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On the Mixture of Burr XII and Weibull Distributions

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Abstract: The burr XII and weibull distributions are the most important and the most widely used distributions for the life time distribution as well as the wealth distribution. Burr XII distribution is mainly used to explain the allocation of wealth and wages among the people of the particular society. And weibull distribution is mostly used in the extreme value theory, failure analysis and the reliability engineering. In this paper, the classical properties of the mixture burr XII weibull distribution have been discussed. The estimation of the parameters and the construction of the information matrix are introduced. The var-cov matrix and the interval estimation of the parameters are also proposed in the paper. Cumulative distribution function, hazard rate, failure rate, inverse hazard function, odd function and the cumulative hazard function, rth moment, moment generating function characteristic function, moments, mean and variance, Renyi and Beta entropies, mean deviation from mean and mean time between failures(MTBF)have also been discussed.

Keywords: weibull distribution, burr XII, mixture distribution, Renyi and beta entropy, MTBF

1. INTRODUCTION

Mixture distributions are being extensively used by many researchers for the discussion, survey estimation and the application. In the paper [9] it was proposed that the inverse weibull mixture models having the weights in negative form can show the outcome of the system under the certain kind of the conditions. From [8] it was also provided with the discussion about the property of aging of the failure rate models having one mode which also includes the inverse weibull distribution. Identifiability provides the different demonstration for the mixture distributions. If we want to organize future data into different groups of the mixture distributions then the lack of identifiability is a severe kind of difficulty. Identifiability of mixtures was completely discussed by different researchers which includes these papers [14, 17,3].In [13] it was discussed that the mixture of the two inverse weibull distributions and discussed in detail the properties and the estimation of the parameters. The technique of the EM algorithm in the Monte Carlo estimation was used for the calculation of the estimates of the five parameters. In his study, the behavior of the median and mode was examined using different values of the parameters. Also the behavior of the failure rate was examined using the graphs. In the empirical financial data the use of the mixture normal distribution is well recognized. In [10] it was proposed about the mixture of the normal distribution to have room for the skewness and the non normality of the financial time series data. The study was conducted on the Bursa Malaysia stock market financial data. The estimation was performed by using the formal maximum likelihood method through EM algorithm. Numerous studies have been conducted on modeling the assets return by using the mixture of the normal. In the work mentioned in [11] it was proposed first time the utilization of the mixture of the normal distribution for handling the heavy tail data. In the research paper of [7] it was proposed in his study that when all the regimes have the same mean, the mixture of the normal distribution is leptokurtic. The normal mixtures are flexible to accommodate different forms of continuous distributions and can capture the characteristics of the leptokurtic and multimodal deviation of financial time series data. Paper mentioned in [15] proposed in his study that it is possible to handle the appropriate mixture of distributions by a variety of techniques, and this includes graphical methods, maximum likelihood, the method of moments, and

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the Bayesian approach. Newly developments have been achieved in fitting the mixture models by the maximum likelihood estimation technique and because of the presence of the associated statistical theory this method has become the first preference. In the research paper mentioned [5] it was proposed for the first time that the important characteristic of the EM algorithm which is being utilized in the context of the mixture models as the useful tool for making the maximum likelihood problems simplified.

2. BURR DISTRIBUTION

Burr distribution is the renowned distribution in the probability. It was first developed by [4]. mathematical properties and computational methods of estimating the maximum likelihood to the life time censored data was proposed. It is commonly used for the modeling of the income data.

2.1. BURR XII DISTRIBUTION

[18] proposed the three parameter Burr XII distribution having the following cumulative distribution function and the probability density function for $x>0$

$$F(x;s,k,c)=1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k} \qquad k,c,s>0 \qquad (2.1.1)$$

And $$f(x;s,k,c)=cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1} \qquad k,c,s>0 \qquad (2.1.2)$$

respectively, where k and c are the shape parameters and s is the scale parameter. If we put $c=1$ then the density function will become unimodal.

Burr distribution is the heavy tailed distribution which means that these distributions have more heavy tails as compared to the exponential distribution.

3. WEIBULL DISTRIBUTION

Weibull distribution is the continuous type of probability distribution which is named after Waloodi Weibull. It was proposed by [10] and it was practically applied by [11] to explain the particle size distribution.

The following is the cumulative distribution function along with the probability density function of the weibull distribution for the random variable $X \geq 0$

$$F(x;\alpha,\beta)=1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \qquad \alpha,\beta>0 \qquad (3.1)$$

And $$f(x;\alpha,\beta)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \qquad \alpha,\beta>0 \qquad (3.2)$$

where α and β are the shape and scale parameters respectively of the distribution. The weibull distribution is also regarded as the heavy tailed distribution. It is widely used in reliability engineering and the failure analysis.

4. Probability Density Function

The probability density function of the mixture of burr XII weibull distribution has the following form

$$f(x;k,s,c,\alpha,\beta)=p_1 f_1(x;k,s,c)+p_2 f_2(x;\alpha,\beta) \qquad (4.1)$$

where p_1 and p_2 are the mixing proportions and $p_1+p_2=1$

$f_1(x;k,s,c)$ is the pdf of the Burr XII distribution and $f_2(x;\alpha,\beta)$ is the pdf of the weibull distribution which

So the mixture of the these probability densities is given as

$$f(x;k,s,c,\alpha,\beta)=p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha} \quad (4.2)$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

4.1. Special cases of Mixture of Burr XII Weibull distribution

If we set $\alpha=1$ in the expression of (4.2), it will become mixture of Burr XII exponential distribution as follow

$$f(x;k,s,c,\alpha,\beta)=p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{1}{\beta}e^{-\left(\frac{x}{\beta}\right)} \quad (4.1.1)$$

and if we set $\alpha=2$ in the expression (4.2), it will become the mixture of Burr XII Rayleigh distribution as

$$f(x;k,s,c,\alpha,\beta)=p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{2}{\beta^2}xe^{-\left(\frac{x}{\beta}\right)^2} \quad (4.1.2)$$

So mixture of Burr XII exponential and Rayleigh distributions are the special cases of mixture of Burr XII Weibull distribution

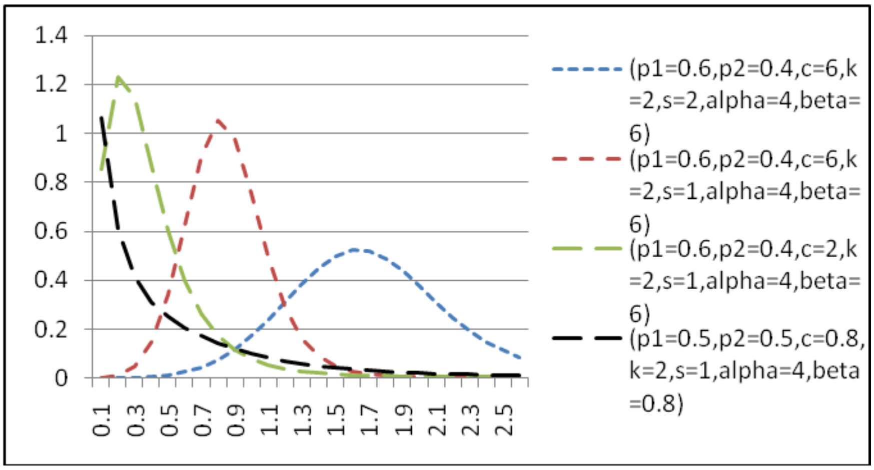

Figure 4.1Graph for the Probability density of the mixture of Burr XII Weibull Distribution for p1=0.6, p$_2$=0.4, c=0.8, 2 and 6, k=2, s=1 and 2, α =4, β =0.8 and 6

From the figure 4.1 it can be seen that if we set the parameter values of p1=0.6,p2=0.4,c=6,k=2,s=2,alpha=4,beta=6, the distribution of mixture of Burr XII Weibull resembles the normal distribution. With p1=0.6,p2=0.4,c=6,k=2,s=1,alpha=4,beta=6, the shape of the mixture of Burr XII Weibull distribution approximately resembles to the Weibull distribution with parameters alpha = 1 and beta=5. It can been observed if the parameter values are set as p1=0.5,p2=0.5,c=0.8,k=2,s=1,alpha=4,beta=0.8 the shape of the mixture distribution takes the shape of two parameter burr distribution with c= 0.5 and k=2.

5. Area under the Curve

Since Burr XII and Weibull distribution are the complete probability density function mentioned in the literature so their mixture will also be the complete pdf

6. Cumulative Distribution Function

The cumulative distribution function for the mixture of burr XII weibull distribution can be expressed in the following form

$$F(x;k,s,c,\alpha,\beta)=p_1 F_1(x;k,s,c)+p_2F_2(x;\alpha,\beta) \quad (6.1)$$

Where $F_1(x;k,s,c)$ is the cdf of the Bur XII distribution and has the following form

$$F_1(x;k,s,c)=1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k} \qquad x>0\ ,\ k,c,s>0 \quad (6.2)$$

And $F_2(x;\alpha,\beta)$ is the cdf of the Weibull distribution and can be expressed in the following form

$$F_2(x;\alpha,\beta)=1-e^{-\left(\frac{x}{\beta}\right)^\alpha} \qquad x>0\ ,\ \alpha,\beta>0 \quad (6.3)$$

So the expression (4.1) becomes

$$F(x;k,s,c,\alpha,\beta)=p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right] \quad (6.4)$$

Where $x>0$ and $k,s,c,\alpha,\beta>0,\ p_1+p_2=1$

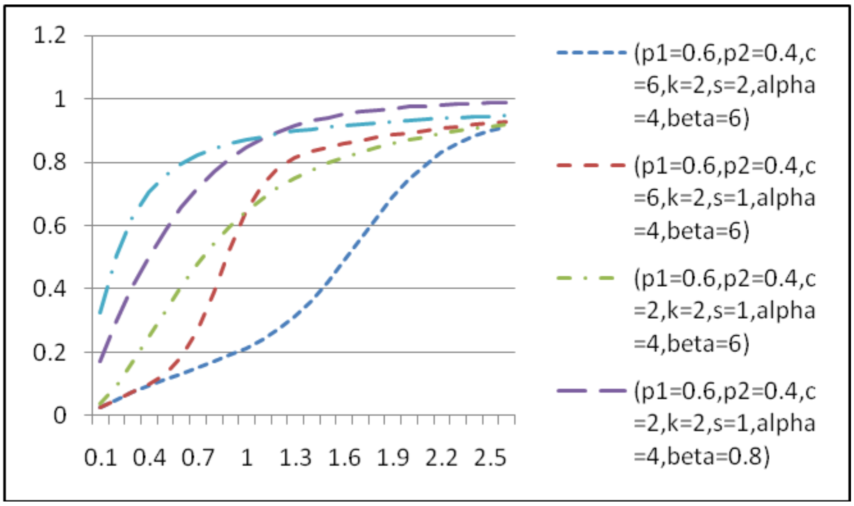

Figure 4.2. Graph for the Cumulative Distribution Function of the mixture of burr XII and Weibull distributions for p1=0.5 and 0.6, p_2=0.4 and 0.5, c=0.8, 2 and 6, k=2, s=1 and 2, α =4, β =0.8 and 6

From the figure 4.2, it can be observed that cumulative distribution function is showing the increasing trend as the time increases. It also depicts that as the value of shape parameter c increases, it is similar to the cdf of the weibull distribution.

7. Reliability Function

The reliability function also known as the survival function is the characteristic of the random variable which is associated with the failure of some system within a given time. It defines the probability that the system will continue to survive beyond the specific time. It is defined as

$$R(x)=1-F(x) \quad (7.1)$$

The reliability or the survival function for the mixture of Burr XII weibull distribution is expressed in the following form by putting (6.4) in (7.1)

$$R(x)=1-\left(p_1\left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)+p_2\left\{1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right\}\right) \tag{7.2}$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

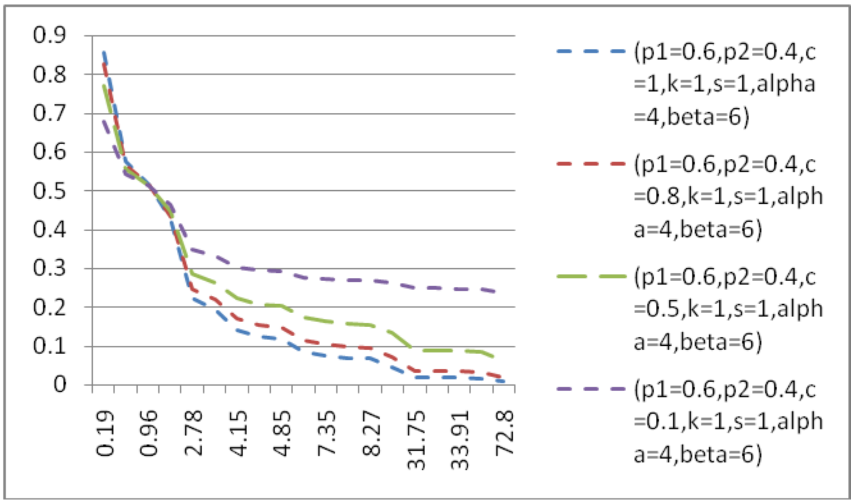

Figure 7.1 Graph for the Reliability Function of the mixture of burr XII and Weibull distributions for p1=0.5 and 0.6, p2=0.4 and 0.5, c=0.8, 2 and 6, k=2, s=1 and 2, α =4, β =0.8 and 6

From the figure 7.1, it can be observed that if we decrease the parametric value of s, the probability of survival has also the decreasing trend. By decreasing the shape parametric value c, the probability of survival provides the decreasing trend

8. Hazard Function

Hazard function is defined as the ratio of the probability density function and reliability function and is expressed in the following form

$$h(x)=\frac{f(x)}{R(x)} \tag{8.1}$$

The hazard rate for the mixture distribution can be obtained by putting (4.2) and (7.2) in (8.1) and can be defined in the following form

$$h(x)=\frac{p_1cks^{-c}x^{c-1}1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}}{1-\left(p_1\left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)+p_2\left(1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)\right)} \tag{8.2}$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

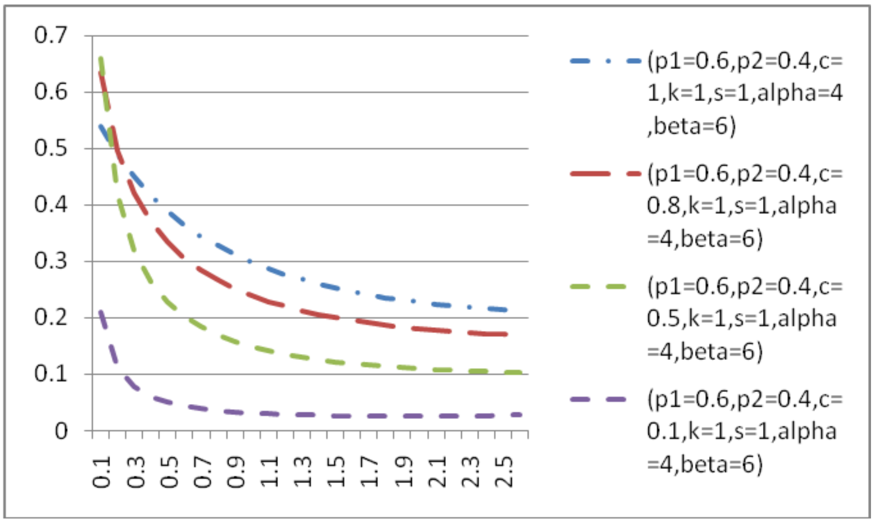

Figure 8.1 Graph for the Hazard Function of the mixture of burr XII and Weibull distributions for p1= 0.6, p2=0.4, c=0.1, 0.5, 0.8 and 1, k=1, s=1, α =4, β = 6

9. Cumulative Hazard Function

The cumulative hazard function can be defined as

$$\wedge(x) = -\log R(x) \tag{9.1}$$

The cumulative hazard function for the mixture distribution can be obtained by putting (7.2) in (9.1) and takes the following form

$$\wedge(x) = -\log\left[1-\left(p_1\left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)+p_2\left(1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)\right)\right] \tag{9.2}$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

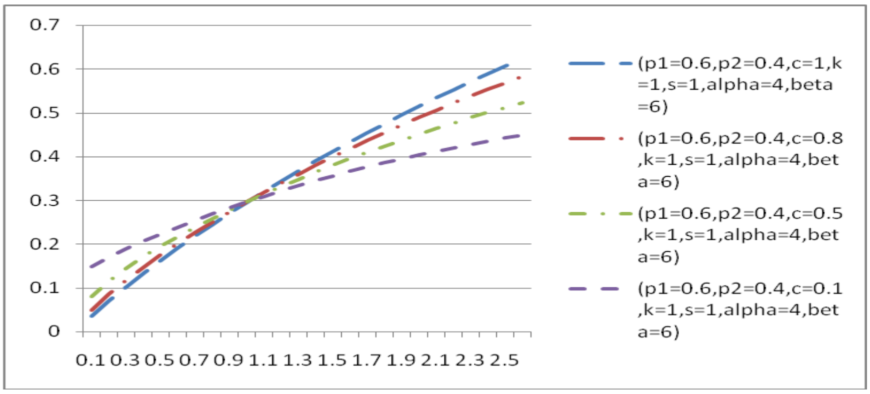

Figure 9.1 Graph for the Cumulative Hazard Function of the mixture of burr XII and Weibull distributions for p1= 0.6, p2=0.4, c=0.1, 0.5, 0.8 and 1, k=1, s=1, α =4, β = 6 10.

Figure 9.1 shows the increasing pattern of the cumulative hazard rate as time increases. The graph also shows that as the value of shape parameter c increases the cumulative hazard rate also increases. Therefore there exists the direct relationship between shape parameter c and cumulative hazard function

10. Reversed Hazard Function

Reversed hazard rate can be defined as the ratio of the probability density function and the cumulative distribution function i.e

$$r(x)=\frac{f(x)}{F(x)} \qquad (10.1)$$

The reversed hazard rate for the mixture distribution can be obtained by putting (2.2) and (4.4) in (8.1) and has the following form

$$r(x)=\frac{p_1 cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}}{p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]} \qquad (10.2)$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

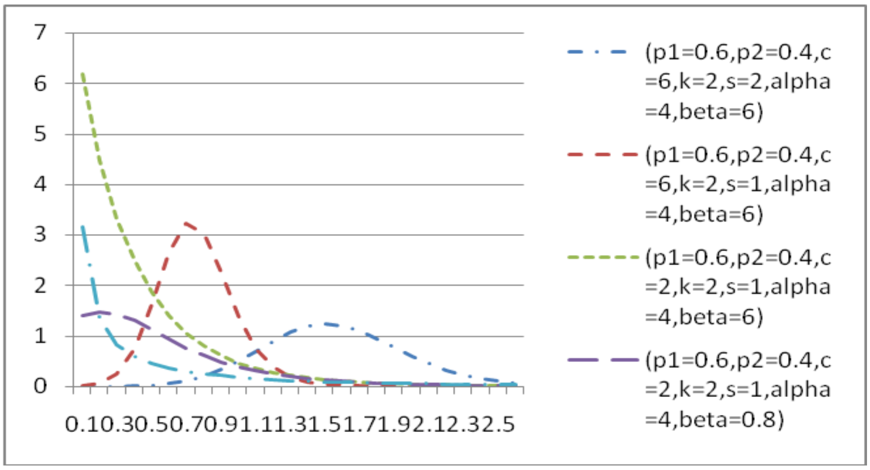

Figure 10.1 Graph for the Reversed Hazard Function of the mixture of burr XII and Weibull distributions for p1=0.5 and 0.6, p2=0.4 and 0.5, c=0.8, 2 and 6, k=2, s=1 and 2, α =4, β =0.8 and 6

11. Odds Function

The odds function denoted by $O(x)$ is the ratio of cumulative distribution function and the reliability function and has the following form:

$$O(x)=\frac{F(x)}{R(x)} \qquad (11.1)$$

The odds function for the mixture distribution can be obtained by putting (6.4) and (7.2) in (11.1) and following expression is obtained.

$$O(x)=\frac{p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]}{1-\left(p_1\left(1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right)+p_2\left\{1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right\}\right)} \quad (11.2)$$

Where $x>0$ and $k,s,c,\alpha,\beta>0$, $p_1+p_2=1$

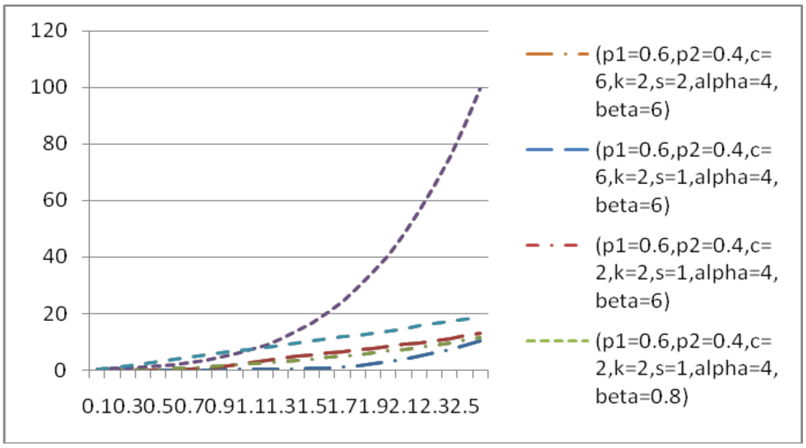

Figure 11.1 Graph for the Odd Function of the mixture of burr XII and Weibull distributions for p1=0.5 and 0.6, p2=0.4 and 0.5, c=0.8, 2 and 6, k=2, s=1 and 2, α =4, β =0.8 and 6

12. rth Moment about Origin

rth moment for the real valued function can be defined as

$$\mu_r'=E\left(x^r\right)$$

$$\mu_r'=\int x^r f(x)dx$$

$$\mu_r'=ks\int_0^\infty x^r\left(p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)dx \quad (12.1)$$

$$\mu_r'=p_1s^r\frac{\Gamma\left(k-\frac{r}{c}\right)\Gamma\left(\frac{r}{c}+1\right)}{\Gamma k}+p_2\beta^r\Gamma\left(1+\frac{r}{\alpha}\right) \quad (12.2)$$

13. Raw Moments about Origin

Putting r = 1, 2, 3 and 4 in (12.2) first four raw moments are

$$\mu_1' = p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right) = Mean \tag{13.1}$$

$$\mu_2' = p_1 s^2 \frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma(k)} + p_2\beta^2\,\Gamma\left(1+\frac{2}{\alpha}\right) \tag{13.2}$$

$$\mu_3' = p_1 s^3 \frac{\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}+1\right)}{\Gamma(k)} + p_2\beta^3\,\Gamma\left(1+\frac{3}{\alpha}\right) \tag{13.3}$$

$$\mu_4' = p_1 s^4 \frac{\Gamma\left(k-\frac{4}{c}\right)\Gamma\left(\frac{4}{c}+1\right)}{\Gamma(k)} + p_2\beta^4\,\Gamma\left(1+\frac{4}{\alpha}\right) \tag{13.4}$$

14. Moments about Mean

$$\mu_1 = 0 \tag{14.1}$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = Variance$$

$$\mu_2 = p_1 s^2 \frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma(k)} + p_2\beta^2\,\Gamma\left(1+\frac{2}{\alpha}\right) - \left(p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right) \right) = \text{Variance} \tag{14.2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + (\mu_1')^3$$

$$\mu_3 = p_1 s^3 \frac{\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}+1\right)}{\Gamma(k)} + p_2\beta^3\,\Gamma\left(1+\frac{3}{\alpha}\right) - 3\left[p_1 s^2 \frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma(k)} + p_2\beta^2\,\Gamma\left(1+\frac{2}{\alpha}\right) \right]\left[p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right) \right]$$

$$+\left[p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right) \right]^3 \tag{14.3}$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4 \tag{14.4}$$

$$\mu_4 = p_1 s^4 \frac{\Gamma\left(k-\frac{4}{c}\right)\Gamma\left(\frac{4}{c}+1\right)}{\Gamma(k)} + p_2\beta^4\,\Gamma\left(1+\frac{4}{\alpha}\right) - 4\left(p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right)\right)\left(p_1 s^3 \frac{\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}+1\right)}{\Gamma(k)} + p_2\beta^3\,\Gamma\left(1+\frac{3}{\alpha}\right)\right)$$
$$+6\left(p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right)\right)^2\left(p_1 s^2 \frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma(k)} + p_2\beta^2\,\Gamma\left(1+\frac{2}{\alpha}\right)\right)$$
$$-3\left(p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma(k)} + p_2\beta\,\Gamma\left(1+\frac{1}{\alpha}\right)\right)^4 \quad (14.5)$$

15. Measure of Skewness and Kurtosis

Skewness is the measure of extent that the distribution leans to the one side of the mean and kurtosis is used to measure the flatness or peakedness of the probability curve.

Skewness and kurtosis are denoted by β_1 and β_2 respectively and is expressed in the following form

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \quad (15.1)$$

And

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad (15.2)$$

putting (14.3) and (14.2) in the (15.1) the expression of β_1 can be obtained and by putting (12.5) and (14.2) in (15.2) the expression for the β_2 can be obtained.

16. Moment Generating Function

The moment generating function can be defined as

$$M_x(t) = E(e^{tx})$$

Where $E(e^{tx}) = \int e^{tx} f(x)\,dx$ (16.1)

Putting the equation (2.2) in (16.1) we get

$$E(e^{tx}) = \int_0^\infty e^{tx}\left(p_1 c k s^{-c} x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1} + p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)dx \quad (16.2)$$

$$E(e^{tx}) = p_1 \sum_{j=0}^{\infty} \frac{(t\,s)^j}{\lfloor j} \frac{\Gamma\left(k-\frac{j}{c}\right)\Gamma\left(\frac{j}{c}+1\right)}{\Gamma(k)} + p_2 \sum_{j=0}^{\infty} \frac{(t\,\beta)^j}{\lfloor j} \Gamma\left(1+\frac{j}{\alpha}\right) \qquad (16.3)$$

17. Characteristic Function

Characteristic function is obtained by replacing t in moment generating function with it and is expressed as below

$$\varphi_x(t) = E\left(e^{itx}\right) \qquad (17.1)$$

Its expression can be obtained by replacing t with it in the expression of moment generating function of the mixture distribution in (16.3) which is

$$E(e^{itx}) = p_1 \sum_{j=0}^{\infty} \frac{(it\,s)^j}{\lfloor j} \frac{\Gamma\left(k-\frac{j}{c}\right)\Gamma\left(\frac{j}{c}+1\right)}{\Gamma(k)} + p_2 \sum_{j=0}^{\infty} \frac{(it\,\beta)^j}{\lfloor j} \Gamma\left(1+\frac{j}{\alpha}\right) \qquad (17.2)$$

18. Renyi Entropy

The measure of variation of the uncertain situations for the random variable X can be calculated by Renyi entropy [24] which can be expressed by the following relation

$$\delta_r(x) = \frac{1}{1-r} \log\left\{\int f^r(x)dx\right\} \qquad (18.1)$$

Where $f^r(x) = \left[p_1 cks^{-c} x^{c-1} \left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k-1} + p_2 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \right]^r$

By using binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The integral will take the form

$$= \sum_{n=0}^{r} \binom{r}{n} \int_0^{\infty} \left[p_1 cks^{-c} x^{c-1} \left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k-1} \right]^{r-n} \left[p_2 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \right]^n dx \qquad (18.2)$$

Since they are independent, the integral can be regarded as the product of the two integrals and the final expression for the Reyni entropy is given below

$$\delta_r(x)=\frac{1}{1-r}\log\left[\frac{\left(p_1cks^{-c}\right)^{r-n}(s)^{(r-n)(c-1)+1}}{c}\frac{\Gamma\left(\frac{(r-n)(c-1)+1}{c}\right)\Gamma\left(\frac{crk+r-n-1}{c}\right)}{\Gamma\left(\frac{(r-n)(c-1)+1}{c}+\frac{crk+r-n-1}{c}\right)}+\frac{p_2{}^{n}\alpha^{n-1}\beta^{1-n}}{n}\Gamma\left(\frac{1-n}{n\alpha}+1\right)\right] \tag{18.3}$$

19. β - Entropy

β -entropy is defined as one parameter generalization of the Shannon entropy. β -entropy can be defined as

$$H_{\bar\beta}=\frac{1}{\bar\beta-1}\left[1-\int_0^\infty f^{\bar\beta}(x)\right]dx,\qquad for\ \bar\beta\neq 1 \tag{19.1}$$

β -entropy for the mixture distribution is given by

$$H_{\bar\beta}=\frac{1}{\bar\beta-1}\int_0^\infty\left[p_1cks^{-c}x^{c-1}\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{\bar\beta}dx$$

By using binomial expansion

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

The integral will take the form

$$=\sum_{n=0}^{\bar\beta}\binom{\bar\beta}{n}\int_0^\infty\left[p_1cks^{-c}x^{c-1}\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k-1}\right]^{\bar\beta-n}\left[p_2\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^n dx \tag{19.2}$$

Since they are independent, the integral can be regarded the product of the two integrals and the expression for the β -entropy is as follow

$$H_{\bar\beta}=\frac{1}{\bar\beta-1}\left[\frac{\left(p_1cks^{-c}\right)^{\bar\beta-n}(s)^{(\bar\beta-n)(c-1)+1}}{c}\frac{\Gamma\left(\frac{(\bar\beta-n)(c-1)+1}{c}\right)\Gamma\left(\frac{c\bar\beta k+\bar\beta-n-1}{c}\right)}{\Gamma\left(\frac{(\bar\beta-n)(c-1)+1}{c}+\frac{c\bar\beta k+\bar\beta-n-1}{c}\right)}+\frac{p_2{}^{n}\alpha^{n-1}\beta^{1-n}}{n}\Gamma\left(\frac{1-n}{n\alpha}+1\right)\right] \tag{19.3}$$

20. Mean deviation from Mean

The amount of spread in the population is measured to some extent by the help of mean deviation from mean. It is called mean deviation from mean.

$$\partial_1(x)=\int_0^\infty |x-\mu|\, f(x)\,dx \tag{20.1}$$

Where μ = E(x). It can be calculated by the following relationship

$$\partial_1(x)=\int_0^\mu (\mu-x)\, f(x)\,dx+2\int_\mu^\infty (x-\mu)\, f(x)\,dx$$

$$\partial_1(x)=2\int_0^\mu (\mu-x)\, f(x)\,dx$$

$$=2\left\{\mu F(\mu)-\int_0^\mu x\, f(x)dx\right\} \tag{20.2}$$

Solving the integrals , the final expression is as follow

$$\partial_1(x)=2\left[\begin{array}{l}\left(p_1 s\dfrac{\left\lceil k-\frac{1}{c}\right.\left\lceil \frac{1}{c}+1\right.}{\lceil k} + p_2\beta\left\lceil 1+\dfrac{1}{\alpha}\right.\right)\left[p_1\left[1-\left\{1+\left(\dfrac{\mu}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-e^{-\left(\frac{\mu}{\beta}\right)^\alpha}\right]\right] \\ -\left[\dfrac{p_1 ks\sum_{j=0}^{\infty}(-1)^j\binom{k+1}{j}}{\frac{1}{c}+j+1}\left\{\left(\dfrac{\mu}{s}\right)^{\frac{j+1}{c}+1}\right\}+\dfrac{p_2\beta\sum_{i=0}^{\infty}(-1)^i}{\frac{1}{\alpha}+i+1(\lfloor i)}\left\{\left(\dfrac{\mu}{\beta}\right)^{\frac{i+1}{\alpha}+1}\right\}\right]\end{array}\right] \tag{20.3}$$

21. Mean Time between Failures

Mean time between failures denoted by MTBF is regarded as the predicted elapsed time between failures of the process during operation. It can be calculated as the average time between the failures of process. Statistically it can be calculated as

$$\text{MTBF} = \int_0^\infty t\, f(t)\,dt$$

For the proposed mixture model it can be calculated as

$$\text{MTBF} = \int_0^\infty t\left[p_1 cks^{-c}t^{c-1}\left[1+\left(\frac{t}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\beta^\alpha}t^{\alpha-1}e^{-\left(\frac{t}{\beta}\right)^\alpha}\right]dt \tag{21.1}$$

$$\text{MTBF} = p_1 s\frac{\left\lceil k-\frac{1}{c}\right.\left\lceil \frac{1}{c}+1\right.}{\lceil k} + p_2\beta\left\lceil 1+\frac{1}{\alpha}\right. \tag{21.2}$$

22. Bonferoni and Lorenz Curve

The Bonferroni and Lorenz curves have wide applications not only in the field to the study income and poverty, but also in various fields like reliability, insurance and medicine.

It is calculated by the following expression

$$B_F[F(x)] = \frac{1}{\mu F(x)} \int_0^x u\, f(u)\, du \tag{22.1}$$

Where

$$\int_0^x u\, f(u)\, du = \int_0^x u \left[p_1 c k s^{-c} u^{c-1} \left[1 + \left(\frac{u}{s} \right)^c \right]^{-k-1} + p_2 \frac{\alpha}{\beta^\alpha} u^{\alpha-1} e^{-\left(\frac{u}{\beta}\right)^\alpha} \right] du \tag{22.2}$$

The expression will take the following final form expression for Bonferoni curve

$$B_F[F(x)] = \frac{\dfrac{p_1 k s \sum_{j=0}^{\infty} (-1)^j \binom{k+1}{j}}{\frac{1}{c} + j + 1} \left[\left(\frac{x}{s} \right)^{\frac{1}{c}+j+1} \right] + \dfrac{p_2 \beta \sum_{i=0}^{\infty} (-1)^i}{\left(\frac{1}{\alpha} + i + 1\right)\left(\lfloor i\right)} \left[\left(\frac{x}{\beta} \right)^{\frac{1}{\alpha}+i+1} \right]}{\left(p_1 s \dfrac{\left\lceil k - \frac{1}{c} \right. \left\lceil \frac{1}{c} + 1 \right.}{\lceil k} + p_2 \beta \left\lceil 1 + \frac{1}{\alpha} \right. \right) \left(p_1 \left[1 - \left\{ 1 + \left(\frac{x}{s} \right)^c \right\}^{-k} \right] + p_2 \left[1 - e^{-\left(\frac{x}{\beta}\right)^\alpha} \right] \right)} \tag{22.3}$$

where $x > 0$ and $k, s, c, \alpha, \beta > 0$, $p_1 + p_2 = 1$

Lorenz curve is used in economics as an inequality measure for the wealth and size. It can be expressed as

$$L(z) = \frac{\int_0^z x\, f(x) dx}{\mu} \tag{22.4}$$

Where

$$\int_0^z x\, f(x) dx = \int_0^z x \left[p_1 c k s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s} \right)^c \right]^{-k-1} + p_2 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \right] dx \tag{22.5}$$

The final expression for Lorenz Curve is

$$L(z) = \frac{\dfrac{p_1 k s \sum_{j=0}^{\infty} (-1)^j \binom{k+1}{j}}{\frac{1}{c} + j + 1} \left[\left(\frac{z}{s} \right)^{\frac{1}{c}+j+1} \right] + \dfrac{p_2 \beta \sum_{i=0}^{\infty} (-1)^i}{\left(\frac{1}{\alpha} + i + 1\right)\left(\lfloor i\right)} \left[\left(\frac{z}{\beta} \right)^{\frac{1}{\alpha}+i+1} \right]}{p_1 s \dfrac{\left\lceil k - \frac{1}{c} \right. \left\lceil \frac{1}{c} + 1 \right.}{\lceil k} + p_2 \beta \left\lceil 1 + \frac{1}{\alpha} \right.} \tag{22.6}$$

23. Maximum Likelihood Estimation (MLE)

The likelihood function is expressed as below

$$L(\underline{X})=\prod_{i=1}^{n}\left(p_1 cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}+p_2\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)$$

The log likelihood function is given as

$$\ln L(\underline{X})=\sum_{i=1}^{n}\ln\left(p_1 cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}+p_2\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right) \quad (23.1)$$

Partially differentiating the equation (23.1) with respect to $k, s, c, \alpha\, and\, \beta$ the estimates can be obtained but the most suitable technique for estimating the parameters is EM algorithm.

The MLE for $\underline{\theta}$ can be obtained by solving the above mentioned system of nonlinear equations $I(\underline{\theta})=0$. The solution of these nonlinear equations is not in closed form. For testing of hypothesis and the estimation of the confidence interval on the parameters of the model, the information matrix is required. The Fisher (1921) information matrix for mixture distribution can be formed

$$I(\underline{\theta})=\begin{vmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} \\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} \\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} \\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} \\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} \end{vmatrix} \quad (23.2)$$

Where

$$I_{11}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k^2}\right] \quad I_{12}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\partial s}\right] \quad I_{13}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\partial c}\right] \quad I_{14}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\partial \alpha}\right]$$

$$I_{15}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\partial \beta}\right] \quad I_{22}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s^2}\right] \quad I_{23}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\partial c}\right] \quad I_{24}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\partial \alpha}\right]$$

$$I_{25}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\partial \beta}\right] \quad I_{33}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c^2}\right] \quad I_{34}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c\partial \alpha}\right] \quad I_{35}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c\partial \beta}\right]$$

$$I_{44}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \alpha^2}\right] \quad I_{45}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \alpha\partial \beta}\right] \quad I_{55}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \beta^2}\right]$$

And Var-Cov matrix can be obtained as follow

$$V=I^{-1}(\underline{\theta})$$

(23.3)

$$N_5(\underline{\theta}, J(\underline{\theta})^{-1})$$

Since MLE's are asymptotically normal so

$$\hat{\theta}_i \sim N(\theta_i, V_i)$$

100 (1- α) % confidence interval for θ_i is $\hat{\theta}_i \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_i}$ Applying the usually large sample approximation that is MLE of $\underline{\theta}$ which can be treated as approximately $N_5(\underline{\theta}, J(\underline{\theta})^{-1})$ where $J(\underline{\theta})^{-1} = E(\underline{\theta})$.Under all the conditions when they are satisfied for parameter the asymptotic distribution $\sqrt{n}(\hat{\underline{\theta}} - \underline{\theta})$ is $N_5(0, J(\underline{\theta})^{-1})$ where $J(\underline{\theta})^{-1} = \lim_{n\to\infty} n^{-1}(I_n(\underline{\theta})$ is the unit information matrix.

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