Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.12785/jsapl/020102

Martingale Method for Ruin Probabilityin a Generalized Risk Process under Rates of Interest with Homogenous Markov Chain Premiums and Homogenous Markov Chain Interests

*Phung Duy Quang**

Department of Mathematics, Foreign Trade University, 91- Chua Lang, Ha noi, Viet Nam

Received: 16 May 2014, Revised: 1 Jul. 2014, Accepted: 7 Jul. 2014
Published online: 1 Jan. 2015

Abstract: This paper gives upper bounds for ruin probabilities of generalized risk processes under rates of interest with homogenous Markov chain premiums and Hemogenenous Markov chain Interests. We assume that premium and rate of interest take a countable number of non-negative values. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by the Martingale approach.

Keywords: Supermartingale, Optional stopping theorem, Ruin probability, Homogeneous Markov chain.

1 Introduction

In recent years, the classical risk process has been extended to more practical and real situations. For most of the investigations treated in risk theory, it is very significant to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital. Teugels and Sundt [9,10] considered the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang [12] built both exponential and non exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu and Wang [11] given upper bounds for ruin probabilities in a risk model with interest force and independent premiums and claims with Markov chain interest rate. Cai [1] studied the ruin probabilities in two risk models, with independent premiums and claims with the interest rates is formed a sequence of i.i.d random variables. In Cai [2], the author studied the ruin probabilities in two risk models, with independent premiums and claims and used a first order autoregressive process to model the rates of in interest. In Cai and Dickson [3], the authors given Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims. Fenglong Guo and Dingcheng Wang [4] used recursive technique to build Lundberg inequalities for ruin probabilities in two discrete- time risk process with the premiums, claims and rates of interest have autoregressive oving average (ARMA) dependent structures simultaneously. P. D. Quang [5] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and premiums, Markov chain claims. P. D. Quang [6] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates, Markov chain claims and Markov chain premiums. P. D. Quang [7] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent premiums, Markov chain claims and Markov chain interests. P. D. Quang [8] also used recursive approach to build upper bounds for ruin probabilities in a risk model with interest force and Markov chain premiums, Markov chain claims, while the interest rates follow a first-order autoregressive processes.

In this paper, we study the models considered by Cai and Dickson [3] to the case homogenous markov chain premiums, homogenous markov chain interests and independent claims. The main difference between the model in our paper and the

* Corresponding author e-mail: quangmathftu@yahoo.com

one in Cai and Dickson [3] is that premiums, interests in our model are assumed to follow homogeneous Markov chains. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by the Martingale approach.

2 The Model and the Basic Assumptions

In this paper, we study the discrete time risk models with $X=\{X_n\}_{n\geq 0}$ are premiums, $Y=\{Y_n\}_{n\geq 0}$ are claims, $I=\{I_n\}_{n\geq 0}$ are interests and X,Y and I are assumed to be independent. To establish probability inequalities for ruin probabilities of these models, we consider two style of premium collections. On one hand of the premiums are collected at the begining of each period then the surplus process $\left\{U_n^{(1)}\right\}_{n\geq 1}$ with initial surplus $U_o^{(1)}=u>0$ can be written as

$$U_n^{(1)}=U_{n-1}^{(1)}(1+I_n)+X_n-Y_n, \tag{1}$$

which can be rearranged as

$$U_n^{(1)}=u.\prod_{k=1}^{n}(1+I_k)+\sum_{k=1}^{n}(X_k-Y_k)\prod_{j=k+1}^{n}(1+I_j). \tag{2}$$

On the other hand, if the premiums are collected at the end of each period, then the surplus process $\left\{U_n^{(2)}\right\}_{n\geq 1}$ with initial surplus $U_o^{(2)}=u>0$ can be written as

$$U_n^{(2)}=(U_{n-1}^{(2)}+X_n)(1+I_n)-Y_n, \tag{3}$$

which is equivalent to

$$U_n^{(2)}=u.\prod_{k=1}^{n}(1+I_k)+\sum_{k=1}^{n}\left[X_k(1+I_k)-Y_k\right]\prod_{j=k+1}^{n}(1+I_j). \tag{4}$$

where throughout this paper, we denote $\prod\limits_{t=a}^{b}x_t=1$ and $\sum\limits_{t=a}^{b}x_t=0$ if $a>b$.

In this paper, we consider models (1) and (3), in which $X=\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain, X_n take values in a set of non - negative numbers $G_X=\{x_1,x_2,...,x_m,...\}$ with $X_o=x_i$ and

$$p_{ij}=P\left[X_{m+1}=x_j\middle|X_m=x_i\right],(m\in N),x_i,x_j\in G_X,$$

where $0\leq p_{ij}\leq 1, \sum\limits_{j=1}^{+\infty}p_{ij}=1$.

We also assume that $I=\{I_n\}_{n\geq 0}$ is homogeneous Markov chain, I_n take values in a set of non - negative numbers $G_I=\{i_1,i_2,...,i_n,...\}$ with $I_o=i_r$ and

$$q_{rs}=P\left[I_{m+1}=i_s|X_m=i_r\right],(m\in N),i_r,i_s\in G_I,$$

where $0\leq q_{rs}\leq 1, \sum\limits_{s=1}^{+\infty}q_{rs}=1$.

In addition, $Y=\{Y_n\}_{n\geq 0}$ is sequence of independent and identically distributed non negative continuous random variables with the same distribution function $F(y)=P(Y_o\leq y)$.

Based on the previous assumptions, we define the finite time and ultimate ruin probabilities in model (1) respectively, by

$$\psi_n^{(1)}(u,x_i,i_r)=P\left(\bigcup_{k=1}^{n}(U_k^{(1)}<0)\middle|U_o^{(1)}=u,X_o=x_i,I_o=i_r\right), \tag{5}$$

$$\psi^{(1)}(u,x_i,i_r)=\lim_{n\to\infty}\psi_n^{(1)}(u,x_i,i_r)=P\left(\bigcup_{k=1}^{\infty}(U_k^{(1)}<0)\middle|U_o^{(1)}=u,X_o=x_i,I_o=i_r\right). \tag{6}$$

Similarly, we define the finite time and ultimate ruin probabilities in model (3) respectively, by

$$\psi_n^{(2)}(u,x_i,i_r) = P\left(\bigcup_{k=1}^{n}(U_k^{(2)}<0)\middle| U_o^{(2)}=u, X_o=x_i, I_o=i_r\right), \tag{7}$$

$$\psi^{(2)}(u,x_i,i_r) = \lim_{n\to\infty}\psi_n^{(2)}(u,x_i,i_r) = P\left(\bigcup_{k=1}^{\infty}(U_k^{(2)}<0)\middle| U_o^{(2)}=u, X_o=x_i, I_o=i_r\right). \tag{8}$$

In this paper, we derive probability inequalities for $\psi^{(1)}(u,x_i,i_r)$ and $\psi^{(2)}(u,x_i,i_r)$ by the Martingale approach.

3 Upper Bounds for Ruin Probability by the Martingale approach

To establish probability inequalities for ruin probabilities of model (1), we first prove the following Lemma.
Lemma 3.1. Let model (1).
If any $x_i \in G_X, i_r \in G_I$,

$$E\left((Y_1-X_1)(1+I_1)^{-1}\middle| X_o=x_i, I_o=i_r\right)<0 \text{ and } P\left((Y_1-X_1)(1+I_1)^{-1}>0\middle| X_o=x_i, I_o=i_r\right)>0, \tag{9}$$

then there exists a unique positive constant R_{ir} satisfying:

$$E\left(e^{R_{ir}(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right)=1. \tag{10}$$

Proof. Define

$$f_{ir}(t) = E\left\{e^{t(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right\}-1,\ t\in(0,+\infty)$$

We have

$$f'_{ir}(t) = E\left\{(Y_1-X_1)(1+I_1)^{-1}e^{t(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right\},$$
$$f''_{ir}(t) = E\left\{\left[(Y_1-X_1)(1+I_1)^{-1}\right]^2 e^{t(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right\}.$$

Hence $f''_{ir}(t)\geq 0$.
This implies that

$$f_{ir}(t) \text{ is a convex function with } f_{ir}(0)=0 \tag{11}$$

and

$$f'_{ir}(0) = E\left\{(Y_1-X_1)(1+I_1)^{-1}\middle| X_o=x_i, I_o=i_r\right\}<0. \tag{12}$$

By $P\left((Y_1-X_1)(1+I_1)^{-1}>0\middle| X_o=x_i, I_o=i_r\right)>0$, we can find some constant $\delta_{ir}>0$ such that

$$P\left((Y_1-X_1)(1+I_1)^{-1}>\delta_{ir}>0\middle| X_o=x_i, I_o=i_r\right)>0.$$

Then, we get

$$f_{ir}(t) = E\left\{e^{t(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right\}-1$$
$$\geq E\left(\left\{e^{t(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o=x_i, I_o=i_r\right\}.1_{\{(Y_1-X_1)(1+I_1)^{-1}>\delta_{\mathrm{ir}}|X_o=x_i,I_o=i_r\}}\right)-1$$
$$\geq e^{t\delta_{ir}}.P\left\{(Y_1-X_1)(1+I_1)^{-1}>\delta_{ir}\middle| X_o=x_i, I_o=i_r\right\}-1.$$

This implies that

$$\lim_{t\to+\infty} f_{ir}(t) = +\infty. \tag{13}$$

From (11), (12) and (13) there exists a unique positive constant R_{ir} satisfying (10).
This completes the proof.
Let: $R_o = \inf\left\{R_{ir} > 0 : E\left(e^{R_{ir}(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o = x_i, I_o = i_r\right) = 1, x_i \in G_X, i_r \in G_I\right\}$.

Remark 3.1. $E\left(e^{R_o(Y_1-X_1)(1+I_1)^{-1}}\middle| X_o = x_i, I_o = i_r\right) \leq 1$.

To establish probability inequalities for ruin probabilities of model (1), we prove the following Theorem.

Theorem 3.1. Let model (1). Under the condition of Lemma 3.1 and $R_o > 0$, then for any $u > 0$, $x_i \in G_X, i_r \in G_I$,

$$\psi^{(1)}(u, x_i, i_r) \leq e^{-R_o u}. \tag{14}$$

Proof. Consider the process $\left\{U_n^{(1)}\right\}$ is given by (2), we let

$$V_n^{(1)} = U_n^{(1)} \prod_{j=1}^{n} (1+I_j)^{-1} = u + \sum_{j=1}^{n} (X_j - Y_j) \prod_{t=1}^{j} (1+I_t)^{-1}, \tag{15}$$

and $S_n^{(1)} = e^{-R_o V_n^{(1)}}$. Thus, we have

$$S_{n+1}^{(1)} = S_n^{(1)} e^{-R_o(X_{n+1}-Y_{n+1})\prod\limits_{t=1}^{n+1}(1+I_t)^{-1}}.$$

With any $n \geq 1$, we have

$$\begin{aligned}
&E\left(S_{n+1}^{(1)}\middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \\
&= S_n^{(1)} E\left(e^{-R_o(X_{n+1}-Y_{n+1})\prod\limits_{t=1}^{n+1}(1+I_t)^{-1}}\middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \\
&= S_n^{(1)} E\left(e^{-R_o(X_{n+1}-Y_{n+1})(1+I_{n+1})^{-1}\prod\limits_{t=1}^{n}(1+I_t)^{-1}}\middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right).
\end{aligned}$$

From $0 \leq \prod\limits_{t=1}^{n} (1+I_t)^{-1} \leq 1$ and Jensen's inequality implies

$$\begin{aligned}
&S_n^{(1)} E\left(e^{-R_o(X_{n+1}-Y_{n+1})(1+I_{n+1})^{-1}\prod\limits_{t=1}^{n}(1+I_t)^{-1}}\middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right) \\
&\leq S_n^{(1)} E\left(e^{-R_o(X_{n+1}-Y_{n+1})(1+I_{n+1})^{-1}}\middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right)^{\prod\limits_{t=1}^{n}(1+I_t)^{-1}}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&E\left(e^{-R_o(X_{n+1}-Y_{n+1})(1+I_{n+1})^{-1}}\middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right) \\
&= E\left(e^{-R_o(X_{n+1}-Y_{n+1})(1+I_{n+1})^{-1}}\middle| X_n, I_n\right) \\
&= E\left(e^{-R_o(X_1-Y_1)(1+I_1)^{-1}}\middle| X_o, I_o\right) \leq 1.
\end{aligned}$$

Thus, we have

$$E\left(S_{n+1}^{(1)}\middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \leq S_n^{(1)}.$$

Hence, $\left\{S_n^{(1)}, n = 1, 2, ...\right\}$ is a supermartingale with respect to the σ - filtration

$$\Im_n^{(1)} = \sigma\left\{X_1, ..., X_n, Y_1, ..., Y_n, I_1, ..., I_n\right\}$$

Define

$$T_{ir}^{(1)} = \min\left\{ n : V_n^{(1)} < 0 \middle| U_o^{(1)} = u, X_o = x_i, I_o = i_r \right\},$$

with $V_n^{(1)}$ is given by (15).
Hence, $T_{ir}^{(1)}$ is a stopping time and $n \wedge T_{ir}^{(1)} = \min(n, T_{ir}^{(1)})$ is a finite stopping time.
Therefore, from the optional stoppling theorem for supermartingales, we have

$$E\left(S^{(1)}_{n\wedge T_{ir}^{(1)}}\right) \leq E(S_o^{(1)}) = e^{-R_o u}.$$

This implies that

$$e^{-R_o u} \geq E\left(S^{(1)}_{n\wedge T_{ir}^{(1)}}\right) \geq E\left(S^{(1)}_{n\wedge T_{ir}^{(1)}}.1_{(T_{ir}^{(1)} \leq n)}\right)$$
$$= E\left(S^{(1)}_{T_{ir}^{(1)}}.1_{(T_{ir}^{(1)} \leq n)}\right) = E\left(e^{-R_o V^{(1)}_{T_{ir}^{(1)}}}.1_{(T_{ir}^{(1)} \leq n)}\right). \tag{16}$$

From $V^{(1)}_{T_{ir}^{(1)}} < 0$ then (16) becomes

$$e^{-R_o u} \geq E\left(1_{(T_{ir}^{(1)} \leq n)}\right) = P(T_{ir}^{(1)} \leq n). \tag{17}$$

In addition,

$$\psi_n^{(1)}(u, x_i, i_r) = P\left(\bigcup_{k=1}^{n} (U_k^{(1)} < 0) \middle| U_o^{(1)} = u, X_o = x_i, I_o = i_r\right)$$
$$= P\left(\bigcup_{k=1}^{n} (V_k^{(1)} < 0) \middle| U_o^{(1)} = u, X_o = x_i, I_o = i_r\right) = P(T_{ir}^{(1)} \leq n). \tag{18}$$

Combining (17) and (18) imply that

$$\psi_n^{(1)}(u, x_i, i_r) \leq e^{-R_o u}. \tag{19}$$

This complete the proof.
Similarly to Lemma 3.1, we have Lemma 3.2.
Lemma 3.2. Let model (3). Any $x_i \in G_X, i_r \in G_Y$, if

$$E\left(Y_1(1+I_1)^{-1} - X_1 \middle| X_o = x_i, I_o = i_r\right) < 0$$

and

$$P\left(Y_1(1+I_1)^{-1} - X_1 > 0 \middle| X_o = x_i, I_o = i_r\right) > 0, \tag{20}$$

then, there exists a unique positive constant R_{ir} satisfying

$$E\left(e^{R_{ir}\left[Y_1(1+I_1)^{-1} - X_1\right]} \middle| X_o = x_i, I_o = i_r\right) = 1. \tag{21}$$

Let $\overline{R}_o = \inf\left\{R_{ir} > 0 : E\left(e^{R_{ir}(Y_1(1+I_1)^{-1} - X_1)} \middle| X_o = x_i, I_o = i_r\right) = 1, x_i \in G_X, i_r \in G_I\right\}$.
Remark 3.2. $E\left(e^{\overline{R}_o(Y_1(1+I_1)^{-1} - X_1)} \middle| X_o = x_i, I_o = i_r\right) \leq 1.$
Similarly, we establish probability inequalities for ruin probabilities of model (3) by proving the following Theorem.
Theorem 3.2. Let model (3). Under the conditions of Lemma 3.2 and $\overline{R}_o > 0$,then for any $u > 0, x_i \in G_X, i_r \in G_r$,

$$\psi^{(2)}(u, x_i, i_r) \leq e^{-\overline{R}_o u} \tag{22}$$

Proof.
Consider the process $\left\{U_n^{(2)}\right\}$ given by (4), we let

$$V_n^{(2)} = U_n^{(2)} \prod_{j=1}^{n} (1+I_j)^{-1} = u + \sum_{j=1}^{n} (X_j(1+I_j) - Y_j) \prod_{t=1}^{j} (1+I_t)^{-1}, \tag{23}$$

and $S_n^{(2)} = e^{-\overline{R}_o V_n^{(2)}}$. Thus, we have

$$S_{n+1}^{(2)} = S_n^{(2)} e^{-\overline{R}_o \left(X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1}\right) \prod_{t=1}^{n} (1+I_t)^{-1}}.$$

With any $n \geq 1$, we have

$$\begin{aligned}
&E\left(S_{n+1}^{(2)} \middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \\
&= S_n^{(2)} E\left(e^{-\overline{R}_o (X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1}) \prod_{t=1}^{n} (1+I_t)^{-1}} \middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \\
&= S_n^{(2)} E\left(e^{-\overline{R}_o (X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1}) \prod_{t=1}^{n} (1+I_t)^{-1}} \middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right).
\end{aligned}$$

From $0 \leq \prod_{t=1}^{n} (1+I_t)^{-1} \leq 1$ and Jensen's inequality implies

$$\begin{aligned}
&E\left(S_n^{(2)} \middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \\
&\leq S_n^{(2)} E\left(e^{-\overline{R}_o (X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1})} \middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right)^{\prod_{t=1}^{n} (1+I_t)^{-1}}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&E\left(e^{-\overline{R}_o (X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1})} \middle| X_1, X_2, ..., X_n, I_1, I_2, ..., I_n\right) \\
&= E\left(e^{-\overline{R}_o (X_{n+1} - Y_{n+1}(1+I_{n+1})^{-1})} \middle| X_n, I_n\right) \\
&= E\left(e^{-\overline{R}_o (X_1 - Y_1(1+I_1)^{-1})} \middle| X_o, I_o\right) \leq 1.
\end{aligned}$$

Thus, we have

$$E\left(S_{n+1}^{(2)} \middle| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n\right) \leq S_n^{(2)}$$

Hence, $\left\{S_n^{(2)}, n = 1, 2, ...\right\}$ is a supermartingale with respect to the σ - filtration

$$\mathfrak{I}_n^{(2)} = \sigma\left\{X_1, ..., X_n, Y_1, ..., Y_n, I_1, ..., I_n\right\}.$$

Define

$$T_{ir}^{(2)} = \min\left\{n : V_n^{(2)} < 0 \middle| U_o^{(2)} = u, X_o = x_i, I_o = i_r\right\}$$

with $V_n^{(2)}$ is given by (23).
Hence, $T_{ir}^{(2)}$ is a stopping and $n \wedge T_{ir}^{(2)} = \min(n, T_{ir}^{(2)})$ is a finite stopping time.
Therefore, from the optional stopping theorem for supermartingales, we have

$$E\left(S_{n \wedge T_{ir}^{(2)}}^{(2)}\right) \leq E(S_o^{(2)}) = e^{-\overline{R}_o u}.$$

This implies that

$$e^{-\overline{R}_o u} \geq E\left(S^{(2)}_{n\wedge T^{(2)}_{ir}}\right) \geq E\left(S^{(2)}_{n\wedge T^{(2)}_{ir}}.1_{(T^{(2)}_{ir}\leq n)}\right)$$
$$= E\left(S^{(2)}_{T^{(2)}_{ir}}.1_{(T^{(2)}_{ir}\leq n)}\right) = E\left(e^{-R_o V^{(2)}_{T^{(2)}_{ir}}}.1_{(T^{(2)}_{ir}\leq n)}\right). \quad (24)$$

From $V^{(2)}_{T^{(2)}_{ir}} < 0$ then (24) becomes

$$e^{-\overline{R}_o u} \geq E\left(1_{(T^{(2)}_{ir}\leq n)}\right) = P(T^{(2)}_{ir} \leq n). \quad (25)$$

In addition,

$$\psi^{(2)}_n(u,x_i,i_r) = P\left(\bigcup_{k=1}^{n}(U^{(2)}_k < 0)\middle| U^{(2)}_o = u, X_o = x_i, I_o = i_r\right)$$
$$= P\left(\bigcup_{k=1}^{n}(V^{(2)}_k < 0)\middle| U^{(2)}_o = u, X_o = x_i, I_o = i_r\right)$$
$$= P(T^{(2)}_{ir} \leq n). \quad (26)$$

Combining (25) and (26) imply that

$$\psi^{(2)}_n(u,x_i,i_r) \leq e^{-\overline{R}_o u}. \quad (27)$$

Thus, (22) follows by letting $n \to \infty$ in (27).
This completes the proof.

4 Conclusion

Our main results in this paper, Theorem 3.1 and Theorem 3.2 give upper bounds for $\psi^{(1)}_n(u,x_i,i_r)$ and $\psi^{(2)}_n(u,x_i,i_r)$ by the Martingale approach with homogenous Markov chain premiums and Hemogenenous Markov chain Interests. To obtain Therem 3.1 and Theorem 3.2, first, we obtain important preliminary results, Lemma 3.1 and Lemma 3.2, which give Lundbergs constants.

There remain many open issues - e.g.
(a) extending results of this article to consider $X = \{X_n\}_{n\geq 0}$ and $I = \{I_n\}_{n\geq 0}$ are homogenous Markov chains, $Y = \{Y_n\}_{n\geq 0}$ is a first - order autoregressive process;
(b) building numerical examples for $\psi^{(1)}_n(u,x_i,r_r)$ and $\psi^{(2)}_n(u,x_i,r_r)$ by the martingale approach;
(c) Let $\tau_m := \inf\left\{k \geq 1 | U^{(m)}_k < 0\right\}$ $(m = 1,2)$ be the time of ruin. Can we calculate or estimate quantities such as $E(\tau_m)$.
Further research in some of these direction is in progress.

Acknowledgement

The authors would like to thank the Editor and the reviewers for their helpful comment on an earlier version of the manuscript which have led to an improverment of this paper.

References

[1] J. Cai, Discrete time risk models under rates of interest. *Probability in the Engineering and Informational Sciences*, **16**, (2002), 309-324.
[2] J. Cai, Ruin probabilities with dependent rates of interest, *Journal of Applied Probability*, **39**, (2002), 312-323.

[3] J. Cai and C.M.D. Dickson, Ruin Probabilities with a Markov chain interest model. *Insurance: Mathematics and Economics*, **35**, (2004), 513-525.
[4] F. Guo and D Wang, Ruin Probabilities with Dependent Rates of Interest and Autoregressive Moving Average Structures, *Internatinal Journal of Mathematical and Computational Sciences*, **6**(2012), 191-196.
[5] P. D. Quang, Upper Bounds for Ruin Probability in a Generalized Risk Process under interest force with homogenous Markov chain claims, *Asian Journal of Mathematics and Statistics*, Vol 7, No.**1**, (2014), 1-11.
[6] P. D. Quang, Upper Bounds for Ruin Probability in a Generalized Risk Process under Rates of Interest with homogenous Markov chain claims and homogenous Markov chain premiums, *Applied Mathematical Sciences*. Vol 8, No. **29**, (2014), 1445-1454.
[7] P. D. Quang, Upper Bounds for Ruin Probability in a Generalized Risk Process under Rates of Interest with homogenous Markov chain claims and homogenous Markov chain Interests, *American Journal of Mathematics and Statistics*, Vol 4, No. **1**, (2014), 21-29.
[8] P.D. Quang, Ruin Probability in a Generalized Risk Process under Rates of Interest with Dependent Structures, *Journal of Statistics Applications & Probability Letters*, Vol.3, No.**3**, (2014), 1-9.
[9] B. Sundt and J. L. Teugels, Ruin estimates under interest force, *Insurance: Mathematics and Economics*, **16**, (1995), 7-22.
[10] B. Sundt and J. L. Teugels, The adjustment function in ruin estimates under interest force. *Insurance: Mathematics and Economics*, **19**, (1997), 85-94.
[11] L. Xu and R. Wang, Upper bounds for ruin probabilities in an autoregressive risk model with Markov chain interest rate, *Journal of Industrial and Management optimization*, Vol.2 No.**2**, (2006), 165- 175.
[12] H. Yang, Non-exponetial bounds for ruin probability with interest effect included, *Scandinavian Actuarial Journal*, **2**, (1999), 66-79.