_k} < (\frac{\delta}{3D})^M\}$$

such that $Q^c \in I$.
Since $\{k \in \mathbb{N} : f(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \|)^{p_k} \geq \delta\} \in I$. Then, we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where $S = \{k \in \mathbb{N} : f(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \|)^{p_k} < (\frac{\delta}{3D})^M\}$.
Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{k \in \mathbb{N} : f(\| \bar{A}_k - \bar{A}_0 \|)^{p_k} < \delta\}$.

Therefore, for each $k \in U^c$, we have
$\{k \in \mathbb{N} : f(\| \bar{A}_k - \bar{A}_0 \|)^{p_k} < \delta\} \supseteq [\{k \in \mathbb{N} : f(\| \bar{A}_k - \bar{A}_k^{(q_0)} \|)^{p_k} < (\frac{\delta}{3})^M\}$

$$\cap\{k \in \mathbb{N} : f(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \|)^{p_k} < (\frac{\delta}{3})^M\}$$

$$\cap\{k \in \mathbb{N} : f(\| \bar{A}_{q_0} - \bar{A}_0 \|)^{p_k} < (\frac{\delta}{3})^M\}] \tag{15}$$

Then, the result follows from (15).
Since the inclusions $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}}, f, p) \subset \ell_\infty(\bar{\mathscr{A}}, f, p)$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}}, f, p) \subset \ell_\infty(\bar{\mathscr{A}}, f, p)$ are strict so in view of Theorem (2.3) we have the following result.

Theorem 2.4. The spaces $\mathscr{M}^I_{\mathscr{C}}(\bar{\mathscr{A}}, f, p)$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}}, f, p)$ are nowhere dense subsets of $\ell_\infty(\bar{\mathscr{A}}, f, p)$.

Theorem 2.5. The spaces $\mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}}, f, p)$ are both solid and monotone.

Proof. We shall prove the result for $\mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$. For $\mathscr{M}^I_{\mathscr{C}_\circ}(\bar{\mathscr{A}}, f, p)$, the result follows similarly.
For, let $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$ and (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.
Since $|\alpha_k|^{p_k} \leq \max\{1, |\alpha_k|^H\} \leq 1$, for all $k \in \mathbb{N}$,
we have

$$f(\| \alpha_k \bar{A}_k \|)^{p_k} \leq f(\| \bar{A}_k \|)^{p_k},\ \text{for all k} \in \mathbb{N}.$$

which further implies that

$$\{k \in \mathbb{N} : f(\| \bar{A}_k \|)^{p_k} \geq \varepsilon\} \supseteq \{k \in \mathbb{N} : f(\| \alpha_k \bar{A}_k \|)^{p_k} \geq \varepsilon\}.$$

Thus, $\alpha_k(\bar{A}_k) \in \mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$.
Therefore, the space $\mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$ is solid and hence by lemma(1.7), it is monotone.

Theorem 2.6. Let $H = \sup_k p_k < \infty$ and I be an admissible ideal. Then, the following are equivalent.
(a) $(\bar{A}_k) \in \mathscr{C}^I(\bar{\mathscr{A}}, f, p)$;
(b) there exists $(\bar{B}_k) \in \mathscr{C}(\bar{\mathscr{A}}, f, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I.
(c) there exists $(\bar{B}_k) \in \mathscr{C}(\bar{\mathscr{A}}, f, p)$ and $\bar{C}_k \in \mathscr{C}^I_0(\bar{\mathscr{A}}, f, p)$ such that $\bar{A}_k = \bar{B}_k + \bar{C}_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f(\| \bar{B}_k - \bar{A} \|)^{p_k} \geq \varepsilon\} \in I$
(d) there exists a subset $K = \{k_1 < k_2 < k_3 < k_4 ...\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} f(\| \bar{A}_{k_n} - \bar{A} \|)^{p_{k_n}} = 0$

Proof. (a) implies (b).
Let $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}^I(\bar{\mathscr{A}}, f, p)$. Then, there exists interval number $\bar{A}$ such that the set

$$\{k \in \mathbb{N} : f(\| \bar{A}_k - \bar{A} \|)^{p_k} \geq \varepsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : f(\| \bar{A}_k - \bar{A} \|)^{p_k} \geq t^{-1}\} \in I$$

Define a sequence $(\bar{B}_k)$ as

$$\bar{B}_k = \bar{A}_k, \quad \text{for all k} \leq m_1.$$

For $m_t < k \leq m_{t+1},\ t \in \mathbb{N}$

$$\bar{B}_k = \begin{cases} \bar{A}_k, \text{ if } f\left(\| \bar{A}_k - \bar{A} \|\right)^{p_k} < t^{-1}, \\ A, otherwise. \end{cases}$$

Then $\bar{B}_k \in \mathscr{C}(\bar{\mathscr{A}}, f, p)$ and from the inclusion

$$\{k \leq m_t : \bar{A}_k \neq \bar{B}_k\} \subseteq \{k \leq m_t : f\left(\| \bar{A}_k - \bar{A} \|\right)^{p_k} \geq \varepsilon\} \in I.$$

we get $\bar{A}_k = \bar{B}_k$ for a.a.k.r.I.
(b) implies (c). For $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}^I(\bar{\mathscr{A}}, f, p)$, then, there exists $(\bar{B}_k) \in \mathscr{C}(\bar{\mathscr{A}}, f, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : \bar{A}_k \neq \bar{B}_k\}$, then $K \in I$.
Define $\bar{C}_k$ as follows.

$$\bar{C}_k = \begin{cases} \bar{A}_k - \bar{B}_k, \text{if } k \in K, \\ O, \ \text{ if } k \notin K. \end{cases}$$

Then $\bar{C}_k \in \mathscr{C}_0^I(\bar{\mathscr{A}}, f, p)$ and $\bar{B}_k \in \mathscr{C}(\bar{\mathscr{A}}, f, p)$.
(c) implies (d). Suppose (c) holds. Let $\varepsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : f\left(\| \bar{C}_k \|\right)^{p_k} \geq \varepsilon\} \in I.$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < k_4 \ldots\} \in \pounds(I).$$

Then we have

$$\lim_{k \to \infty} f\left(\| \bar{A}_{k_n} - \bar{A} \|\right)^{p_{k_n}} = 0.$$

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < k_4 \ldots\} \in \pounds(I)$ and

$$\lim_{k \to \infty} f\left(\| \bar{A}_{k_n} - \bar{A} \|\right)^{p_{k_n}} = 0.$$

Then for any $\varepsilon > 0$, and Lemma (II), we have

$$\{k \in \mathbb{N} : f\left(\| \bar{A}_k - \bar{A} \|\right)^{p_k} \geq \varepsilon\} \subseteq K^c \cup \{k \in K : f\left(\| \bar{A}_k - \bar{A} \|\right)^{p_k} \geq \varepsilon\}.$$

Thus, $(\bar{A}_k) \in \mathscr{C}^I(\bar{\mathscr{A}}, f, p)$

Theorem 2.7. Let f_1 and f_2 be two modulus functions satisfying $\Delta_2 -$ Condition and $p = (p_k) \in \ell_\infty$ be a sequence of positive real numbers, then
(a) $\mathscr{X}(\bar{\mathscr{A}}, f_2, p) \subseteq \mathscr{X}(\bar{\mathscr{A}}, f_1 f_2, p)$
(b) $\mathscr{X}(\bar{\mathscr{A}}, f_1, p) \cap (\bar{\mathscr{A}}, f_2, p) \subseteq \mathscr{X}(\bar{\mathscr{A}}, f_1 + f_2, p)$
for $\mathscr{X} = \mathscr{C}^I$, $\mathscr{C}_\circ^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$

Proof.(a) Let $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_2, p)$ be any arbitrary element.
Then, the set

$$\left\{k \in \mathbb{N} : f_2\left(\| \bar{A}_k \|\right)^{p_k} \geq \varepsilon\right\} \in I \tag{16}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_1(t) < \varepsilon,\ 0 \leq t \leq \delta$.
Let us denote

$$\bar{B}_k = f_2\left(\| \bar{A}_k \|\right) \tag{17}$$

and consider

$$\lim_k f_1\left(\bar{B}_k\right)^{p_k} = \lim_{\bar{B}_k \leq \delta, k \in \mathbb{N}} f_1\left(\bar{B}_k\right)^{p_k} + \lim_{\bar{B}_k > \delta, k \in \mathbb{N}} f_1\left(\bar{B}_k\right)^{p_k}$$

Now, since f_1 is an modulus function , we have

$$\lim_{\bar{B}_k \leq \delta, k \in \mathbb{N}} f_1\left(\bar{B}_k\right)^{p_k} \leq f_1(2)^{p_k} \lim_{\bar{B}_k \leq \delta, k \in \mathbb{N}} (\bar{B}_k)^{p_k} \tag{18}$$

For $\bar{B}_k > \delta$, we have

$$\bar{B}_k < \frac{\bar{B}_k}{\delta} < 1 + \frac{\bar{B}_k}{\delta}$$

Now, since f_1 is non-decreasing and modulus, it follows that

$$f_1(\bar{B}_k) < f_1(1 + \frac{\bar{B}_k}{\delta}) < \frac{1}{2} f_1(2) + \frac{1}{2} f_1(\frac{2\bar{B}_k}{\delta})$$

Again, since f_1 satisfies $\Delta_2 -$ Condition, we have

$$f_1\left(\bar{B}_k\right) < \frac{1}{2} K \frac{(\bar{B}_k)}{\delta} f_1(2) + \frac{1}{2} K \frac{(\bar{B}_k)}{\delta} f_1(2)$$

Thus, $f_1\left(\bar{B}_k\right) < K \frac{(\bar{B}_k)}{\delta} f_1(2)$
Hence,

$$\lim_{\bar{B}_k > \delta, k \in \mathbb{N}} f_1\left(\bar{B}_k\right)^{p_k} \leq \max\{1, (K\delta^{-1} f_1(2))^H\} \lim_{\bar{B}_k > \delta, k \in \mathbb{N}} (\bar{B}_k)^{p_k},\ H = \max\{1, \sup_k p_k\} \tag{19}$$

Therefore, from (17), (18) and (19), we have
$(\bar{A}_k) \in \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1 f_2, p)$
Thus, $\mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_2, p) \subseteq \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1 f_2, p)$. Hence, $\mathscr{X}(\bar{\mathscr{A}}, f_2, p) \subseteq \mathscr{X}(\bar{\mathscr{A}}, f_1 f_2, p)$ for $\mathscr{X} = \mathscr{C}_\circ^I$.
For $\mathscr{X} = \mathscr{C}^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$ the inclusions can be established similarly.
(b). Let $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1, p) \cap \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_2, p)$. Let $\varepsilon > 0$ be given. Then the sets

$$\left\{k \in \mathbb{N} : f_1\left(\| \bar{A}_k \|\right)^{p_k} \geq \varepsilon\right\} \in I; \tag{20}$$

and

$$\left\{k \in \mathbb{N} : f_2\left(\| \bar{A}_k \|\right)^{p_k} \geq \varepsilon\right\} \in I. \tag{21}$$

Therefore, from (20) and (21), we have

$$\left\{k \in \mathbb{N} : (f_1 + f_2)\left(\| \bar{A}_k \|\right)^{p_k} \geq \varepsilon\right\} \in I$$

Thus, $\bar{\mathscr{A}} = (\bar{A}_k) \in \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1 + f_2, p)$
Hence, $\mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1, p) \cap \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_2, p) \subseteq \mathscr{C}_\circ^I(\bar{\mathscr{A}}, f_1 + f_2, p)$
For $\mathscr{X} = \mathscr{C}^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$, the inclusions are similar.
For $f_2(x) = x$ and $f_1(x) = f(x)$, $\forall\, x \in [0, \infty)$, we have the following corollary.

Corollary 2.8.$\mathscr{X}(\bar{\mathscr{A}}, p) \subseteq \mathscr{X}(\bar{\mathscr{A}}, f, p)$ for $\mathscr{X} = \mathscr{C}^I$, $\mathscr{C}_\circ^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$

Theorem 2.9. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, p) \supseteq \overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, q)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.

Proof. Let $\liminf_{k\in K} \frac{p_k}{q_k} > 0$ and $(\overline{A_k}) \in \mathscr{M}^I_{\mathscr{C}_\circ}(\overline{\mathscr{A}}, f, q)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$ for sufficiently large $k \in K$.
Since $(\overline{A_k}) \in \mathscr{M}^I_{\mathscr{C}_\circ}(\overline{\mathscr{A}}, f, q)$. For a given $\varepsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : f\left(\| \overline{A_k} \| \right)^{q_k} \geq \varepsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k\in\mathbb{N} : f\left(\| \overline{A_k} \| \right)^{p_k} \geq \varepsilon\} \subseteq \{k\in\mathbb{N} : f\left(\| \overline{A_k} \| \right)^{\beta q_k} \geq \varepsilon\} \in I.$$

Therefore, $(\overline{A_k}) \in \mathscr{M}^I_{\mathscr{C}_\circ}(\overline{\mathscr{A}}, f, p)$.
The converse part of the result follows obviously.

Theorem 2.10. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, q) \supseteq \overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, p)$ if and only if $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.

Proof. The proof follows similarly as the proof Theorem [2.9].

Theorem 2.11. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $\overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, q) = \overline{\mathscr{M}^I_{\mathscr{C}_\circ}}(\mathscr{A}, f, p)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$ and $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K \in \pounds(I)$.

Proof. On combining Theorem (2.9.) and (2.10.) we get the desired result.

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