Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.12785/jant/030106

Common Fixed Point Results for Generalized Contractions on Ordered Partial Metric Spaces

Reza Arab[*]

Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran

Received: 15 Oct. 2014, Revised: 1 Dec. 2014, Accepted: 4 Dec. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in ordered partial metric spaces. Some coincidence and common fixed point results for these mapping are presented. An example is given to illustrate our obtained results.

Keywords: partially ordered set, partial metric, common fixed point, coincidence point, generalized contraction.

1 Introduction and Preliminaries

In spite of its simplicity, the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. Existence of a fixed point for contraction type mappings in partially metric spaces and its applications has been considered recently by many authors[1,2,4,5,7,11,21,25, 28,30,32,37]. Consistent with [6,24], the following definitions and results will be needed in the sequel.

Definition 1.[24] *A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow R^+$ such that for all $x,y,z \in X$:*

(P_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(P_2) $p(x,x) \leq p(x,y)$,
(P_3) $p(x,y) = p(y,x)$,
(P_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark. It is clear that, if $p(x,y) = 0$, then from (P_1) and (P_2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Example 1.[24] Let a function $p : R^+ \times R^+ \longrightarrow R^+$ be defined by $p(x,y) = max\{x,y\}$ for any $x,y \in R^+$. Then, (R^+, p) is a partial metric space.

Example 2.[24] If $X = \{[a,b] : a,b \in R, a \leq b\}$, then $p : X \times X \longrightarrow R^+$ defined by $p([a,b],[c,d]) = max\{b,d\} - min\{a,c\}$ defines a partial metric on X.

If p is a partial metric on X, then the function $p^s : X \times X \longrightarrow R^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \qquad (1)$$

is a metric on X.

Definition 2.[24,26,27] *Let (X, p) be a partial metric space. Then*

(i) *A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\longrightarrow\infty} p(x,x_n)$.*
(ii) *A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\longrightarrow\infty} p(x_m,x_n)$.*
(iii) *A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is $p(x,x) = \lim_{n,m\longrightarrow\infty} p(x_m,x_n)$.*

Remark. It is easy to see that, every closed subset of a complete partial metric space is complete.

Example 3.[20] If $X = [0,1] \cup [2,3]$ and define $p : X \times X \longrightarrow [0,\infty)$ by

$$p(x,y) = \begin{cases} max\{x,y\} & if\ \ \{x,y\} \cap [2,3] \neq \emptyset, \\ |x-y| & if\ \ \{x,y\} \subset [0,1]. \end{cases}$$

Then (X, p) is a complete partial metric space.

Lemma 1.[24,25,26] *Let (X, p) be a partial metric space. Then*

* Corresponding author e-mail: mathreza.arab@iausari.ac.ir

(a) $\{x_n\}$ *is a Cauchy sequence in* (X,p) *if and only if it is a Cauchy sequence in the metric space* (X,p^s).
(b) *A partial metric space* (X,p) *is complete if and only if the metric space* (X,p^s) *is complete. Furthermore,*

$$\lim_{n\longrightarrow\infty} p^s(x_n,x)=0$$

if and only if

$$p(x,x)=\lim_{n\longrightarrow\infty} p(x,x_n)=\lim_{n,m\longrightarrow\infty} p(x_m,x_n).$$

Lemma 2.[4] *A mapping* $f:X\longrightarrow X$ *is said to be continuous at* $a\in X$, *if for every* $\varepsilon>0$, *there exists* $\delta>0$ *such that* $f(B(a,\delta))\subset B(f(a),\varepsilon)$.

The following result is easy to check.

Lemma 3. *Let* (X,p) *be a partial metric space.* $T:X\longrightarrow X$ *is continuous if and only if given a sequence* $\{x_n\}\subseteq X$ *and* $x\in X$ *such that* $p(x,x)=\lim_{n\longrightarrow\infty} p(x_n,x)$, *then* $p(Tx,Tx)=\lim_{n\longrightarrow\infty} p(Tx_n,Tx)$.

Lemma 4.[33] *Consider* $X=[0,\infty)$ *endowed with the partial metric* $p:X\times X\longrightarrow[0,\infty)$ *defined by* $p(x,y)=max\{x,y\}$ *for all* $x,y\geq 0$. *Let* $F:X\longrightarrow X$ *be a non-decreasing function. If* F *is continuous with respect to the standard metric* $d(x,y)=|x-y|$ *for all* $x,y\geq 0$, *then* F *is continuous with respect to the partial metric* p.

Definition 3. *Let* X *be a set,* T *and* g *are selfmaps of* X. *A point* x *in* X *is called a coincidence point of* T *and* g *if* $Tx=gx$. *We shall call* $w=Tx=gx$ *a point of coincidence of* T *and* g.

Definition 4.[12] *Let* $(X,\leq)$ *be a partially ordered set and* $F,g:X\longrightarrow X$ *are mappings of* X *into itself. One says* F *is g-non-decreasing if for* $x,y\in X$, *we have*

$$g(x)\leq g(y)\Longrightarrow F(x)\leq F(y).$$

Definition 5.[33] *Let* (X,p) *be a partial metric space and* $T,g:X\longrightarrow X$ *are mappings of* X *into itself. We say that the pair* $\{T,g\}$ *is partial compatible if the following conditions is hold:*

(b_1) $p(x,x)=0$ *then* $p(gx,gx)=0$,
(b_2) $\lim_{n\longrightarrow\infty} p(T(gx_n),g(Tx_n))=0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $Tx_n\longrightarrow t$ *and* $gx_n\longrightarrow t$ *for some* $t\in X$.

Alber and Guerre-Delabriere [3] defined weakly contractive mappings on a Hilbert spaces and established a fixed point theorem for such maps.

Definition 6.[3] *Let* (X,d) *be a metric space. A selfmapping* f *on* X *is said to be weakly contractive if*

$$d(fx,fy)\leq d(x,y)-\varphi(d(x,y)) \tag{2}$$

for all $x,y\in X$, *where* $\varphi:[0,\infty)\longrightarrow[0,\infty)$ *is a continuous and nondecreasing function such that* $\varphi(t)=0$ *if and only if* $t=0$.

Aydi [8] obtained the following result.

Theorem 1.[8] *Let* $(X,\leq_X)$ *be a partially ordered set and let* p *be a partial metric on* X *such that* (X,p) *is complete. Let* $f:X\longrightarrow X$ *be a nondecreasing map with respect to* $\leq_X$. *Suppose that the following conditions hold: for* $y\leq x$, *we have*
(i)

$$p(f(x),f(y)\leq p(x,y)-\varphi(p(x,y)), \tag{3}$$

where $\varphi:[0,\infty)\longrightarrow[0,\infty)$ *is a continuous and non-decreasing function such that it is positive in* $[0,\infty)$, $\varphi(0)=0$ *and* $\lim_{t\longrightarrow\infty}\varphi(t)=\infty$;
(ii) there exist $x_0\in X$ *such that* $x_0\leq_X fx_0$;
(iii) f *is continuous in* (X,p), *or;*
(iv) if a non-decreasing sequence $\{x_n\}$ *converges to* $x\in X$, *then* $x_n\leq_X x$ *for all* n. *Then* f *has a fixed point* $u\in X$. *Moreover,* $p(u,u)=0$.

Choudhury [13] introduced the following definition.

Definition 7. *A mapping* $T:X\rightarrow X$, *where* (X,d) *is a metric space is said to be weakly* $C-$*contractive (or a weak* $C-$*contraction) if for all* $x,y\in X$,

$$d(Tx,Ty)\leq\frac{1}{2}[d(x,Ty)+d(y,Tx)]-\phi(d(x,Ty),d(y,Tx)),$$

Where $\phi:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ *is a continuous function such that* $\phi(x,y)=0$ *if and only if* $x=y=0$.

2 Main results

Set $\Psi[0,\infty)=\{\psi:[0,\infty)\longrightarrow[0,\infty):\psi$ is continuous and nondecreasing mapping with $\psi(t)=0$ if and only if $t=0\}$. Our first main result is the following.

Theorem 2. *Let* $(X,\leq)$ *be a partially ordered set and suppose there is a partial metric* p *on* X *such that* (X,p) *is a complete partial metric space. Assume there is a continuous function* $\varphi:[0,\infty)\longrightarrow[0,\infty)$ *with* $\varphi(t)<t$ *for each* $t>0$ *and also suppose* $T,g:X\longrightarrow X$ *are such that* $TX\subseteq gX$, T *is a* $g-$*non-decreasing and for every two elements* $x,y\in X$ *which* gx *and* gy *are comparable, we have*

$$\psi(p(Tx,Ty))\leq\psi(M(x,y))-\phi(p(gx,gy),p(gx,Tx)), \tag{4}$$

where

$$M(x,y)=max\{\varphi(p(gx,gy)),\varphi(p(gx,Tx)),\varphi(p(gy,Ty)),\varphi(\frac{p(gx,Ty)+p(gy,Tx)}{2})\}, \tag{5}$$

$\psi\in\Psi[0,\infty)$ *and* $\phi:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ *is continuous mapping such that* $\phi(x,y)=0$ *if and only if* $x=y=0$. *Also suppose either*

(i) T,g *are two continuous self-mappings of* X *and* $\{T,g\}$ *is partial compatible or* *(ii) if gX is closed and $\{g(x_n)\} \subseteq X$ is a non-decreasing sequence with $g(x_n) \longrightarrow g(x)$ in $g(X)$ then $g(x_n) \le g(x),\ \ g(x) \le g(g(x)) \quad \forall\, n$*

holds.

If there exists an $x_0 \in X$ with $g(x_0) \le T(x_0)$ then T and g have a coincidence point.

Proof. Note that if $T,\ g$ have a coincidence point z, then $p(Tz,Tz) = p(gz,gz) = 0$. Indeed, assume that $p(gz,gz) > 0$. Then from (4) with $x = y = z$, we have

$$\begin{aligned}\psi(p(gz,gz)) &= \psi(p(Tz,Tz)) \\ &\le \psi(M(z,z)) - \phi(p(gz,gz),p(gz,Tz)),\end{aligned} \tag{6}$$

where

$$\begin{aligned}M(z,z) &= max\{\varphi(p(gz,gz)),\varphi(p(gz,Tz)),\varphi(p(gz,Tz)),\\ &\qquad \varphi(\frac{p(gz,Tz)+p(gz,Tz)}{2})\}\\ &= max\{\varphi(p(gz,gz)),\varphi(p(gz,gz)),\varphi(p(gz,gz)),\\ &\qquad \varphi(\frac{p(gz,gz)+p(gz,gz)}{2})\}\\ &= \varphi(p(gz,gz)).\end{aligned}$$

Then we have

$$\begin{aligned}\psi(p(gz,gz)) &= \psi(p(Tz,Tz))\\ &\le \psi(\varphi(p(gz,gz))) - \phi(p(gz,gz),p(gz,gz))\\ &\le \psi(p(gz,gz)) - \phi(p(gz,gz),p(gz,gz))\end{aligned}$$

$\phi(p(gz,gz),p(gz,gz)) \le 0$, a contradiction. Hence $p(Tz,Tz) = p(gz,gz) = 0$.
Let x_0 be an arbitrary point of X such that $g(x_0) \le T(x_0)$. Since $TX \subseteq gX$ we can choose $x_1 \in X$ so that $g(x_1) = T(x_0)$. Again from $TX \subseteq gX$ we can choose $x_2 \in X$ so that $g(x_2) = T(x_1)$. Since $g(x_0) \le T(x_0) = g(x_1)$ and T is $g-$non-decreasing, we have $T(x_0) \le T(x_1)$.
Continuing this process we can choose a sequence $\{x_n\}$ in X such that

$$g(x_{n+1}) = T(x_n) \quad n = 0,1,2,...,$$

$$T(x_0) \le T(x_1) \le T(x_2) \le ... \le T(x_n) \le T(x_{n+1}) \le \cdots$$

Therefore,

$$g(x_1) \le g(x_2) \le g(x_3) \le ... \le g(x_n) \le g(x_{n+1}) \le \cdots \tag{7}$$

If there exists $n \in \mathbb{N}$ such that $p(Tx_n,Tx_{n+1}) = 0$, then by (p_1) and (p_2) we have $gx_{n+1} = Tx_n = Tx_{n+1}$. Hence x_{n+1} is a coincidence of T and g. So we assume that $p(Tx_n,Tx_{n+1}) > 0$, for all $n \in \mathbb{N}$. We will show that

$$p(Tx_n,Tx_{n+1}) < p(Tx_{n-1},Tx_n) \quad \forall\, n \ge 1. \tag{8}$$

From (4) with $x = x_n$ and $y = x_{n+1}$, we have

$$\begin{aligned}&\psi(p(Tx_n,Tx_{n+1}))\\ &\le \psi(M(x_n,x_{n+1})) - \phi(p(gx_n,gx_{n+1}),p(gx_n,Tx_n))\\ &= \psi(M(x_n,x_{n+1})) - \phi(p(Tx_{n-1},Tx_n),p(Tx_{n-1},Tx_n)),\end{aligned} \tag{9}$$

where

$$\begin{aligned}&M(x_n,x_{n+1})\\ &= max\{\varphi(p(gx_n,gx_{n+1})),\varphi(p(gx_n,Tx_n)),\varphi(p(gx_{n+1},Tx_{n+1})),\\ &\varphi(\frac{p(gx_n,Tx_{n+1})+p(gx_{n+1},Tx_n)}{2})\}\\ &= max\{\varphi(p(Tx_{n-1},Tx_n)),\varphi(p(Tx_{n-1},Tx_n)),\varphi(p(Tx_n,Tx_{n+1})),\\ &\varphi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2})\}\\ &= max\{\varphi(p(Tx_{n-1},Tx_n)),\\ &\varphi(p(Tx_n,Tx_{n+1})),\varphi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2})\}.\end{aligned}$$

– If $M(x_n,x_{n+1}) = \varphi(p(Tx_{n-1},Tx_n))$, by (9) and using the fact that $\varphi(t) < t$ for $t > 0$, we have

$$\begin{aligned}&\psi(p(Tx_n,Tx_{n+1}))\\ &\le \psi(\varphi(p(Tx_{n-1},Tx_n))) - \phi(p(Tx_{n-1},Tx_n),p(Tx_{n-1},Tx_n))\\ &\le \psi(\varphi(p(Tx_{n-1},Tx_n)))\\ &< \psi(p(Tx_{n-1},Tx_n)).\end{aligned}$$

Then (8) holds.

– If $M(x_n,x_{n+1}) = \varphi(p(Tx_n,Tx_{n+1}))$, then we have

$$\begin{aligned}&\psi(p(Tx_n,Tx_{n+1}))\\ &\le \psi(\varphi(p(Tx_n,Tx_{n+1}))) - \phi(p(Tx_{n-1},Tx_n),p(Tx_{n-1},Tx_n))\\ &\le \psi(\varphi(p(Tx_n,Tx_{n+1}))),\end{aligned}$$

since we suppose that $p(Tx_n,Tx_{n+1}) > 0$ and as $\varphi(t) < t$ for $t > 0$, then $p(Tx_n,Tx_{n+1}) \le \varphi(p(Tx_n,Tx_{n+1}))$ it is impossible.

– If $M(x_n,x_{n+1}) = \varphi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2})$, we get

$$\begin{aligned}&\psi(p(Tx_n,Tx_{n+1}))\\ &\le \psi(\varphi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2}))\\ &\qquad -\phi(p(Tx_{n-1},Tx_n),p(Tx_{n-1},Tx_n))\\ &\le \psi(\varphi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2}))\\ &\le \psi(\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2}),\end{aligned}$$

thus, we have

$$p(Tx_n,Tx_{n+1}) \le \frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2}.$$

On the other hand, by the triangular inequality in partial metric space, we have

$$\frac{p(Tx_{n-1},Tx_{n+1})+p(Tx_n,Tx_n)}{2} \le \frac{p(Tx_{n-1},Tx_n)+p(Tx_n,Tx_{n+1})}{2},$$

so, we have

$$p(Tx_n,Tx_{n+1}) \le \frac{p(Tx_{n-1},Tx_n)+p(Tx_n,Tx_{n+1})}{2},$$

which implies that

$$p(Tx_n,Tx_{n+1}) \le p(Tx_{n-1},Tx_n).$$

Therefore, we proved that (8) holds. Then, the sequence $\{p(Tx_n,Tx_{n+1})\}$ of real numbers is monotone decreasing. Hence there exists a real number $\delta \geq 0$ such that,

$$\lim_{n\longrightarrow\infty} p(Tx_n,Tx_{n+1}) = \delta. \tag{10}$$

We show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then from continuity ϕ and (9) gives that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} \psi(p(Tx_n,Tx_{n+1}))\\ &\leq \lim_{n\longrightarrow\infty} \psi(\varphi(p(Tx_{n-1},Tx_n)))\\ &\qquad - \lim_{n\longrightarrow\infty} \phi(p(Tx_{n-1},Tx_n),p(Tx_{n-1},Tx_n)),\end{aligned}$$

which implies that

$$\psi(\delta) \leq \psi(\varphi(\delta)) - \phi(\delta,\delta),$$

which is possible only when $\delta = 0$. Therefore, we proved that

$$\lim_{n\longrightarrow\infty} p(Tx_n,Tx_{n+1}) = 0. \tag{11}$$

From $p(Tx_n,Tx_n),\ p(Tx_{n+1},Tx_{n+1}) \leq p(Tx_n,Tx_{n+1})$ and (11), we have

$$\lim_{n\longrightarrow\infty} p(Tx_n,Tx_n) = \lim_{n\longrightarrow\infty} p(Tx_{n+1},Tx_{n+1}) = 0. \tag{12}$$

From (11), (12) and (1), we have

$$\lim_{n\longrightarrow\infty} p^s(Tx_n,Tx_{n+1}) = 0. \tag{13}$$

Now, we prove that

$$\lim_{m,n\to\infty} p(Tx_m,Tx_n) = 0.$$

If not, then there exists an $\varepsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that $p(Tx_{n(k)},Tx_{m(k)}) \geq \varepsilon$ and $p(Tx_{n(k)-1},Tx_{m(k)}) < \varepsilon$. Then we have

$$\begin{aligned}\varepsilon &\leq p(Tx_{n(k)},Tx_{m(k)})\\ &\leq p(Tx_{n(k)},Tx_{n(k)-1}) + p(Tx_{n(k)-1},Tx_{m(k)})\\ &\qquad - p(Tx_{n(k)-1},Tx_{n(k)-1})\\ &< p(Tx_{n(k)},Tx_{n(k)-1}) + \varepsilon - p(Tx_{n(k)-1},Tx_{n(k)-1}).\end{aligned} \tag{14}$$

Taking $k \to \infty$ in (14) and using (11) and (12) we get

$$\lim_{k\to\infty} p(Tx_{n(k)},Tx_{m(k)}) = \varepsilon. \tag{15}$$

Thus from the definition p we have

$$\begin{aligned}&p(Tx_{m(k)},Tx_{n(k)})\\ &\leq p(Tx_{m(k)},Tx_{m(k)-1}) + p(Tx_{m(k)-1},Tx_{n(k)-1})\\ &\qquad + p(Tx_{n(k)-1},Tx_{n(k)})\\ &\qquad - p(Tx_{m(k)-1},Tx_{m(k)-1}) - p(Tx_{n(k)-1},Tx_{n(k)-1}),\end{aligned} \tag{16}$$

$$\begin{aligned}&p(Tx_{m(k)-1},Tx_{n(k)-1})\\ &\leq p(Tx_{m(k)-1},Tx_{m(k)}) + p(Tx_{m(k)},Tx_{n(k)})\\ &\qquad + p(Tx_{n(k)},Tx_{n(k)-1})\\ &- p(Tx_{m(k)},Tx_{m(k)}) - p(Tx_{n(k)},Tx_{n(k)}).\end{aligned} \tag{17}$$

Taking $k \to \infty$ in (16) and (17) and using (12), (13) and (15) we get

$$\begin{aligned}\lim_{k\to\infty} p(Tx_{n(k)},Tx_{m(k)}) &= \lim_{k\to\infty} p(Tx_{n(k)-1},Tx_{m(k)-1})\\ &= \varepsilon.\end{aligned} \tag{18}$$

$$\begin{aligned}&p(Tx_{n(k)-1},Tx_{m(k)}) + p(Tx_{m(k)-1},Tx_{n(k)})\\ &\leq p(Tx_{n(k)-1},Tx_{m(k)-1}) + p(Tx_{m(k)-1},Tx_{m(k)})\\ &- p(Tx_{m(k)-1},Tx_{m(k)-1}) + p(Tx_{m(k)-1},Tx_{n(k)-1})\\ &+ p(Tx_{n(k)-1},Tx_{n(k)}) - p(Tx_{n(k)-1},Tx_{n(k)-1}).\end{aligned} \tag{19}$$

Now using inequality (4), we have

$$\begin{aligned}\psi(\varepsilon) &\leq \psi(p(Tx_{n(k)},Tx_{m(k)}))\\ &\leq \psi(M(x_{n(k)},x_{m(k)}))\\ &\qquad - \phi(p(gx_{n(k)},gx_{m(k)}),p(gx_{n(k)},Tx_{n(k)}))\\ &\leq \psi(M(x_{n(k)},x_{m(k)}))\\ &\qquad - \phi(p(Tx_{n(k)-1},Tx_{m(k)-1}),p(Tx_{n(k)-1},Tx_{n(k)})),\end{aligned} \tag{20}$$

where

$$\begin{aligned}&M(x_{n(k)},x_{m(k)})\\ &= max\{\varphi(p(gx_{n(k)},gx_{m(k)})),\varphi(p(gx_{n(k)},Tx_{n(k)})),\\ &\qquad \varphi(p(gx_{m(k)},Tx_{m(k)})),\\ &\qquad \varphi(\frac{p(gx_{n(k)},Tx_{m(k)}) + p(gx_{m(k)},Tx_{n(k)})}{2})\}\\ &= max\{\varphi(p(Tx_{n(k)-1},Tx_{m(k)-1})),\\ &\qquad \varphi(p(Tx_{n(k)-1},Tx_{n(k)})),\varphi(p(Tx_{m(k)-1},Tx_{m(k)})),\\ &\qquad \varphi(\frac{p(Tx_{n(k)-1},Tx_{m(k)}) + p(Tx_{m(k)-1},Tx_{n(k)})}{2})\}.\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using(18) and (19), we obtain

$$\begin{aligned}\lim_{k\longrightarrow\infty} M(x_{n(k)},x_{m(k)}) &= max\{\varphi(\varepsilon),\varphi(0),\varphi(0),\varphi(\frac{\varepsilon+\varepsilon}{2})\}\\ &= \varphi(\varepsilon).\end{aligned}$$

As $k \longrightarrow \infty$, inequality (20) becomes,

$$\begin{aligned}\psi(\varepsilon) &\leq \psi(\varphi(\varepsilon)) - \phi(\varepsilon,0)\\ &< \psi(\varepsilon) - \phi(\varepsilon,0)\end{aligned}$$

which is a contradiction by virtue of a property of ϕ and ψ.

Thus, we obtain that $\lim_{m,n\to\infty} p(Tx_m,Tx_n) = 0$, i.e., $\{Tx_n\}$ is a cauchy sequence in (X,p), and hence in the metric space (X,p^s) by Lemma 1. Since (X,p) be a complete partial metric space, then, from lemma 1, (X,p^s) is also complete, so the sequence $\{Tx_n\}$ converges in the metric space (X,p^s), so there exist z in X such that

$$\lim_{n\to\infty} p^s(Tx_n,z) = \lim_{n\to\infty} p^s(gx_{n+1},z) = 0.$$

Again, from Lemma 1, we get

$$\begin{aligned}p(z,z) &= \lim_{n\longrightarrow\infty} p(Tx_n,z) = \lim_{n\longrightarrow\infty} p(gx_{n+1},z)\\ &= \lim_{n,m\longrightarrow\infty} p(Tx_n,Tx_m) = 0.\end{aligned} \tag{21}$$

because $\lim_{n,m\longrightarrow\infty} p(Tx_n,Tx_m)=0$.
Suppose that the assumption (i) holds. Now we show that z is a coincidence point of T and g.
Since T and g are continuous, from (21) and using Lemma 3, we get

$$p(Tz,Tz)=\lim_{n\longrightarrow\infty} p(T(Tx_n),Tz) \quad \text{and} \quad p(gz,gz)=\lim_{n\longrightarrow\infty} p(g(Tx_{n+1}),gz). \tag{22}$$

Since T and g are partial compatible mappings, this implies that

$$p(gz,gz)=0 \text{ and } \lim_{n\longrightarrow\infty} p(T(g(x_{n+1})),g(T(x_{n+1})))=0. \tag{23}$$

The condition (p_4), we obtain

$$\begin{aligned} p(Tz,gz) &\le p(Tz,T(Tx_n))+p(T(Tx_n),g(Tx_{n+1})) \\ &\quad + p(g(Tx_{n+1}),gz)-p(T(Tx_n),T(Tx_n)) \\ &\quad - p(g(Tx_{n+1}),g(Tx_{n+1})) \\ &= p(Tz,T(Tx_n))+p(T(gx_{n+1}),g(Tx_{n+1})) \\ &\quad + p(g(Tx_{n+1}),gz) \\ &\quad - p(T(Tx_n),T(Tx_n))-p(g(Tx_{n+1}),g(Tx_{n+1})). \end{aligned}$$

Letting $n\longrightarrow\infty$ in the above inequality and using (22)and (23), we have

$$p(Tz,gz)\le p(Tz,Tz)+p(gz,gz)=p(Tz,Tz). \tag{24}$$

Now, we will prove that $p(Tz,gz)=0$. Suppose that this is not the case. Then, from (4) with y =x=z, we get

$$\psi(p(Tz,Tz))\le\psi(M(z,z))-\phi(p(gz,gz),p(gz,Tz)),$$

where

$$\begin{aligned} M(z,z) &= max\{\varphi(p(gz,gz)),\varphi(p(gz,Tz)),\varphi(p(gz,Tz)), \\ &\qquad \varphi(\frac{p(gz,Tz)+p(gz,Tz)}{2})\} \\ &= \varphi(p(gz,Tz))<p(gz,Tz). \end{aligned}$$

Therefore, from (24) and the above inequality, we have

$$p(gz,Tz)<p(gz,Tz),$$

a contradiction. Hence $p(gz,Tz)=0$ which implies that $Tz=gz$, that is, z is a coincidence point of T and g.
Suppose now that (ii) holds. Since $\{Tx_n\}\subseteq gX$ and gX is closed, there exists $x\in X$ such that $z=gx$. From (7) and hypothesis (ii), we have

$$gx_n\le gx \quad for\ all\ n,\ gx\le g(gx). \tag{25}$$

Now, we claim that x is a coincidence point of T and g. We have

$$\begin{aligned} p(gx,Tx) &\le p(gx,gx_{n+1})+p(gx_{n+1},Tx)-p(gx_{n+1},gx_{n+1}) \\ &= p(z,gx_{n+1})+p(Tx_n,Tx)-p(gx_{n+1},gx_{n+1}), \\ p(gx_n,Tx) &\le p(gx_n,gx)+p(gx,Tx)-p(gx,gx) \\ &= p(gx_n,z)+p(gx,Tx)-p(gx,gx). \end{aligned}$$

Taking $n\longrightarrow\infty$ in the above inequality, we have

$$p(gx,Tx)\le\lim_{n\longrightarrow\infty} p(Tx_n,Tx), \tag{26}$$

$$\lim_{n\longrightarrow\infty} p(gx_n,Tx)\le p(gx,Tx). \tag{27}$$

By property of ψ and using (26), we have

$$\begin{aligned} \psi(p(gx,Tx)) &\le \lim_{n\longrightarrow\infty}\psi(p(Tx,Tx_n)) \\ &\le \lim_{n\longrightarrow\infty}[\psi(M(x,x_n))-\phi(p(gx,gx_n),p(gx,Tx))] \\ &\le \lim_{n\longrightarrow\infty}\psi(M(x,x_n))-\phi(0,p(gx,Tx)) \\ &= \psi(\lim_{n\longrightarrow\infty} M(x,x_n))-\phi(0,p(gx,Tx)) \end{aligned}$$

where

$$\begin{aligned} &\lim_{n\longrightarrow\infty} M(x,x_n) \\ &= \lim_{n\longrightarrow\infty}[max\{\varphi(p(gx,gx_n)),\varphi(p(gx,Tx)), \\ &\qquad \varphi(p(gx_n,Tx_n)),\varphi(\frac{p(gx,Tx_n)+p(gx_n,Tx)}{2})\}] \\ &= \varphi(p(gx,Tx)), \end{aligned}$$

hence

$$\psi(p(gx,Tx))\le\psi(p(gx,Tx))-\phi(0,p(gx,Tx)).$$

which is possible only when $p(gx,Tx)=0$, which implies that $Tx=gx$, that is, x is a coincidence point of T and g.

Theorem 3. *Adding to the hypotheses of Theorem 2 the following condition:*
if T and g commute at their coincidence points, we obtain the uniqueness of the common fixed point of T and g.

Proof. Suppose that T and g commute at x. Set $y=Tx=gx$. Then

$$Ty=T(gx)=g(Tx)=gy, \tag{28}$$

from (4) we get

$$\psi(p(Tx,Ty))\le\psi(M(x,y))-\phi(p(gx,gy),p(gx,Tx)) \tag{29}$$

where

$$\begin{aligned} M(x,y) &= max\{\varphi(p(gx,gy)),\varphi(p(gx,Tx)),\varphi(p(gy,Ty)), \\ &\qquad \varphi(\frac{p(gx,Ty)+p(gy,Tx)}{2})\} \\ &= max\{\varphi(p(Tx,Ty)),\varphi(p(Tx,Tx)),\varphi(p(Ty,Ty)), \\ &\qquad \varphi(\frac{p(Tx,Ty)+p(Ty,Tx)}{2})\} \\ &= \varphi(p(Tx,Ty)). \end{aligned}$$

Suppose that $p(Tx,Ty)>0$, from (29), we get

$$\begin{aligned} \psi(p(Tx,Ty)) &\le \psi(M(x,y))-\phi(p(gx,gy),p(gx,Tx)) \\ &= \psi(\varphi(p(Tx,Ty)))-\phi(p(gx,gy),p(gx,Tx)) \\ &\le \psi(\varphi(p(Tx,Ty))), \end{aligned}$$

by property of ψ and φ, we have

$$p(Tx,Ty)\le\varphi(p(Tx,Ty))<p(Tx,Ty),$$

which is a contradiction. Hence $p(Tx,Ty)=0$, that is, $p(y,Ty)=0$. Therefore,

$$Ty = gy = y. \tag{30}$$

Thus we proved that T and g have a common fixed point.
Uniqueness: Let v and w be two common fixed points of T and g. (i.e) $v=Tv=gv$ and $w=Tw=gw$. Using inequality (4), we have

$$\psi(p(Tw,Tv)) \le \psi(M(w,v)) - \phi(p(gw,gv),p(gw,Tw)),$$

where

$$\begin{aligned} M(w,v) &= max\{\varphi(p(gw,gv)),\varphi(p(gw,Tw)),\\ &\qquad \varphi(p(gv,Tv)),\varphi(\frac{p(gw,Tv)+p(gv,Tw)}{2})\}\\ &= \varphi(p(w,v)). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(p(w,v)) &= \psi(p(Tw,Tv))\\ &\le \psi(\varphi(p(w,v))) - \phi(p(gw,gv),p(gw,Tw)) \end{aligned}$$

which is possible only when $w=v$. Hence T and g have an unique common fixed point.

*Example 4.*Let $X=[0,1]$ be endowed with usual order and let p be the complete partial metric on X defined by $p(x,y)=max\{x,y\}$ for all $x,y\in X$. Let $T,g:X\longrightarrow X$ and $\psi,\varphi:[0,\infty)\longrightarrow[0,\infty)$ and $\phi:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ be given by $Tx=\frac{x^3}{3x+9}$, $gx=\frac{x^2}{x+3}$, $\psi(t)=t$, $\phi(s,t)=\frac{s+t}{6}$ and $\varphi(t)=\frac{t}{2}$. Clearly ψ is continuous and nondecreasing, $\psi(t)=0$ if and only if $t=0$. ϕ and φ are continuous and $\phi(s,t)=0$ if and only if $t=0$. We show that condition (4) is satisfied.
If $x,y\in X$ with $x\le y$, then we have

$$\begin{aligned} \psi(p(Tx,Ty)) &= \psi(max\{\frac{x^3}{3x+9},\frac{y^3}{3y+9}\})\\ &= max\{\frac{x^3}{3x+9},\frac{y^3}{3y+9}\}\\ &\le \frac{1}{3}max\{\frac{x^2}{x+3},\frac{y^2}{y+3}\}\\ &= \frac{1}{3}p(gx,gy) = \frac{2}{3}\varphi(p(gx,gy))\\ &\le \psi(M(x,y)) - \phi(p(gx,gy),p(gx,Tx)). \end{aligned}$$

Note that, T and g satisfy all the conditions given in Theorem 2. Moreover, 0 is a unique common fixed point of T and g.
If we replace p by p^s in (4) of Theorem 2, then T and g do not satisfy (4) of Theorem 2, because

$$\begin{aligned} p^s(x,y) &= 2p(x,y)-p(x,x)-p(y,y) = 2max\{x,y\}-x-y\\ &= |x-y|, \end{aligned}$$

and

$$\psi(p^s(T1,T0)) = \psi(p^s(\frac{1}{12},0)) = \frac{1}{12},$$

$$\begin{aligned} &M(1,0)\\ &= max\{\varphi(p^s(g1,g0)),\varphi(p^s(g1,T1)),\\ &\qquad \varphi(p^s(g0,T0)),\varphi(\frac{p^s(g1,T0)+p^s(g0,T1)}{2})\}\\ &= max\{\varphi(p^s(\frac{1}{4},0)),\varphi(p^s(\frac{1}{4},\frac{1}{12})),\\ &\qquad \varphi(p^s(0,0)),\varphi(\frac{p^s(\frac{1}{4},0)+p^s(0,\frac{1}{12})}{2})\}\\ &= max\{\varphi(\frac{1}{4}),\varphi(\frac{1}{6}),\varphi(0),\varphi(\frac{\frac{1}{4}+\frac{1}{12})}{2})\}\\ &= \frac{1}{8}, \end{aligned}$$

$$\begin{aligned} \phi(p^s(g1,g0),p^s(g1,T1)) &= \phi(p^s(\frac{1}{4},0),p^s(\frac{1}{4},\frac{1}{12}))\\ &= \phi(\frac{1}{4},\frac{1}{6}) = \frac{5}{72}, \end{aligned}$$

$$\psi(M(1,0)) - \phi(p^s(g1,g0),p^s(g1,T1)) = \frac{4}{72}.$$

Now, we will show that many results can be deduced from our previous obtained results.
An immediate consequence of Theorem 2 are the following results.

Corollary 1.*Let $(X,\le)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Assume there is a continuous function $\varphi:[0,\infty)\longrightarrow[0,\infty)$ with $\varphi(t)<t$ for each $t>0$ and suppose $T:X\longrightarrow X$ be a non-decreasing function for all comparable $x,y\in X$, we have*

$$\psi(p(Tx,Ty)) \le \psi(M(x,y)) - \theta(max\{p(x,y),p(x,Tx)\}),$$

where

$$\begin{aligned} M(x,y) &= max\{\varphi(p(x,y)),\varphi(p(x,Tx)),\\ &\qquad \varphi(p(y,Ty)),\varphi(\frac{p(x,Ty)+p(y,Tx)}{2})\}, \end{aligned}$$

$\psi\in\Psi[0,\infty)$ and $\theta:[0,\infty)\longrightarrow[0,\infty)$ is continuous mapping such that $\theta(t)=0$ if and only if $t=0$. Also suppose either

$(i)T$ is continuous or
$(ii)X$ has the following proprety :
if a non-decreasing sequence $x_n\longrightarrow x$, then $x_n\le x\quad\forall\, n$.

If there exists an $x_0\in X$ with $x_0\le T(x_0)$ then have a unique fixed point $x\in X$. Moreover, $p(x,x)=0$.

*Proof.*In Theorem 2, taking $\phi(x,y)=\theta(max\{x,y\})$ for all $x,y\in[0,\infty)$, we get Corollary 1.

Corollary 2.*Let $(X,\le)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Assume there is a continuous function $\varphi:[0,\infty)\longrightarrow[0,\infty)$ with $\varphi(t)<t$ for*

each $t > 0$ and suppose $T : X \longrightarrow X$ be a non-decreasing function for all comparable $x, y \in X$, we have

$$\psi(p(Tx,Ty)) \leq \psi(M(x,y)) - \theta(p(x,y) + p(x,Tx)),$$

where

$$M(x,y) = max\{\varphi(p(x,y)), \varphi(p(x,Tx)), \varphi(p(y,Ty)), \varphi(\frac{p(x,Ty)+p(y,Tx)}{2})\},$$

$\psi \in \Psi[0,\infty)$ and $\theta : [0,\infty) \longrightarrow [0,\infty)$ is continuous mapping such that $\theta(t) = 0$ if and only if $t = 0$. Also suppose either

$(i)T$ is continuous or
$(ii)X$ has the following proprety :
if a non-decreasing sequence $x_n \longrightarrow x$, then $x_n \leq x \quad \forall\, n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then have a unique fixed point $x \in X$. Moreover, $p(x,x) = 0$.

*Proof.*In Theorem 2, taking $\phi(x,y) = \theta(x+y)$ for all $x, y \in [0,\infty)$, we get Corollary 2.

Corollary 3.*Let $(X, \leq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Assume there is a continuous function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\varphi(t) < t$ for each $t > 0$ and suppose $T : X \longrightarrow X$ be a non-decreasing function for all comparable $x, y \in X$, we have*

$$\psi(p(Tx,Ty)) \leq \psi(M(x,y)) - \phi(p(x,y), p(x,Tx)),$$

where

$$M(x,y) = max\{\varphi(p(x,y)), \varphi(p(x,Tx)), \varphi(p(y,Ty)), \varphi(\frac{p(x,Ty)+p(y,Tx)}{2})\},$$

$\psi \in \Psi[0,\infty)$ and $\phi : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is continuous mapping such that $\phi(x,y) = 0$ if and only if $x = y = 0$. Also suppose either

$(i)T$ is continuous or
$(ii)X$ has the following proprety :
if a non-decreasing sequence $x_n \longrightarrow x$, then $x_n \leq x \quad \forall\, n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then have a unique fixed point $x \in X$. Moreover, $p(x,x) = 0$.

*Remark.*The following condition

$$\psi(p(Tx,Ty)) \leq \psi[\varphi(max\{p(gx,gy), p(gx,Tx), p(gy,Ty), \frac{p(gx,Ty)+p(gy,Tx)}{2}\})] - \phi(p(gx,gy), p(gx,Tx)), \tag{31}$$

implies condition (4). We observe also that condition (31) is equivalent to condition (4) if we suppose that φ is a non-decreasing function.

References

[1] T. Abdeljawad, E. Karapinar, K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces. Applied Mathematics Letters. 2011, 24(11):1900-1904.
[2] A. Aghajani, R. Arab, Some fixed point results for generalized contractions on partial metric spaces, European Journal of Scientific Research Volume 107 No 1 July, (2013)13-24.
[3] Ya.I. Alber, S. Guerre-Delabrere, Principle of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich (Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhuser Verlag, Basel, 1997, pp. 722.
[4] I. Altun, A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces. Fixed Point Theory Appl 2011, 10.
[5] I. Altun, F. Sola, H. Simsek, Generalized contractions on Partial metric spaces, Topology and its Applications.157(2010) 2778-2785.
[6] I. Altun, H. Simsek, Some fixed point theorems on dualistic partial metric spaces. J Adv Math Stud. 1, 18 (2008).
[7] R. Arab, M. Rabbani. Coupled coincidence and common fixed point theorems for mappings in partially ordered metric spaces, Math. Sci. Lett., 3, 81?87 (2014).
[8] H. Aydi, Some fixed point results in ordered partial metric spaces, Accepted in J. Nonlinear Sci. Appl, (2011).
[9] I. Beg, A. R. Butt, Coupled fixed points of set valued mappings in partially ordered metric spaces, J. Nonlinear Sci. Appl. 3 (3) (2010), 179-185.
[10] TG. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal: Theorey Methods Appl. 65, 1379-1393 (2006). doi:10.1016/j.na.2005.10.017
[11] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, Partial metric spaces. Am. Math. Mon. 116, 708-718(2009)
[12] L. Ćirić, N.Cakić, M. Rajovic, JS. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl 2008(Article ID 131294), 11 (2008).
[13] B. S. Choudhury, unique fixed point theorem for weak $C-$contractive mapping, Kathmandu University J. Sci. , Engng. Technology, 5(2009), No.1,613.
[14] D. Dorić, Common fixed point for generalized (ψ, ϕ)-weak contractions, Applied Mathematics Letters 22 (2009) 1896.1900.
[15] P. N. Dutta, B. S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory and Applications (2008) Article ID 406368.
[16] M. H. Escardo, PCF extended with real numbers, Theoret. Comput. Sci. 162 (1996) 79-115.
[17] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis. 65 (2006) 1379-1393.
[18] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009), 3403-3410.
[19] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7 (1999) 7183.
[20] D. Ilić, V. Pavlović, and V. Rakočcević, Some new extensions of Banachs contraction principle to partial metric space, Appl. Math. Lett., vol. 24, no. 8, pp. 13261330, 2011.

[21] E. Karapinar, M. E. Inci, Fixed point theorems for operators on partial metric spaces. Appl Math Lett 2011, 24(11):1894-1899.

[22] NV. Luong, NX. Thuan, Coupled fixed point in partially ordered metric spaces and applications. Nonlinear Anal: Theorey Methods Appl. 74, 983-992 (2011). doi:10.1016/j.na.2010.09.055

[23] S. G. Matthews, Partial metric topology. Research Report 212. Department of Computer Science, University of Warwick 1992.

[24] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728,1994, pp. 183-197.

[25] S. Oltra, O. Valero, Banachs fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste 36 (2004) 17-26.

[26] S. J. ONeill, Two topologies are better than one, Tech. report, University of Warwick, Coven-try, UK, http : //www.dcs.warwick.ac.uk/reports/283.html,1995.

[27] S. J. ONeill, Partial metrics, valuations and domain theory, in: Proc. 11th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 806, 1996, pp. 304-315.

[28] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[29] B. H. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis (2001) 2683-2693.

[30] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces. Fixed Point Theory Appl. 2010(Article ID 493298), 6 (2010)

[31] S. Romaguera, M. Schellekens, Partial metric monoids and semivaluation spaces, Topology Appl. 153 (5-6) (2005), 948-962.

[32] S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, Math. Structures Comput. Sci. 19 (3) (2009), 541-563.

[33] B. Samet, M. Rajovic, R. Lazovi, R. Stoiljkovic, Common Fixed Point Results For Nonlinear Contractions in Ordered Partial Metric Spaces.Fixed Point Theory Appl. 2011.doi:10.1186/1687-1812-2011-71

[34] M. P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004), 135-149.

[35] R. Sumitra, V. Rhymend Uthariaraj, R. Hemavathy, Common Fixed Point and Invariant Approximation Theorems for Mappings Satisfying Generalized Contraction Principle,Journal of Mathematics Research,Vol. 2, No. 2, May (2010).

[36] H. Sheng Ding, L. Li, coupled fixed point theorems in partialy orderd con metric spaces, Faculty of Sciences and Mathematics, University of Niš, Serbia, (2011), 137-149

[37] O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. Gen. Topol, 6 (2) (2005) 229-240.

Reza Arab His PhD thesis entitled “Fixed and coupled fixed point theorems on complete metric spaced and applications” last year. He has a great experience in teaching Mathematics as a lecturer in Islamic Azad university, Sari branch, since 1999 and now He is a head of Department of Mathematics. His favorite field of study is Mathematical Analysis and He is really interested in Fixed point theorems and Measure of non-compactness. As you can see in most of his papers, He has tried to apply Fixed point theorems to prove the existence and uniqueness of various kinds of applied problems such as Integral equations and Differential equations.