Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040203

Infinite Log-Concavity and r-Factor

Zahid Raza and Anjum Ali*

Department of Mathematics, National University of Computer and Emerging Sciences, Lahore Campus, Pakistan.

Received: 6 Jun. 2014, Revised: 20 Nov. 2014, Accepted: 27 Nov. 2014
Published online: 1 May 2015

Abstract: Uminsky and Yeats [D. Uminsky, and K. Yeats, electronic Journal of Combinatorics **14**, 1-13 (2007)], studied the properties of the *log-operator* $\mathscr{L}$ on the subset of the finite symmetric sequences and prove the existence of an infinite region $\mathscr{R}$, bounded by parametrically defined hypersurfaces such that any sequence corresponding a point of $\mathscr{R}$ is *infinitely log-concave*. We study the properties of a new operator $\mathscr{L}_r$ and redefine the hypersurfaces which generalizes the one defined by Uminsky and Yeats. We show that any sequence corresponding a point of the region $\mathscr{R}_r$, bounded by the new generalized parametrically defined r-factor hypersurfaces, is *Generalized r-factor infinitely log concave*. We also give an improved value of $r_\circ$ found by McNamara and Sagan [P. R. W. McNamara and B. E. Sagan, Adv. App. Math., **44**, 1-15 (2010)], as the log-concavity criterion using the new *log-operator*.

Keywords: infinitely log-concave, hypersurfaces, generalized r-factor infinitely log concave, log-concavity criterion

1 Introduction

A sequence $(a_k) = a_0, a_1, a_2, \dots$ of real numbers is said to be *log-concave* or *1-fold log-concave iff* the new sequence (b_k) defined by the $\mathscr{L}$ operator $(b_k) = \mathscr{L}(a_k)$ is non negative for all $k \in \mathbb{N}$, where $b_k = {a_k}^2 - a_{k-1}a_{k+1}$. A sequence (a_k) is said to be *2-fold log-concave iff* $\mathscr{L}^2(a_k) = \mathscr{L}(\mathscr{L}(a_k)) = \mathscr{L}(b_k)$ is non negative for all $k \in \mathbb{N}$, where $\mathscr{L}(b_k) = {b_k}^2 - b_{k-1}b_{k+1}$ and the sequence (a_k) is said to be *i-fold log-concave iff* $\mathscr{L}^i(a_k)$ is non negative for all $k \in \mathbb{N}$, where

$$\mathscr{L}^i(a_k) = [\mathscr{L}^{i-1}(a_k)]^2 - [\mathscr{L}^{i-1}(a_{k-1})]\,[\mathscr{L}^{i-1}(a_{k+1})].$$

(a_k) is said to be *infinitely log-concave iff* $\mathscr{L}^i(a_k)$ is non negative for all $i \geq 1$. Binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots$ along any row of Pascal's triangle are log concave for all $n \geq 0$. Boros and Moll [3] conjectured that binomial coefficients along any row of Pascal's triangle are *infinitely log-concave* for all $n \geq 0$. This was later confirmed by McNamara and Sagan [2] for the n^{th} rows of Pascal's triangle for $n \leq 1450$ and complete proof in [4]. for more details about the log concave and other related stuff see [5] and [6].

McNamara and Sagan [2] defined a stronger version of *log-concavity*.
A sequence $(a_k) = a_0, a_1, a_2, \dots$ of real numbers is said to be *r-factor log-concave iff*

$$a_k^{\,2} \geq r\, a_{k-1}\, a_{k+1} \tag{1}$$

for all $k \in \mathbb{N}$. Thus *r-factor log-concave* sequence implies *log-concavity* if $r \geq 1$. We are interested only in *log-concave* sequences, so from here onward, value of r used would mean $r \geq 1$ unless otherwise stated.

We first define a new operator $\mathscr{L}_r$ and then using this operator, we define *Generalized r-factor infinite log-concavity* which is a bit more stronger version of *log-concavity*. Define the real operator $\mathscr{L}_r$ and the new sequence (b_k) such that $(b_k) = \mathscr{L}_r(a_k)$, where $b_k = \mathscr{L}_r(a_k) = {a_k}^2 - r\, a_{k-1}\, a_{k+1}$.

Then (a_k) is said to be *r-factor log-concave* (or *Generalized r-factor 1-fold log-concave) iff* (b_k) is non negative for all $k \in \mathbb{N}$.
This again defines (1) alternatively using $\mathscr{L}_r$ operator. (a_k) is said to be *Generalized r-factor 2-fold log-concave iff* $\mathscr{L}_r^2(a_k) = \mathscr{L}_r(\mathscr{L}_r(a_k)) = \mathscr{L}_r(b_k)$ is non negative for all $k \in \mathbb{N}$, where

$$\mathscr{L}_r(b_k) = {b_k}^2 - r\, b_{k-1}\, b_{k+1}$$
$$\text{or } \mathscr{L}_r^2(a_k) = [\mathscr{L}_r(a_k)]^2 - r\,[\mathscr{L}_r(a_{k-1})]\,[\mathscr{L}_r(a_{k+1})]$$

(a_k) is said to be *Generalized r-factor i-fold log-concave iff* $\mathscr{L}_r^i(a_k)$ is non negative for all $k \in \mathbb{N}$, where $\mathscr{L}_r^i(a_k) = [\mathscr{L}_r^{i-1}(a_k)]^2 - r\,[\mathscr{L}_r^{i-1}(a_{k-1})]\,[\mathscr{L}_r^{i-1}(a_{k+1})]$
(a_k) is said to be *Generalized r-factor infinite log-concave iff* $\mathscr{L}_r^i(a_k)$ is non negative for all $i \geq 1$.

Uminsky and Yeats [1] studied the properties of the *log-operator* $\mathscr{L}$ on the subset of the finite symmetric

* Corresponding author e-mail: zahid.raza@nu.edu.pk

sequences of the form

$$\{\ldots,0,0,1,x_\circ,x_1,\ldots,x_n,\ldots,x_1,x_\circ,1,0,0,\ldots\},$$

$$\{\ldots,0,0,1,x_\circ,x_1,\ldots,x_n,x_n,\ldots,x_1,x_\circ,1,0,0,\ldots\}.$$

The first sequence above is referred as odd of length $2n+3$ and second as even of length $2n+4$. Any such sequence corresponds to a point $(x_0,x_1,x_2,\ldots,x_n)$ in $\mathbb{R}^{n+1}$. They prove the existence of an infinite region $\mathscr{R}\subset\mathbb{R}^{n+1}$, bounded by $n+1$ parametrically defined hypersurfaces such that any sequence corresponding a point of $\mathscr{R}$ is *infinitely log concave*.

In the first part of this paper, we study the properties of the *Generalized r-factor log-operator* $\mathscr{L}_r$ on these finite symmetric sequences and redefine the parametrically defined hypersurfaces which generalizes the one defined by [1]. We show that any sequence corresponding a point of the region $\mathscr{R}_r$, bounded by the new generalized parametrically defined r-factor hypersurfaces, is *Generalized r-factor infinite log concave*.

In the end, we give an improved value of $r_\circ$ found by McNamara and Sagan [2] as the log-concavity criterion using the new *log-operator* $\mathscr{L}_r$.

Lemma 1.1. Let (a_k) be a r-factor log-concave sequence of non-negative terms. If $\mathscr{L}_r(a_k)$ is Generalized r-factor log-concave, then

$$(r^5)a_{k-2}\,a_{k-1}\,a_{k+1}\,a_{k+2}\;\le\;a_k^4.$$

In general, if $\mathscr{L}^{i+1}(a_k)$ is Generalized r-factor log-concave, then

$$(r^5)\mathscr{L}_r^i(a_{k-2})\,\mathscr{L}_r^i(a_{k-1})\,\mathscr{L}_r^i(a_{k+1})\,\mathscr{L}_r^i(a_{k+2})\;\le\;[\mathscr{L}_r^i(a_k)]^4.$$

Proof. Let $\mathscr{L}_r(a_k)$ is *r-factor log-concave*. Then

$$[\mathscr{L}_r(a_k)]^2\;\ge\;r\,[\mathscr{L}_r(a_{k-1})]\,[\mathscr{L}_r(a_{k+1})]$$

$$\left.\begin{array}{l} a_k^4+(r^2-r)a_{k-1}^2a_{k+1}^2\\ +r^2\,a_{k-1}^2a_ka_{k+2}\\ +r^2a_{k-2}a_ka_{k+1}^2\end{array}\right)\ge 2ra_{k-1}a_k^2a_{k+1}+r^3a_{k-2}a_k^2a_{k+2}.$$

Since (a_k) is *r-factor log concave*, so applying $a_k^2\ge r\,a_{k-1}\,a_{k+1}$, we have

$$(r^5)\;a_{k-2}\;a_{k-1}\;a_{k+1}\;a_{k+2}\;\le\;a_k^4.$$

Similarly, if $\mathscr{L}_r^2(a_k)$ is *Generalized r-factor log-concave, then*

$$(r^5)\;\mathscr{L}_r(a_{k-2})\;\mathscr{L}_r(a_{k-1})\mathscr{L}_r(a_{k+1})\mathscr{L}_r(a_{k+2})\;\le\;[\mathscr{L}_r(a_k)]^4$$

Continuing this way, if $\mathscr{L}_r^{i+1}(a_k)$ is *Generalized r-factor log-concave, then*

$$(r^5)\;\mathscr{L}_r^i(a_{k-2})\,\mathscr{L}_r^i(a_{k-1})\,\mathscr{L}_r^i(a_{k+1})\,\mathscr{L}_r^i(a_{k+2})\le[\mathscr{L}_r^i(a_k)]^4.$$

□.

If we can prove conversely, above lemma can be used as an alternative criterion to verify the *r-factor i-fold log-concavity* of a given *r-factor log-concave* sequence. The Generalized r-factor log-operator $\mathscr{L}_r$ equals the log-operator $\mathscr{L}$ for $r=1$, so *Generalized r-factor infinite log-concavity* implies *infinite log-concavity*. Thus, we have the following results:

Lemma 1.2. Let (a_k) be a log-concave sequence of non-negative terms. If $\mathscr{L}(a_k)$ is log-concave, then $a_{k-2}\,a_{k-1}\,a_{k+1}\,a_{k+2}\;\le\;a_k^4$. In general, if $\mathscr{L}^{i+1}(a_k)$ is log-concave, then

$$\mathscr{L}^i(a_{k-2})\,\mathscr{L}^i(a_{k-1})\,\mathscr{L}^i(a_{k+1})\,\mathscr{L}^i(a_{k+2})\;\le\;[\mathscr{L}^i(a_k)]^4$$

Lemma 1.3. Every Generalized r-factor infinitely log-concave sequence (a_k) of non-negative terms is infinitely log-concave.

2 Region of infinite log-concavity and r-factor

One dimensional even and odd sequences $\{1,x,x,1\}$,$\{1,x,1\}$ correspond to a point $x\in\mathbb{R}$. Uminsky and Yeats [1] after applying the *log-operator* $\mathscr{L}$ showed that the positive fixed point for the sequence $\mathscr{L}\{1,x,x,1\}=\left\{1,x^2-x,x^2-x,1\right\}$ is $x=2$ and for $\mathscr{L}\{1,x,1\}=\left\{1,x^2-1,1\right\}$ is $x=\frac{1+\sqrt{5}}{2}$. Also the sequence $\{1,x,x,1\}$ is infinitely log-concave if $x\ge 2$ and $\{1,x,1\}$ is infinitely log-concave if $x\ge\frac{1+\sqrt{5}}{2}$. For detail see [1]

Now if we apply the Generalized r-factor log operator $\mathscr{L}_r$, instead of applying the log operator $\mathscr{L}$, then after a simple calculation we see that the positive fixed point for the sequence $\mathscr{L}_r\{1,x,x,1\}=\left\{1,x^2-rx,x^2-rx,1\right\}$ is $x=1+r$ and for $\mathscr{L}_r\{1,x,1\}=\left\{1,x^2-r,1\right\}$ is $x=\frac{1+\sqrt{1+4r}}{2}$. Also the sequence $\{1,x,x,1\}$ is Generalized r-factor infinitely log-concave if $x\ge 1+r$ and $\{1,x,1\}$ is Generalized r-factor infinitely log-concave if $x\ge\frac{1+\sqrt{1+4r}}{2}$. This agrees with the results obtained by Uminsky and Yeats for $r=1$.

2.1 Leading terms analysis using r-factor log-concavity

Consider the even sequence of length $2n+4$

$$S=\Big\{1,a_\circ x,a_1x^{1+d_1},a_2x^{1+d_1+d_2},\ldots,a_nx^{1+d_1+\cdots+d_n},\\ a_nx^{1+d_1+\cdots+d_n},\ldots,a_1x^{1+d_1},a_\circ x,1\Big\}\qquad(2)$$

If we apply $\mathscr{L}_r$ operator on s, instead of applying $\mathscr{L}$, then

$$\mathscr{L}_r(s) = \Big\{1, x(a_\circ^2 x - ra_1 x^{d_1}), x^{2+d_1}(a_1^2 x^{d_1} - ra_2 a_\circ x^{d_2}),$$
$$x^{2+2d_1+d_2}(a_2^2 x^{d_2} - ra_3 a_1 x^{d_3}), \dots,$$
$$x^{2+2d_1+\cdots+2d_{n-1}+d_n}(a_n^2 x^{d_n} - ra_n a_{n-1}),$$
$$x^{2+2d_1+\cdots+2d_{n-1}+d_n}(a_n^2 x^{d_n} - ra_n a_{n-1}), \dots, 1\Big\}$$

where, $0 \le d_n \le d_{n-1} \le \cdots \le d_1 \le 1$. The $(n-1)$ faces are defined by $d_1 = 1,\ d_j = d_{j+1},$ for $0 < j < n$, and $d_n = 0$, they define the boundaries of what will be our open region of convergence, for detail see [1]

For $\mathbf{d_1 = 1}$. The leading terms of $\mathscr{L}_r(s)$ are $\{1, (a_\circ^2 - ra_1)x^2, a_1^2 x^4, a_2^2 x^{4+2d_2}, \dots, a_n^2 x^{4+2d_2+\cdots+2d_n}, a_n^2 x^{4+2d_2+\cdots+2d_n}, \dots, 1\}$ matching the coefficients of leading terms in $\mathscr{L}_r(s)$ with the coefficients of s. So that the leading terms of $\mathscr{L}_r$ have the same form as s itself for some new x, we have the positive values

$$a_\circ = \frac{1+\sqrt{1+4r}}{2}, \text{ and } a_i = 1 \text{ for } 0 < i \le n. \qquad (3)$$

This agrees with the values, $a_\circ = \frac{1+\sqrt{5}}{2}$, and $a_i = 1$ for $0 < i \le n$, obtained by Uminsky and Yeats [1] for $r = 1$.

For $\mathbf{d_j = d_{j+1}}$. The leading terms of $\mathscr{L}_r(s)$ are

$$\Big\{1, a_\circ^2 x^2, a_1^2 x^{2+2d_1}, a_2^2 x^{2+2d_1+2d_2}, \dots,$$
$$(a_j^2 - ra_{j-1}a_{j+1})x^{2+2d_1+\cdots+2d_j}, a_{j+1}^2 x^{2+2d_1+\cdots+2d_{j-1}+4d_j},$$
$$\dots, a_n^2 x^{2+2d_1+\cdots+2d_n}, a_n^2 x^{2+2d_1+\cdots+2d_n}, \dots, 1\Big\}$$

comparing the coefficients, we get the positive values

$$a_i = 1 \text{ for } i \ne j, \text{ and } a_j = \frac{1+\sqrt{1+4r}}{2}. \qquad (4)$$

This gives the values for $r = 1$, $a_i = 1$ for $i \ne j$, and $a_j = \frac{1+\sqrt{5}}{2}$, same as in [1].

For $\mathbf{d_n = 0}$. The leading terms of $\mathscr{L}_r(s)$ are

$$\{1, a_\circ^2 x^2, a_1^2 x^{2+2d_1}, a_2^2 x^{2+2d_1+2d_2}, \dots, a_{n-1}^2 x^{2+2d_1+\cdots+2d_{n-1}},$$
$$(a_n^2 - ra_n a_{n-1})x^{2+2d_1+\cdots+2d_{n-1}},$$
$$(a_n^2 - ra_n a_{n-1})x^{2+2d_1+\cdots+2d_{n-1}}, \dots, 1\}$$

comparing the coefficients, we get the values

$$a_i = 1 \text{ for } 0 \le i < n, \text{ and } a_n = 1 + r. \qquad (5)$$

This again agrees with the values, $a_i = 1$ for $0 \le i < n$, and $a_n = 2$, obtained in [1] for $r = 1$.

Similarly for the odd sequence of length $2n+3$

$$s = \{1, a_\circ x, a_1 x^{1+d_1}, a_2 x^{1+d_1+d_2}, \dots,$$
$$a_n x^{1+d_1+\cdots+d_n}, \dots, a_1 x^{1+d_1}, a_\circ x, 1\} \qquad (6)$$

Applying $\mathscr{L}_r$ operator

$$\mathscr{L}_r(s) = \{1, x(a_\circ^2 x - ra_1 x^{d_1}), x^{2+d_1}(a_1^2 x^{d_1} - ra_2 a_\circ x^{d_2}),$$
$$x^{2+2d_1+d_2}(a_2^2 x^{d_2} - ra_3 a_1 x^{d_3}), \dots,$$
$$x^{2+2d_1+\cdots+2d_{n-1}}(a_n^2 x^{2d_n} - ra_{n-1}^2), \dots, 1\}$$

For $\mathbf{d_1 = 1}$ and $\mathbf{d_j = d_{j+1}}$. This is equivalent to the even case, see (3), (4). So we only analyze for $\mathbf{d_n = 0}$. The leading terms of $\mathscr{L}_r(s)$ are

$$\{1, a_\circ^2 x^2, a_1^2 x^{2+2d_1}, \dots, a_{n-1}^2 x^{2+2d_1+\cdots+2d_{n-1}},$$
$$(a_n^2 - ra_{n-1}^2)x^{2+2d_1+\cdots+2d_{n-1}}, \dots, 1\}$$

so equating the coefficients, we get,

$$a_i = 1 \text{ for } 0 \le i < n, \text{ and } a_n = \frac{1+\sqrt{1+4r}}{2}. \qquad (7)$$

This again agrees with the values for $r = 1$, as obtained in [1]. The even sequence (2) and the odd sequence (6) correspond to the point $\left(a_\circ x, a_1 x^{1+d_1}, \dots, a_n x^{1+d_1+\cdots+d_n}\right) \in \mathbb{R}^{n+1}$. Hence from (3), (4), (5) and (7) the redefined and generalized parametrically defined Hypersurfaces are

$$\mathscr{H}_\circ = \Big\{\Big(\frac{1+\sqrt{1+4r}}{2}x, x^2, x^{2+d_2}, \dots, x^{2+d_2+\cdots+d_n}\Big) : 1 \le x,$$
$$1 > d_2 > \cdots > d_n > 0\Big\}$$

$$\mathscr{H}_j = \Big\{\Big(x, x^{1+d_1}, \dots, \frac{1+\sqrt{1+4r}}{2}x^{1+d_1+\cdots+d_j}, x^{1+d_1+\cdots+d_{j-1}+2d_j},$$
$$\dots, x^{1+d_1+\cdots+d_{j-1}+2d_j+d^{j+2}+\cdots+d_n}\Big) :$$
$$1 \le x,\ 1 > d_1 > \cdots > d_j > d_{j+2} > \cdots > d_n > 0\Big\}$$

The hypersurfaces $\mathscr{H}_j$ are same for $0 \le j < n$ in both even and odd cases, while $\mathscr{H}_n$ is different.

In even case:

$$\mathscr{H}_n = \Big\{\Big(x, x^{1+d_1}, \dots, x^{1+d_1+\cdots+d_{n-1}}, (1+r)x^{1+d_1+\cdots+d_{n-1}}\Big)$$
$$: 1 \le x,\ 1 > d_1 > \cdots > d_{n-1} > 0\Big\}$$

In odd case:

$$\mathscr{H}_n = \Big\{\Big(x, x^{1+d_1}, \dots, x^{1+d_1+\cdots+d_{n-1}}, \frac{1+\sqrt{1+4r}}{2}x^{1+d_1+\cdots+d_{n-1}}\Big)$$
$$: 1 \le x,\ 1 > d_1 > \cdots > d_{n-1} > 0\Big\}$$

Hence, the r-factor hypersurfaces for $r = 1$ agrees with the hypersurfaces obtained in [1].

So from here onward we consider $\mathscr{R}_r$ to be the region of Generalized r-factor infinite log-concavity and is bounded by the new generalized r-factor hypersurfaces. Also any sequence $\{\dots,0,0,1,x_\circ,x_1,\dots,x_n,x_n,\dots,x_1,x_\circ,1,0,0,\dots\}$ is in $\mathscr{R}_r$ *iff* $(x_\circ,x_1,\dots,x_n)\in\mathscr{R}_r$ and with the positive increasing coordinates defined as greater in the i^{th} coordinate than $\mathscr{H}_i$. In this case we say that above sequence lies on the correct side of $\mathscr{H}_i$. Next, we present the r-factor log-concavity version of the Lemma (3.2) of [1].

Lemma 2.1.1. Let the sequence

$$s=\{1,x,x^{1+d_1},x^{1+d_1+d_2},\dots,x^{1+d_1+\cdots+d_n},x^{1+d_1+\cdots+d_n},\dots,x,1\}$$

be r-factor 1-log-concave for $x>0$. Then $1\geq d_1\geq\cdots\geq d_n\geq 0$.

In Lemma (3.3) of Uminsky and Yeats [1] using properties of the triangular numbers and the sequence

$$s=\{1,C^{T(0)}ax_\circ,C^{T(1)}a^2x_1,C^{T(2)}a^3x_2,\dots,\\ C^{T(n)}a^{n+1}x_n,C^{T(n)}a^{n+1}x_n,\dots,1\} \tag{8}$$

proved the existence of the log-concavity region $\mathscr{R}$ by applying log-operator $\mathscr{L}$ for $a>2C^{T(n-1)-T(n)}$ and for $0<C<\frac{2}{1+\sqrt{5}}$. Sequence s (8) is not the only sequence for which $\mathscr{R}$ is non-empty. One can also prove it by some other numbers such as Pentagon numbers and figurate numbers.

If we choose C such that $0<C<\frac{2\sqrt{r}}{1+\sqrt{1+4r}}$, then applying the Generalized r-factor log-operator $\mathscr{L}_r$ on the sequence (8), we can easily prove the existence of the Generalized r-factor log-concavity region $\mathscr{R}$ for $a>(1+r)C^{T(n-1)-T(n)}$. Let $\tilde{P}(n)$ denotes the n^{th} pentagonal number, then

$$\tilde{P}(n)=\frac{n(3n-1)}{2}=\tilde{P}(n-1)+3n-2$$

Define $P(n)=2\tilde{P}(n)$ for $n\geq 0$, we can easily have

$$P(n+1)+P(n-1)=2P(n)+6 \tag{9}$$
$$P(n+1)+P(n-1)>2P(n) \tag{10}$$
$$C^{P(n+1)+P(n-1)}<C^{2P(n)}\quad\text{for all}\quad C<1 \tag{11}$$

$$\text{Also } P(0)-\frac{P(1)}{2}=-1\ \because\ \tilde{P}(0)=0\text{ and }\tilde{P}(1)=1 \tag{12}$$

Hence the Generalized r-factor log-concavity version of Lemma (3.3) [1] is given below:

Lemma 2.1.2. The Generalized r-factor infinite log-concavity region $\mathscr{R}_r$ is non-empty and unbounded.

Proof. Let us consider any r-factor log-concave sequence. $q=\{\dots,0,0,1,x_\circ,x_1,\dots,x_n,\dots,x_1,x_\circ,1,0,0,\dots\}$. Choose C such that

$$0<C<\frac{2\sqrt{r}}{1+\sqrt{1+4r}}<1 \tag{13}$$

and consider the following sequence

$$s=\{1,C^{P(0)}ax_\circ,C^{P(1)}a^2x_1,C^{P(2)}a^3x_2,\dots,\\ C^{P(n)}a^{n+1}x_n,C^{P(n)}a^{n+1}x_n,\dots,1\} \tag{14}$$

for $a>(1+r)C^{P(n-1)-P(n)}>C^{P(n-1)-P(n)}$. Now using r-factor log-concavity of q, we have

$$C^{2P(0)}a^2x_\circ^2 = a^2x_\circ^2\geq a^2rx_1>rC^{P(1)}a^2x_1 \tag{15}$$

$$\begin{aligned} C^{2P(j)}a^{2j+2}x_j^2 &\geq C^{2P(j)}a^{2j+2}(rx_{j-1}x_{j+1})\ \forall\, 0<j>n\\ &= r\,C^{2P(j)}\ a^jx_{j-1}\ \ a^{j+2}x_{j+1}\\ &> rC^{P(j-1)}a^jx_{j-1}C^{P(j+1)}a^{j+2}x_{j+1}.\ \text{by (11)} \end{aligned} \tag{16}$$

$$\begin{aligned} \text{and } C^{P(n)}a^{n+1}x_n &\geq aC^{P(n)}a^n(rx_{n-1})\\ &> C^{P(n-1)-P(n)}rC^{P(n)}a^nx_{n-1}\ \ \text{by (14)}\\ &> C^{P(n-1)}a^nx_{n-1} \end{aligned} \tag{17}$$

$$\begin{aligned} \text{and so } C^{2P(n)}a^{2n+2}x_n^2 &= C^{P(n)}a^{n+1}x_n\,C^{P(n)}a^{n+1}x_n\\ &> rC^{P(n-1)}a^nx_{n-1}C^{P(n)}a^{n+1}x_n.\ \ \text{by (17)} \end{aligned} \tag{18}$$

From (15),(16),(18), we conclude that s is also r-factor 1-log-concave.

Define $\tilde{x}=C^{P(0)}ax_\circ$ and define $\tilde{d_1}$ such that $\tilde{x}^{1+\tilde{d_1}}=C^{P(1)}a^2x_1$ and continuing, we have $\tilde{x}^{1+\tilde{d_1}+\cdots+\tilde{d_j}}=C^{P(j)}a^{j+1}x_j\Rightarrow 1>\tilde{d_1}>\tilde{d_2}>\cdots>\tilde{d_n}>0$ by lemma (2.1)

For $\mathscr{H}_j$

Choose $x=\tilde{x}$, $d_i=\tilde{d_i}$ for $i\neq j,j+1$ and $d_j=(\tilde{d_j}+\tilde{d_{j+1}})/2$ for hypersurface $\mathscr{H}_j$. Consequently, $1>d_1>\cdots>d_j>d_{j+2}>\cdots>d_n>0$, and so

$$\begin{aligned} C^{P(j)}a^{j+1}x_j &\geq C^{P(j)}a^{j+1}\sqrt{rx_{j-1}x_{j+1}}\\ &=\sqrt{r}\sqrt{C^{2P(j)-P(j+1)-P(j-1)}C^{P(j-1)}a^jx_{j-1}C^{P(j+1)}a^{j+2}x_{j+1}}\\ &=\sqrt{r}\sqrt{C^{-6}x^{1+d_1+\cdots+d_{j-1}}x^{1+d_1+\cdots+d_{j-1}+2d_j}}\ \text{by (9)}\\ &>\sqrt{r}C^{-1}x^{1+d_1+\cdots+d_{j-1}+d_j}\\ &>\frac{1+\sqrt{1+4r}}{2}x^{1+d_1+\cdots+d_{j-1}+d_j}\ \text{by (13)} \end{aligned} \tag{19}$$

Thus s is on the correct side of $\mathscr{H}_j$.

For $\mathscr{H}_\circ$

Choose $x=\tilde{x}$, $d_1=1$ and $d_i=\tilde{d_i}\ \forall\ i>1$. Consequently, $1>d_2>\cdots>d_n>0$, by lemma (2.1)and so

$$\begin{aligned} C^{P(1)}a^2x^1 &= \tilde{x}^{1+\tilde{d_1}}=\tilde{x}^2=x^2\\ \Rightarrow\ \ a^2x_1 &= C^{-P(1)}x^2 \end{aligned} \tag{20}$$

also $C^{P(j)}a^{j+1}x^j=\tilde{x}^{1+\tilde{d_1}+\cdots+\tilde{d_j}}=x^{2+d_2+\cdots+d_j}$

Now we check

$$\begin{aligned} C^{P(0)}ax_\circ &\geq C^{P(0)}\sqrt{ra^2x_1} \\ &= \sqrt{r}\,C^{P(0)}\sqrt{C^{-P(1)}x^2} \quad \text{by (20)} \\ &= \sqrt{r}\,C^{-1}\,x \quad \text{by (12)} \\ &> \frac{1+\sqrt{1+4r}}{2}\,x \quad \text{by (13)} \qquad (21) \end{aligned}$$

Thus s is on the correct side of $\mathscr{H}_0$.

For $\mathscr{H}_n$

Choose $x=\tilde{x}$, and $d_i=\tilde{d}_i$ for $i<n$, $\tilde{d}_n=d_n=0$ for $\mathscr{H}_n$. Consequently, we have, $1>d_1>\cdots>d_{n-1}>0$,

$$\begin{aligned} C^{P(n)}a^{n+1}x_n &\geq C^{P(n)}a^{n+1}(r\,x_{n-1}) \\ &\geq a\,C^{P(n)-P(n-1)}\,x^{1+d_1+\cdots+d_{n-1}} \\ &> (1+r)\,x^{1+d_1+\cdots+d_{n-1}} \quad \text{by (14)} \qquad (22) \end{aligned}$$

Thus s is on the correct side of $\mathscr{H}_n$. From (19),(21),(22), and by the definition of the region $\mathscr{R}_r$, we conclude that sequence s is in $\mathscr{R}_r$. Hence using *r-factor log-concavity*, $\mathscr{R}_r$ is non-empty and unbounded.□.

Now we present the Generalized r-factor Infinite log-concavity version of the main theorem of [1].

Theorem 2.1.3. Any sequence in $\mathscr{R}_r$ is Generalized r-factor Infinite log-concave.

Proof. Let us consider the sequence in $\mathscr{R}_r$

$$s=\Big\{1,x,x^{1+d_1},\ldots,x^{1+d_1+\cdots+d_{j-1}},\frac{1+\sqrt{1+4r}}{2}x^{1+d_1+\cdots+d_j}+\varepsilon, x^{1+d_1+\cdots+d_{j-1}+2d_j},x^{1+d_1+\cdots+2d_j+\cdots+d_n}, x^{1+d_1+\cdots+2d_j+\cdots+d_n},\ldots,1\Big\} \quad x,\varepsilon>0$$

Applying $\mathscr{L}_r$ operator on s and simplifying, we get

$$\begin{aligned} \mathscr{L}_r(s) = \Big\{&1,\ x^2-rx^{1+d_1},\ \ldots, \\ &x^{2+2d_1+\cdots+2d_{j-1}}-r\left(\frac{1+\sqrt{1+4r}}{2}\right)x^{2+2d_1+\cdots+2d_{j-2}+d_{j-1}+d_j} \\ &-\varepsilon\,r\,x^{1+d_1+\cdots+d_{j-2}},\left(\left(\frac{1+\sqrt{1+4r}}{2}\right)^2-r\right)x^{2+2d_1+\cdots+2d_j}+\varepsilon^2 \\ &-\varepsilon\left(1+\sqrt{1+4r}\right)x^{1+d_1+\cdots+d_j},x^{2+2d_1+\cdots+2d_{j-1}+4d_j} \\ &-r\left(\frac{1+\sqrt{1+4r}}{2}\right)x^{2+2d_1+\cdots+3d_j+d_{j+2}} \\ &-r\,\varepsilon\left(x^{1+d_1+\cdots+2d_j+d_{j+2}}\right),\ldots,x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} \\ &-r\left(x^{2+2d_1+\cdots+4d_j+\cdots+2d_{n-1}+d_n}\right),x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} \\ &-r\left(x^{2+2d_1+\cdots+4d_j+\cdots+2d_{n-1}+d_n}\right),\ldots,1\Big\} \end{aligned}$$

Since

$$\left(\frac{1+\sqrt{1+4r}}{2}\right)^2-r=\frac{1+\sqrt{1+4r}}{2}, \qquad (23)$$

so by using x^2 in place of x in the definition of $\mathscr{H}_j$ and applying Lemma(3.4) of [1], we conclude that both s and $\mathscr{L}_r(s)$ are on the same side of $\mathscr{H}_j$ which are larger in the j^{th} coordinate. Hence result is true for hypersurface $\mathscr{H}_j$.

Similarly, for $x,\varepsilon>0$ consider the sequence

$$s=\Big\{1,\frac{1+\sqrt{1+4r}}{2}x+\varepsilon,x^2,\ldots, x^{2+d_2+\cdots+d_n},x^{2+d_2+\cdots+d_n},\ldots,1\Big\}$$

After applying $\mathscr{L}_r$ operator on s and simplifying, we get

$$\begin{aligned} &\mathscr{L}_r(s)= \\ &\Big\{1,\left(\left(\frac{1+\sqrt{1+4r}}{2}\right)^2-r\right)x^2+\varepsilon\left(1+\sqrt{1+4r}\right)x+\varepsilon^2, \\ &x^4-r\left(\frac{1+\sqrt{1+4r}}{2}\right)x^{3+d_2}-r\varepsilon x^{2+d_2},\ldots, \\ &x^{4+2d_2+\cdots+2d_n}-rx^{4+2d_2+\cdots+2d_{n-1}+d_n}, \\ &x^{4+2d_2+\cdots+2d_n}-rx^{4+2d_2+\cdots+2d_{n-1}+d_n},\ldots,1\Big\} \end{aligned}$$

again by (23) and Lemma(3.4) of [1], we conclude that s and $\mathscr{L}_r(s)$ lie on the same side of $\mathscr{H}_\circ$. Hence result is true for $\mathscr{H}_\circ$.

Finally, for $x,\varepsilon>0$, $d_n=0$ consider the sequence

$$s=\Big\{1,x,x^{1+d_1},\ldots,x^{1+d_1+\cdots+d_{n-1}},(1+r)x^{1+d_1+\cdots+d_{n-1}}+\varepsilon, (1+r)x^{1+d_1+\cdots+d_{n-1}}+\varepsilon,\ldots,1\Big\}$$

Applying $\mathscr{L}_r$, we get

$$\begin{aligned} \mathscr{L}_r(s)=\big\{&1,x^2-rx^{1+d_1},x^{2+2d_1}-rx^{2+d_1+d_2},\ldots,x^{2+2d_1+\cdots+2d_{n-1}} \\ &-r(1+r)x^{2+2d_1+\cdots+2d_{n-2}+d_{n-1}}-\varepsilon rx^{1+d_1+\cdots+d_{n-2}}, \\ &\left((1+r)^2-r(1+r)\right)x^{2+2d_1+\cdots+2d_{n-1}}+\varepsilon(r+2)x^{1+d_1+\cdots+d_{n-1}}+\varepsilon^2, \\ &\left((1+r)^2-r(1+r)\right)x^{2+2d_1+\cdots+2d_{n-1}}+ \\ &\varepsilon(r+2)x^{1+d_1+\cdots+d_{n-1}}+\varepsilon^2,\ldots,1\big\} \end{aligned}$$

Since $(1+r)^2-r(1+r)=1+r$, so again by Lemma(3.4) of [1], we conclude that s and $\mathscr{L}_r(s)$ lie on the same side of $\mathscr{H}_n$. Hence the result is true for considering $\mathscr{H}_n$.

Consequently from the above three cases, $s\in\mathscr{R}_r \Rightarrow \mathscr{L}_r(s)\in\mathscr{R}_r$. Hence any sequence in $\mathscr{R}_r$ is Generalized r-factor Infinite log-concave.

In case of the odd sequences, system is equivalent to the even case for $\mathscr{H}_\circ$ and $\mathscr{H}_j$. So we only need to consider for $\mathscr{H}_n$. Let

$$s=\Big\{1,x,x^{1+d_1},\ldots,x^{1+d_1+\cdots+d_{n-1}},\frac{1+\sqrt{1+4r}}{2}x^{1+d_1+\cdots+d_{n-1}}+\varepsilon, x^{1+d_1+\cdots+d_{n-1}},\ldots,1\Big\}$$

be a sequence in $\mathscr{R}_r$. Applying $\mathscr{L}_r$ operator on s and simplifying, we get

$$\mathscr{L}_r(s) = \Bigg\{ 1, x^2 - rx^{1+d_1}, x^{2+2d_1} - rx^{2+d_1+d_2}, \ldots,$$

$$x^{2+2d_1+\cdots+2d_{n-1}} - r\left(\frac{1+\sqrt{1+4r}}{2}\right)x^{2+2d_1+\cdots+2d_{n-2}+d_{n-1}}$$

$$-\varepsilon rx^{1+d_1+\cdots+d_{n-2}}, \left(\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - r\right)x^{2+2d_1+\cdots+2d_{n-1}}$$

$$+\varepsilon\left(1+\sqrt{1+4r}\right)x^{1+d_1+\cdots+d_{n-1}} + \varepsilon^2,$$

$$x^{2+2d_1+\cdots+2d_{n-1}} - r\left(\frac{1+\sqrt{1+4r}}{2}\right)x^{2+2d_1+\cdots+2d_{n-2}+d_{n-1}}$$

$$-\varepsilon rx^{1+d_1+\cdots+d_{n-2}}, \ldots, 1\Bigg\}$$

So by (23) and Lemma(3.4) of [1], we conclude that s and $\mathscr{L}_r(s)$ lie on the same side of $\mathscr{H}_\circ$. Hence any (odd) sequence in $\mathscr{R}_r$ is also Generalized r-factor Infinite log-concave.□.

3 Generalized r-factor infinite log-concavity criterion

We start this section by a Lemma 2.1, proved by McNamara and Sagan [2] using the log-operator $\mathscr{L}$, that is

Lemma 3.1.[Lemma 2.1, [2],] Let (a_k) be a non-negative sequence and let $r_\circ = (3+\sqrt{5})$. Then (a_k) being $r_\circ$-factor log-concave implies that $\mathscr{L}(a_k)$ is too. So in this case (a_k) is infinitely log-concave.

If we apply the Generalized r-factor log-operator $\mathscr{L}_r$, instead of applying the log-operator $\mathscr{L}$, we have the following result:

Lemma 3.2. Let (a_k) be a sequence of non-negative terms and $r = 1+\sqrt{2}$. If (a_k) is Generalized r-factor log-concave, then so is $\mathscr{L}_r(a_k)$ Hence continuing, (a_k) is Generalized r-factor infinitely log-concave sequence.

Proof. Let (a_k) be *r-factor log-concave* sequence of non-negative terms. Now $\mathscr{L}_r(a_k)$ will be r-factor log-concave if and only if

$$[\mathscr{L}_r(a_k)]^2 \geq r[\mathscr{L}_r(a_{k-1})][\mathscr{L}_r(a_{k+1})]$$

$$(a_k^2 - ra_{k-1}a_{k+1})^2 \geq r(a_{k-1}^2 - ra_{k-2}a_k)(a_{k+1}^2 - ra_ka_{k+2})$$

$$2a_{k-1}a_k^2a_{k+1} + r^2a_{k-2}a_k^2a_{k+2} \leq \frac{1}{r}a_k^4 + (r-1)a_{k-1}^2a_{k+1}^2$$

$$+ra_{k-1}^2a_ka_{k+2} + ra_{k-2}a_ka_{k+1}^2 \leq a_k^4 + (r-1)a_{k-1}^2a_{k+1}^2$$

$$+ra_{k-1}^2a_ka_{k+2} + ra_{k-2}a_ka_{k+1}^2 \tag{24}$$

Since (a_k) is *r-factor log concave*, so applying $a_k^2 \geq ra_{k-1}a_{k+1}$, to the L.H.S. of the above inequality, we have

$$2a_{k-1}a_k^2a_{k+1} + r^2a_{k-2}a_k^2a_{k+2} \leq \frac{2}{r}a_k^4 + \frac{1}{r^2}a_k^4 = \left(\frac{2r+1}{r^2}\right)a_k^4$$

So to keep (24) valid, we have $\frac{2r+1}{r^2} = 1 \Rightarrow r^2 - 2r - 1 = 0$. Thus $r = 1+\sqrt{2}$, is the positive root of the above equation. This proves the assertion. Thus, if (a_k) is Generalized r-factor log-concave, then so is $\mathscr{L}_r(a_k)$. Continuing this way, if $\mathscr{L}_r^i(a_k)$ is Generalized r-factor log-concave, then so is $\mathscr{L}_r^{i+1}(a_k)$. This also implies Generalized r-factor infinite log-concavity of the sequence (a_k).□.

Comparing this new value of r, say $r_1 = 1+\sqrt{2}$, with the value of $r_\circ = \frac{3+\sqrt{5}}{2}$ obtained by McNamara and Sagan [2]. We find that the value of $r_1 = 1+\sqrt{2}$ obtained by using Generalized r-factor log-concavity is smaller than obtained by McNamara and Sagan which is $r_\circ = \frac{3+\sqrt{5}}{2}$.

So in this way we get an improved /smaller value of $r = 1+\sqrt{2}$. It is clear that Generalized r-factor log concave operator is more useful and dynamic than the previously used log-operator $\mathscr{L}$. Hence for the new improved value of r, we can restate Lemma (3.1) [2] as:

Lemma 3.3. Let $a_\circ, a_1, \ldots, a_{2m+1}$ be symmetric, nonnegative sequence such that

(i) $a_k^2 \geq r_1a_{k-1}a_{k+1}$ for $k < m$,
(ii) $a_m \geq (1+r)a_{m-1}$ for $r \geq 1$.

Then $\mathscr{L}_{r_1}(a_k)$ has the same properties, which implies that (a_k) is r_1-factor infinitely log-concave.

Using above lemma we now show that Generalized r-factor log-operator $\mathscr{L}_r$ and r-factor hypersurfaces agrees with Theorem (3.2) of [2] for $r = 1$. It also proves theorem (2.1) alternatively.

Theorem 3.4.[Revised Theorem 3.2, [2]] Any sequence corresponding to a point of $\mathscr{R}_r$ is Generalized infinitely r_1-factor log-concave.

Proof. Let (a_k) be a sequence corresponding to a point of $\mathscr{R}$. Then, for (a_k), being on the correct side of $\mathscr{H}_\mathbf{j}$, we have

$$a_j \geq \left(\frac{1+\sqrt{1+4r}}{2}\right)x^{1+d_1+\cdots+d_j}$$

$$\Rightarrow \quad a_j^2 \geq \left(\frac{1+\sqrt{1+4r}}{2}\right)^2 x^{2+2d_1+\cdots+2d_j}$$

$$= \left(\frac{1+2r+\sqrt{1+4r}}{2}\right)a_{j-1}a_{j+1} \text{ for } 0 < j < n,$$

but $r \geq 1$, so above inequality is true for $r = 1$ as well

$$\Rightarrow \quad a_j^2 \geq \left(\frac{3+\sqrt{5}}{2}\right)a_{j-1}a_{j+1} = r_\circ a_{k-1}a_{k+1} \tag{25}$$

$$\Rightarrow \quad a_j^2 \geq \left(1+\sqrt{2}\right)a_{j-1}a_{j+1} = r_1a_{j-1}a_{j+1} \tag{26}$$

Also being on the correct side of $\mathscr{H}_\circ$, we have

$$a_\circ \geq \left(\frac{1+\sqrt{1+4r}}{2}\right) x$$
$$\Rightarrow \quad a_\circ^2 \geq \left(\frac{1+\sqrt{1+4r}}{2}\right)^2 x^2$$
$$= \left(\frac{1+2r+\sqrt{1+4r}}{2}\right) a_1$$

also true for r = 1

$$\Rightarrow \quad a_\circ^2 \geq \left(\frac{3+\sqrt{5}}{2}\right) a_1 = r_\circ a_{-1}\, a_1 \qquad (27)$$
$$\Rightarrow \quad a_\circ^2 \geq \left(1+\sqrt{2}\right) a_1 = r_1 a_{-1} a_1 \qquad (28)$$

Odd Case

Being on the correct side of $\mathscr{H}_{\mathbf{n}}$, we have

$$a_n \geq \left(\frac{1+\sqrt{1+4r}}{2}\right) x^{1+d_1+\cdots+d_{n-1}}$$
$$\Rightarrow \quad a_n^2 \geq \left(\frac{1+\sqrt{1+4r}}{2}\right)^2 x^{2+2d_1+\cdots+2d_{n-1}}$$
$$= \left(\frac{1+2r+\sqrt{1+4r}}{2}\right) a_{n-1} a_{n+1}$$

above inequality is true for $r = 1$

$$\Rightarrow \quad a_n^2 \geq \left(\frac{3+\sqrt{5}}{2}\right) a_{n-1}\, a_{n+1} = r_\circ a_{n-1} a_{n+1} \qquad (29)$$
$$\Rightarrow \quad a_n^2 \geq \left(1+\sqrt{2}\right) a_{n-1} a_{n+1} = r_1 a_{n-1} a_{n+1} \qquad (30)$$

Even Case

Being on the correct side of $\mathscr{H}_{\mathbf{n}}$ is equivalent to

$$a_n \geq (1+r)\, x^{1+d_1+\cdots+d_{n-1}} = (1+r)\, a_{n-1} \qquad (31)$$
$$\Rightarrow \quad a_n \geq 2a_{n-1} \qquad (32)$$

Since for $r = 1$, (25), (27), (29) agrees with Lemma 3.1 (i) and (32) with (ii) of McNamara and Sagan [2]. Thus any sequence in $\mathscr{R}_r$ is infinitely log-concave for $r = 1$. Hence Generalized r-factor log-operator $\mathscr{L}_r$ and r-factor hypersurfaces agrees with the results obtained by [2] for $r = 1$. Also (26), (28), (30)and (31) by Lemma 3 proves theorem (2.1) alternatively.□.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] D. Uminsky, K. Yeats, Electronic J. Combin. **14**, 1-13 (2007).
[2] P. R. W. McNamara and B. E. Sagan, Adv. App. Math. **44**, 1-15 (2010).
[3] G. Boros, V. Moll, Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals, Oxford University Press, Cambridge (2004).
[4] M. Kauers,P. Paule, Proc. Amer. Math. Soc. **135**, 3847-3856 (2007).
[5] B. Francesco Contemp. Math. Amer. Math. Soc., **178**, 71-89 (1994).
[6] R. P. Stanley, Ann. New York Acad. Sci., **576**, 500-535 (1986).

Zahid Raza received M.Sc. from the University of Punjab and Ph.D from Abdus Salam School of Mathematical Sciences Government College University Lahore, Pakistan. He joint National University of Computer & Emerging Sciences, Lahore campus as assistant professor in 2010 Now, he is working as associate professor. His research interests include algebraic combinatorics and elementary number theory. Department of Mathematics, National University of Computer & Emerging Sciences, Lahore campus B-Block Faisal Town, Lahore, Pakistan.

Anjum Ali received his MS from National University of Computer & Emerging Sciences. His research interests are computational mathematics and combinatorial number theory. Department of Mathematics, National University of Computer & Emerging Sciences, Lahore B-Block Faisal Town, Lahore, Pakistan.