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Fixed Point Theorems for Expanding Mappings in Dislocated Metric Space

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Abstract: The aim of this paper is to present fixed point theorems in dislocated metric space. We have proved some unique fixed point results for expanding type of continuous self-mapping and surjective expanding self-map in dislocated metric space. A non-unique fixed point theorem has been obtained for Hardy-Rogers type mapping using expanding mapping in dislocated metric space. Examples are given in the support of our constructed results.

Keywords: Complete dislocated metric space, contraction mapping, expanding mapping, fixed point.

1 Introduction

The concept of dislocated metric space was introduced by Hitzler and Seda [1], [2]. In dislocated metric space the self distance of a point need not to be zero necessarily. They also generalized famous Banach contraction principle in dislocated metric space. Dislocated metric space play a vital role in Topology, Logical Programming and Electronic Engineering. Zeyada et al. [3] developed the notion of complete dislocated quasi-metric spaces and generalized the result of Hitzler [1] in dislocated quasi-metric space. Later on many papers have been published containing fixed point results for different type of contractions defined by [4,5] in dislocated quasi-metric spaces (see [6,7,8,9,10]).

In this article, we have proved some unique and non-unique fixed point results for expanding type mapping in dislocated metric space. A non-unique fixed point theorem have been obtained for Hardy-Rogers type mapping using expanding mapping in dislocated metric space.

2 Preliminaries

Throughout this paper $\mathbb{R}^+$ will represent the set of non-negative real numbers.

Definition 2.1. [3]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions

d_1) $d(x,x) = 0$;
d_2) $d(x,y) = d(y,x) = 0$ implies that $x = y$;
d_3) $d(x,y) = d(y,x)$;
d_4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric (dq-metric) on X.

Clearly every metric is a dislocated metric but the converse is not necessarily true as clear form the following example:

Example 2.2. Let X=$\mathbb{R}^+$ define the distance function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = max\{x,y\}$$

clearly d is dislocated metric but not a metric.

Also every metric and dislocated metric is dislocated quasi-metric but the converse is not true as clear from the following example:

Example 2.3. Let $X = \mathbb{R}^+$ we define the function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = |x-y| + |x| \quad \text{for all } x,y \in X$$

evidently d is dq-metric but not a metric nor dislocated metric.

In our main work we will use the following definitions which can be found in [1].

Definition 2.4. A sequence $\{x_n\}$ in d-metric space (X,d)

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is called Cauchy sequence if for $\varepsilon > 0$ there exist a positive integer $n_0 \in N$ such that for $m,n \geq n_0$, we have $d(x_m,x_n) < \varepsilon$.

Definition 2.5. A sequence $\{x_n\}$ is called d-convergent in (X,d) if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

In this case x is called the d-limit of the sequence $\{x_n\}$.

Definition 2.6. A d-metric space (X,d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.7. Let (X,d) be a d-metric space a mapping $T : X \to X$ is called contraction if there exist $0 \leq \alpha < 1$ such that

$$d(Tx,Ty) \leq \alpha d(x,y) \text{ for all } x,y \in X.$$

Theorem 2.8.[12]. Let (X,d) be a complete metric space. $T : X \to X$ be a self-mapping satisfying the condition,

$$d(Tx,Ty) \leq a\cdot d(x,y) + b\cdot d(x,Tx) + c\cdot d(y,Ty) +$$

$$e\cdot d(x,Ty) + f\cdot d(y,Tx)$$

$\forall\ x,y \in X$ and $a,b,c,e,f \geq 0$ with $a+b+c+e+f < 1$. Then T has a unique fixed point.

Theorem 2.9.[6]. Let (X,d) be a complete dislocated quasi-metric space. $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \leq a\cdot d(x,y) + b\cdot d(x,Tx) + c\cdot d(y,Ty)$$

$\forall\ x,y \in X$ and $a,b,c \geq 0$ with $a+b+c < 1$. Then T has a unique fixed point.

Lemma 2.10. [1]. Limit in d-metric space is unique.

Theorem 2.11. [1]. Let (X,d) be a complete d-metric space $T : X \to X$ be a contraction then T has a unique fixed point.

3 Main Results

In this section, we first prove some unique fixed point results satisfying expanding condition by taking the continuity of self-mapping and then considering surjective self-mapping in the context of dislocated metric space.

Theorem 3.1. Let (X,d) be a complete dislocated metric space let $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \geq a\cdot d(x,y) + b\cdot d(x,Tx) + c\cdot d(y,Ty) \quad (1)$$

$\forall\ x,y \in X$ and $a > 1, b \in \mathbb{R}$ and $c \leq 1$ with $a+b+c > 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ in X by the rule

$$x_0 = Tx_1, x_1 = Tx_2, \ldots\ldots\ldots, x_n = Tx_{n+1}.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X, Consider

$$d(x_n,x_{n-1}) = d(Tx_{n+1},Tx_n).$$

Now by (1) and definition of the sequence

$$d(x_n,x_{n-1}) = d(Tx_{n+1},Tx_n) \geq a\cdot d(x_{n+1},x_n)$$

$$+b\cdot d(x_{n+1},Tx_{n+1}) + c\cdot d(x_n,Tx_n)$$

$$d(x_n,x_{n-1}) \geq a\cdot d(x_{n+1},x_n) + b\cdot d(x_{n+1},x_n) + c\cdot d(x_n,x_{n-1}).$$

By use of symmetric property we have

$$d(x_{n-1},x_n) \geq a\cdot d(x_n,x_{n+1}) + b\cdot d(x_n,x_{n+1}) + c\cdot d(x_{n-1},x_n)$$

$$(1-c)d(x_{n-1},x_n) \geq (a+b)\cdot d(x_n,x_{n+1})$$

$$d(x_n,x_{n+1}) \leq \left(\frac{1-c}{a+b}\right) d(x_{n-1},x_n).$$

Let

$$k = \frac{1-c}{a+b} < 1.$$

So the above inequality become

$$d(x_n,x_{n+1}) \leq k\cdot d(x_{n-1},x_n).$$

Also

$$d(x_{n-1},x_n) \leq k\cdot d(x_{n-2},x_{n-1}).$$

So

$$d(x_n,x_{n+1}) \leq k^2\cdot d(x_{n-2},x_{n-1}).$$

Proceeding in similar way we can get

$$d(x_n,x_{n+1}) \leq k^n\cdot d(x_0,x_1).$$

Taking limit $n \to \infty$, as $k < 1$ so $k^n \to 0$ so

$$d(x_n,x_{n+1}) \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete d-metric space. So there must exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now to show that u is a fixed point of T, since T is continuous So,

$$\lim_{n\to\infty} Tx_n = Tu \ \Rightarrow\ \lim_{n\to\infty} x_{n-1} = Tu \ \Rightarrow\ Tu = u.$$

Hence u is the fixed point of T.

Uniqueness. Let u,v are two distinct fixed points of T. Now to show that $(u,u) = d(v,v) = 0$, putting $x = y = u$ in (1), We have

$$d(Tu,Tu) \geq a\cdot d(u,u) + b\cdot d(u,Tu) + c\cdot d(u,Tu)$$

$$d(u,u) \geq (a+b+c)\cdot d(u,u).$$

Since $a+b+c > 1$, so the above inequality is possible if

$$d(u,u) = 0.$$

Similarly we can show that

$$d(v,v) = 0.$$

Now consider

$$d(u,v) = d(Tu,Tv) \geq a \cdot d(u,v) + b \cdot d(u,Tu) + c \cdot d(v,Tv)$$

$$d(u,v) \geq a \cdot d(u,v) + b \cdot d(u,u) + c \cdot d(v,v)$$

$$d(u,v) \geq a \cdot d(u,v).$$

Since $a > 1$ so the above inequality is possible if $d(u,v) = 0$ similarly we can show that $d(v,u) = 0$ which implies that $u = v$. Hence fixed point of T is unique.

Corollary 3.2. Let (X,d) be a complete dislocated metric space. $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \geq a \cdot d(x,y) + \cdot d(x,Tx)$$

$\forall\, x,y \in X$ and $a > 1, b \in \mathbb{R}$ with $a+b > 1$. Then T has a unique fixed point.

Proof. By putting $c = 0$ in Theorem 3.1 we can get the required result easily.

Corollary 3.3. Let (X,d) be a complete dislocated metric space. $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \geq a \cdot d(x,y)$$

$\forall\, x,y \in X$ and $a > 1$. Then T has a unique fixed point.

Proof. Putting $b = c = 0$ in Theorem 3.1 one can get the required result without any difficulty.

Example 3.4. Let $X = \mathbb{R}^+$ the dislocated distance function defined on X is

$$d(x,y) = \max\{x,y\}$$

for all $x,y \in X$ and the continuous function on X is given by $Tx = 2x$ so for $a = 2, b = -1$ and $c = \frac{1}{3}$ all the conditions of Theorem 3.1 are satisfied. Therefore $x = 0$ is the unique fixed point of T.

Example 3.5. Let $X = \mathbb{R}^+$ the dislocated distance function defined on X is

$$d(x,y) = \max\{x,y\}$$

for all $x,y \in X$ and the continuous function on X is given by $Tx = 2x$ so for $a \geq 2$ all the conditions of Corollary 3.2 are satisfied. Therefore $x = 0$ is the unique fixed point of T.

Theorem 3.6. Let (X,d) be a complete dislocated metric space. $T : X \to X$ be a surjective self-mapping satisfying the condition

$$d(Tx,Ty) \geq a \cdot d(x,y) + b \cdot d(x,Tx) + c \cdot d(y,Ty) \quad (2)$$

$\forall\, x,y \in X$ and $a > 1, b \in \mathbb{R}$ and $c \leq 1$ with $a+b+c > 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ in X by the rule

$$x_0 = Tx_1, x_1 = Tx_2, \ldots\ldots\ldots\ldots, x_n = Tx_{n+1}.$$

Proceeding like Theorem **??**, we obtain that $\{x_n\}$ is a Cauchy sequence in complete dislocated metric space. So there must exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now to show that u is the fixed point of T since T is surjective (onto) mapping so for any $p \in X \;\; Tp = u$. Consider

$$d(x_n,u) = d(Tx_{n+1},Tp) \geq a \cdot d(x_{n+1},p) + b \cdot d(x_{n+1},Tx_{n+1})$$
$$+c \cdot d(p,Tp)$$
$$d(x_n,u) \geq a \cdot d(x_{n+1},p) + b \cdot d(x_{n+1},x_n) + c \cdot d(p,u).$$

Taking limit $n \to \infty$ we get

$$0 \geq (a+c) \cdot d(p,u) \;\; \Rightarrow \;\; d(p,u) = 0 \;\; \Rightarrow \;\; p = u.$$

So $Tp = u$ becomes $Tu = u$. Thus u is the fixed point of T.

Uniqueness. Follows from Theorem 3.1.

Example 3.7. Let $X = \mathbb{R}^+$ the dislocated distance function defined on X is

$$d(x,y) = \max\{x,y\}$$

for all $x,y \in X$ and the surjective continuous function on X is given by $Tx = \frac{5}{2}x$ so for $a = 4, b = -2$ and $c = \frac{3}{4}$ all the conditions of Theorem 3.6 are satisfied. Therefore $x = 0$ is the unique fixed point of T.

Our next theorem is about a non-unique fixed point for Hardy-Rogers type mapping satisfying the expanding condition in the context of dislocated metric space.

Theorem 3.8. Let (X,d) be a complete dislocated metric space. $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \geq a \cdot d(x,y) + b \cdot d(x,Tx) + c \cdot d(y,Ty) + e \cdot d(x,Ty) + f \cdot d(y,Tx) \quad (3)$$

$\forall\, x,y \in X$ with $a+b+c > 1$ and $c \leq 1+e+f$. Then T has a fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ in X by the rule

$$x_0 = Tx_1, x_1 = Tx_2, \ldots\ldots\ldots\ldots, x_n = Tx_{n+1}.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X Consider,

$$d(x_n,x_{n-1}) = d(Tx_{n+1},Tx_n).$$

Now by (3) and definition of the sequence we have

$$d(x_n,x_{n-1}) = d(Tx_{n+1},Tx_n) \geq a \cdot d(x_{n+1},x_n) +$$
$$b \cdot d(x_{n+1},Tx_{n+1}) + c \cdot d(x_n,Tx_n) + e \cdot d(x_{n+1},Tx_n) +$$
$$f \cdot d(x_n,Tx_{n+1})$$

$$d(x_n,x_{n-1}) \geq a\cdot d(x_{n+1},x_n)+b\cdot d(x_{n+1},x_n)+$$
$$c\cdot d(x_n,x_{n-1})+e\cdot d(x_{n+1},x_{n-1})+f\cdot d(x_n,x_n).$$

By using symmetric property we have

$$d(x_{n-1},x_n) \geq a\cdot d(x_n,x_{n+1})+b\cdot d(x_n,x_{n+1})+$$
$$c\cdot d(x_{n-1},x_n)+e\cdot d(x_{n-1},x_{n+1})+f\cdot d(x_n,x_n).$$

Since

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)$$
$$d(x_{n-1},x_{n+1}) \geq d(x_n,x_{n+1})+d(x_{n-1},x_n).$$

Using the above technique in 4th and 5th term of above We have

$$(1-c+e+f)\cdot d(x_{n-1},x_n) \geq (a+b+d+f)\cdot d(x_n,x_{n+1})$$

$$d(x_n,x_{n+1}) \leq \left(\frac{1-c+e+f}{a+b+e+f}\cdot\right) d(x_{n-1},x_n).$$

Now using the restrictions on the constants in the theorem We have let

$$k=\frac{1-c+e+f}{a+b+e+f}<1.$$

So the above inequality become

$$d(x_n,x_{n+1}) \leq k\cdot d(x_{n-1},x_n).$$

Also

$$d(x_{n-1},x_n) \leq k\cdot d(x_{n-2},x_{n-1}).$$

So

$$d(x_n,x_{n+1}) \leq k^2\cdot d(x_{n-2},x_{n-1}).$$

Proceeding in similar way we get

$$d(x_n,x_{n+1}) \leq k^n\cdot d(x_0,x_1).$$

Taking limit $n\to\infty$, as $k<1$ so $k^n\to 0$ so

$$d(x_n,x_{n+1}) \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete d-metric space. So there must exists $u\in X$ such that $\lim_{n\to\infty} x_n = u$. Now to show that u is a fixed point of T since T is continuous so

$$\lim_{n\to\infty} Tx_n = Tu \;\Rightarrow\; \lim_{n\to\infty} x_{n-1} = Tu \;\Rightarrow\; Tu = u.$$

Hence u is the fixed point of T.

Corollary 3.9. Let (X,d) be a complete dislocated metric space. $T: X\to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \geq a\cdot d(x,y)+b\cdot d(x,Tx)+c\cdot d(y,Ty)+e\cdot d(x,Ty)$$

$\forall\, x,y\in X$ with $a+b+c>1$ and $c\leq 1+e$. Then T has a fixed point.

Proof. Putting $f=0$ in Theorem 3.8 one can get the required result easily.

Theorem 3.10. Let (X,d) be a complete dislocated metric space. $T: X\to X$ be a surjective self-mapping satisfying the condition,

$$d(Tx,Ty) \geq a\cdot d(x,y)+b\cdot d(x,Tx)+c\cdot d(y,Ty)$$
$$+e\cdot d(x,Ty)+f\cdot d(y,Tx) \quad (4)$$

$\forall\, x,y\in X$ with $a+b+c>1$, $c\leq 1+e+f$ and $a,c,f>0$. Then T has a fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0=Tx_1, x_1=Tx_2, \ldots\ldots\ldots, x_n=Tx_{n+1}.$$

Proceeding like Theorem 3 we obtain that $\{x_n\}$ is a Cauchy sequence in complete dislocated metric space. So there must exists $u\in X$ such that $\lim_{n\to\infty} x_n = u$. Now to show that u is the fixed point of T. Since T is surjective (onto) mapping so for any $p\in X$ $Tp=u$. Consider

$$d(x_n,u)=d(Tx_{n+1},Tp) \geq a\cdot d(x_{n+1},p)+b\cdot d(x_{n+1},Tx_{n+1})$$
$$+c\cdot d(p,Tp)$$
$$+e\cdot d(x_{n+1},Tp)+f\cdot d(p,Tx_{n+1})$$
$$d(x_n,u) \geq a\cdot d(x_{n+1},p)+b\cdot d(x_{n+1},x_n)+c\cdot d(p,u)$$
$$+e\cdot d(x_{n+1},u)+f\cdot d(p,x_n.$$

Taking limit $n\to\infty$ we get

$$0 \geq (a+c+f)\cdot d(u,p).$$

Since $a,c,f>0$ so the above inequality is possible only if

$$d(u,p)=0 \;\Rightarrow\; u=p \;\Rightarrow\; Tu=u.$$

Thus u is the fixed point of T.

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References

[1] P. Hitzler, Generalized metrics and Topology in Logic Programming Semantics, Ph.D Thesis, National Univeristy of Ireland, University College Cork, (2001).
[2] P. Hitzler and A. K. Seda, Dislocated Topologies, *J. Electr. Engin.*, **51**(2000), 3-7.
[3] F. M. Zeyada, G. H. Hassan and M. A. Ahmad, A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space, *Arabian J. Sci. Engg.*, **31**(2005), 111-114.

[4] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expression, *Indian Journal of Pure and Applied Mathematics*, **6**(1975), 1455-1458.
[5] B. E. Rohades, A comparison of various definitions of contractive mappings, *Transfer. Amer. Soc.*, **226**(1977), 257-290.
[6] C. T. Aage and J. N. Salunke, Some results of fixed point theorem in dislocated quasi-metric space, *Bulletin of Marathadawa Mathematical Society*, **9**(2008), 1-5.
[7] C. T. Aage and J. N. Salunke, The results of fixed points in dislocated and dislocated quasi-metric space, *Applied Mathematical Sciences*, **2**(2008), 2941-2948.
[8] A. Isufati, Fixed point theorem in dislocated quasi-metric spaces, *Applied Mathematical Sciences*, **4**(2010), 217-223.
[9] Mujeeb Ur Rahman and M. Sarwar, Fixed point results in dislocated quasi-metric spaces, *International Mathematical Forum*, **9**(2014), 677-682.
[10] M. Sarwar, Mujeeb Ur Rahman and G. Ali , Some fixed point results in dislocated quasi-metric (dq-metric) Spaces, *Journal of Inequalities and Applications*, 2014, 278:2014, 1-11.
[11] Yan Han and Shaoyuan Xu, Some new theorems of expanding mappings without continuity in cone metric space, *Fixed Point Theory and Applications*, **2013**:3, (2013), ISSN 1687-1812.
[12] G. F. Hardy and T. D. Rogers, A generalization of a fixed point theorem, *Reich Cand. Math. Bull.*, **16**(1973), 201-206.
[13] A. S. Saluja, A. K. Dhakda and D. Magarde, Some fixed point theorems for expansive type mapping in dislocated metric space, *Mathematical Theorey and Modeling*, **3**(2013), 12-15.

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