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A Common Coupled Fixed Point Theorem for Two Pairs of w^*-Compatible Mappings in $G-$Metric Spaces

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Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in $G-$metric spaces. Some coincidence and common fixed point results for these mapping are presented. Our results extend and generalize many known results in the literature.An example is presented to show the ectiveness of our results.

Keywords: Common coupled fixed point, generalized metric space, w^*-compatible mappings.

1 Introduction and Preliminaries

Mustafa and Sims [5] introduced the notion of complete $G-$metric spaces as a generalization of complete metric spaces. For details on $G-$metric spaces, we refer to [5,6,7,8]. The notion of a coupled fixed point in partially ordered metric spaces has been introduced by Bhaskar and Lakshmikantham in (2006)[9].
In this paper ,we prove a common coupled fixed point theorem for two mappings in $G-$metric spaces.

Definition 1.[5] *Let X be a nonempty set and $G : X \times X \times X \longrightarrow \mathbb{R}^+$ a function satisfying the following properties:*

$(G_1)G(x,y,z) = 0$ *if* $x = y = z = 0$,
$(G_2)0 < G(x,x,y)$, *for all* $x,y \in X$ *with* $x \neq y$,
$(G_3)G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $z \neq y$,
$(G_4)G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$, *symmetry in all three variables,*
$(G_5)G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ *for all* $x,y,z,a \in X$

Then the function G is called a generalized metric, or, more specifically, a $G-$metric on X, and the pair (X,G) is called a $G-$metric space.

Definition 2.[5] *Let (X,G) be a $G-$metric space and (x_n) a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{m,n\longrightarrow\infty} G(x,x_n,x_m) = 0$, and we say that the sequence (x_n) is $G-$convergent to x or that (x_n) $G-$converges to x.*
Thus, $x_n \longrightarrow x$ in a $G-$metric space (X,G) if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq k$.

Proposition 11[5] *Let (X,G) be a $G-$metric space. Then the following are equivalent:*

$(1)(x_n)$ *is $G-$convergent to x.*
$(2)G(x_n,x_n,x) \longrightarrow 0$ *as* $n \longrightarrow \infty$.
$(3)G(x_n,x,x) \longrightarrow 0$ *as* $n \longrightarrow \infty$.
$(4)G(x_n,x_m,x) \longrightarrow 0$ *as* $n,m \longrightarrow \infty$.

Definition 3.[5] *Let (X,G) be a $G-$metric space, a sequence (x_n) is called $G-$Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq k$, that is $G(x_n,x_m,x_l) \longrightarrow 0$ as $n,m,l \longrightarrow \infty$.*

Proposition 12[5] *Let (X,G) be a $G-$metric space, then the following statements are equivalent:*

(1)*The sequence (x_n) is $G-$Cauchy.*
(2)*For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq k$.*

Definition 4.[5] *A $G-$metric space (X,G) is called $G-$complete if every $G-$Cauchy sequence in (X,G) is $G-$convergent in (X,G).*

Proposition 13[5] *Let (X,G) be a $G-$metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Example.[5] Let $(\mathbb{R},d)$ be the usual metric space. Define G_s by

$$G_s(x,y,z) = d(x,y)+d(y,z)+d(x,z)$$

for all $x,y,z \in \mathbb{R}$. Then it is clear that $(\mathbb{R},G_s)$ is a $G-$metric space.

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Proposition 14[5] *Let (X,G) be a $G-$metric space. Then $T:X\longrightarrow X$ is $G-$continuous at $x\in X$ if and only if it is $G-$sequentially continuous at x,that is, whenever (x_n) is $G-$convergent to x, $(f(x_n))$ is $G-$convergent to $f(x)$.*

Definition 5.[4] *Let (X,G) be a $G-$metric space. A mapping $F:X\times X\longrightarrow X$ is said to be continuous if for any two $G-$convergent sequences (x_n) and (y_n) converging to x and y respectively, $(F(x_n,y_n))$ is $G-$convergent to $F(x,y)$.*

Definition 6.[3] *An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F:X\times X\longrightarrow X$ if*

$$F(x,y)=x,\ F(y,x)=y.$$

Definition 7.[9] *An element $(x,y)\in X\times X$ is called a coupled coincidence point of a mapping $F:X\times X\longrightarrow X$ and a mapping $g:X\longrightarrow X$ if*

$$F(x,y)=gx,\ F(y,x)=gy.$$

Note that if g is the identity mapping, then Definition 1.7 reduces to Definition 1.6.

Definition 8.[1] *An element $x\in X$ is called a common fixed point of a mapping $F:X\times X\longrightarrow X$ and $g:X\longrightarrow X$ if*

$$F(x,x)=gx=x.$$

Abbas et al. [1] introduced the concept of $w-$compatible and w^*-compatible mappings and utilized this concept to prove an interesting uniqueness theorem of a coupled fixed point for mappings F and g in cone metric spaces.

Definition 9.[1] *Mappings $F:X\times X\longrightarrow X$ and $g:X\longrightarrow X$ are called*

(W_1) *$w-$compatible if $g(F(x,y))=F(gx,gy)$ whenever $gx=F(x,y)$ and $gy=F(y,x)$.*
(W_2) *w^*-compatible if $g(F(x,x))=F(gx,gx)$ whenever $gx=F(x,x)$.*

Let $X=\mathbb{R}^+$, define $F:X\times X\longrightarrow X$ and $g:X\longrightarrow X$ by

$$F(x,y)=\begin{cases}8, & x=1,y=0,\\ 10, & x=0,y=1,\\ 4 & otherwise,\end{cases}$$

$$g(x)=\begin{cases}8, & x=1,\\ 10, & x=0,\\ 5, & x=4,\\ 4, & otherwise.\end{cases}$$

Then it is clear that F and g are $w-$compatible but not w^*-compatible.

Definition 10.[9] *Let X be a nonempty set and $F:X\times X\longrightarrow X$ and $g:X\longrightarrow X$. One says F and g are commutative if for all $x,y\in X$,*

$$F(gx,gy)=g(F(x,y)).$$

2 MAIN RESULTS

Our first result is the following.

Theorem 21*Let (X,G) be a $G-$metric space. Set $T:X\times X\longrightarrow X$ and $g:X\longrightarrow X$. Assume there exist $a_1,a_2,a_3\geq 0$ with $2a_1+3a_2+3a_3<2$ such that*

$$\begin{aligned}&G(T(x,y),T(u,v),T(w,z))\\ &\leq\frac{a_1}{2}[G(gx,gu,gw)+G(gy,gv,gz)]\\ &+\frac{a_2}{2}[G(gx,T(x,y),T(x,y))+G(gu,T(u,v),T(u,v))\\ &+G(gy,gv,gz)]\\ &+\frac{a_3}{2}[G(gx,T(u,v),T(u,v))+G(gu,T(x,y),T(x,y))\\ &+G(gy,gv,gz)],\end{aligned}\tag{1}$$

for all $x,y,u,v,w,z\in X$. If $T(X\times X)\subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be in X. Since $T(X\times X)\subseteq g(X)$, we can choose $x_1,y_1\in X$ such that $gx_1=T(x_0,y_0)$ and $gy_1=T(y_0,x_0)$. Analogously, there exist $x_2,y_2\in X$ such that $gx_2=T(x_1,y_1)$ and $gy_2=T(y_1,x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1}=T(x_n,y_n)\ and\ gy_{n+1}=T(y_n,x_n)\ for\ all\ n\geq 0\tag{2}$$

From by (1), we have

$$\begin{aligned}&G(gx_n,gx_{n+1},gx_{n+1})\\ &=G(T(x_{n-1},y_{n-1}),T(x_n,y_n),T(x_n,y_n))\\ &\leq\frac{a_1}{2}[G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)]\\ &+\frac{a_2}{2}[G(gx_{n-1},T(x_{n-1},y_{n-1}),T(x_{n-1},y_{n-1}))\\ &+G(gx_n,T(x_n,y_n),T(x_n,y_n))+\\ &G(gy_{n-1},gy_n,gy_n)]+\frac{a_3}{2}[G(gx_{n-1},T(x_n,y_n),T(x_n,y_n))\\ &+G(gx_n,T(x_{n-1},y_{n-1}),T(x_{n-1},y_{n-1}))+G(gy_{n-1},gy_n,gy_n)]\\ &=\frac{a_1}{2}[G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)]\\ &+\frac{a_2}{2}[G(gx_{n-1},gx_n,gx_n)+G(gx_n,gx_{n+1},gx_{n+1})\\ &+G(gy_{n-1},gy_n,gy_n)]\\ &+\frac{a_3}{2}[G(gx_{n-1},gx_{n+1},gx_{n+1})+G(gx_n,gx_n,gx_n)\\ &+G(gy_{n-1},gy_n,gy_n)].\end{aligned}$$

Thus, we obtain

$$\begin{aligned}G(gx_n,gx_{n+1},gx_{n+1})\leq\frac{a_1+a_2+a_3}{2-a_2-a_3}&[G(gx_{n-1},gx_n,gx_n)\\ &+G(gy_{n-2},gy_{n-1},gy_{n-1})],\end{aligned}\tag{3}$$

and

$$\begin{aligned}
&G(gy_n, gy_{n+1}, gy_{n+1})\\
&= G(T(y_{n-1}, x_{n-1}), T(y_n, x_n), T(y_n, x_n))\\
&\le \frac{a_1}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)]\\
&+ \frac{a_2}{2}[G(gy_{n-1}, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1}))\\
&+ G(gy_n, T(y_n, x_n), T(y_n, x_n)) +\\
&G(gx_{n-1}, gx_n, gx_n)] + \frac{a_3}{2}[G(gy_{n-1}, T(y_n, x_n), T(y_n, x_n))\\
&+ G(gy_n, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1})) + G(gx_{n-1}, gx_n, gx_n)]\\
&= \frac{a_1}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)]\\
&+ \frac{a_2}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gy_n, gy_{n+1}, gy_{n+1})\\
&+ G(gx_{n-1}, gx_n, gx_n)]\\
&+ \frac{a_3}{2}[G(gy_{n-1}, gy_{n+1}, gy_{n+1}) + G(gy_n, gy_n, gy_n)\\
&+ G(gx_{n-1}, gx_n, gx_n)].
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&G(gy_n, gy_{n+1}, gy_{n+1})\\
&\le \frac{a_1 + a_2 + a_3}{2 - a_2 - a_3}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)].
\end{aligned} \tag{4}$$

From (3) and (4), we have

$$\begin{aligned}
&G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})\\
&\le \frac{2(a_1 + a_2 + a_3)}{2 - a_2 - a_3}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)].
\end{aligned}$$

Set $a_n = G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})$ and $\lambda = \frac{2(a_1+a_2+a_3)}{2-a_2-a_3}$, then the sequence $\{a_n\}$ is decreasing as

$$0 \le a_n \le \lambda a_{n-1} \le \lambda^2 a_{n-2} \le \ldots \le \lambda^n a_0$$

which implies

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} a_n\\
&= \lim_{n \longrightarrow \infty} [G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})] = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} G(gx_n, gx_{n+1}, gx_{n+1}) = 0,\\
&\lim_{n \longrightarrow \infty} G(gy_n, gy_{n+1}, gy_{n+1})] = 0.
\end{aligned} \tag{5}$$

Next, let us prove that $\{gx_n\}$ and $\{gy_n\}$ are $G-$Cauchy sequences. In fact, for $m > n$, we have

$$\begin{aligned}
&G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m)\\
&\le G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})\\
&+ G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + G(gy_{n+1}, gy_{n+2}, gy_{n+2})\\
&+ \ldots + G(gx_{m-1}, gx_m, gx_m) + G(gy_{m-1}, gy_m, gy_m)\\
&= a_n + a_{n+1} + \ldots + a_{m-1}\\
&\le \lambda^n a_0 + \lambda^{n+1} a_0 + \ldots + \lambda^{m-1} a_0 = (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) a_0\\
&\le \frac{\lambda^n}{1-\lambda} a_0.
\end{aligned}$$

Letting $n, m \longrightarrow \infty$, we have

$$\lim_{n,m \longrightarrow \infty} G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) = 0.$$

This imply that $\{gx_n\}$ and $\{gy_n\}$ are $G-$Cauchy sequences in $g(X)$. By $G-$completeness of $g(X)$, there exists $gx, gy \in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to gx and gy, respectively.
We claim that $g(x) = T(x,y)$ and $g(y) = T(y,x)$. Indeed, from (1), we have

$$\begin{aligned}
&G(gx_{n+1}, T(x,y), T(x,y)) = G(T(x_n, y_n), T(x,y), T(x,y))\\
&\le \frac{a_1}{2}[G(gx_n, g(x), g(x)) + G(gy_n, g(y), g(y))]\\
&+ \frac{a_2}{2}[G(gx_n, T(x_n, y_n), T(x_n, x_n)) + G(g(x), T(x,y), T(x,y))\\
&+ G(gy_n, g(y), g(y))] + \frac{a_3}{2}[G(gx_n, T(x,y), T(x,y))\\
&+ G(g(x), T(x_n, y_n), T(x_n, y_n) + G(gy_n, g(y), g(y)))]
\end{aligned}$$

Letting $n \longrightarrow \infty$, and using the fact that G is continuous on its variables, we get that

$$G(g(x), T(x,y), T(x,y)) \le \frac{a_2 + a_3}{2} G(g(x), T(x,y), T(x,y)).$$

Hence $g(x) = T(x,y)$. Similarly, we may show that $g(y) = T(y,x)$. Then, (gx, gy) is a coupled point of coincidence of mappings T and g.
Now we prove that $gx = gy$. By (1), we have

$$\begin{aligned}
&G(g(x), g(y), g(y)) = G(T(x,y), T(y,x), T(y,x))\\
&\le \frac{a_1}{2}[G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))]\\
&+ \frac{a_2}{2}[G(g(x), T(x,y), T(x,y)) + G(g(y), T(y,x), T(y,x))\\
&+ G(g(y), g(x), g(x))]\\
&+ \frac{a_3}{2}[G(g(x), T(y,x), T(y,x)) + G(g(y), T(x,y), T(x,y))\\
&+ G(g(y), g(x), g(x))]\\
&= \frac{a_1 + a_3}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_2 + 2a_3}{2} G(g(y), g(x), g(x)).
\end{aligned}$$

Similarly, we may show that

$$\begin{aligned}
&G(g(y), g(x), g(x))\\
&\le \frac{a_1 + a_2 + 2a_3}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_3}{2} G(g(y), g(x), g(x)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))\\
&\le \frac{2a_1 + a_2 + 3a_3}{2}[G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))]\\
&< G(g(x), g(y), g(y)) + G(g(y), g(x), g(x)).
\end{aligned}$$

which is a contradiction. So $g(x) = g(y)$. We conclude that $T(x,y) = g(x) = g(y) = T(y,x)$. Thus, $(g(x), g(x))$ is a coupled point of coincidence of mappings T and g. Now, if there is another $x_1 \in X$ such that $(g(x_1), g(x_1))$ is a coupled point of coincidence of mappings T and g, then

$$\begin{aligned}
&G(g(x),g(x_1),g(x_1))\\
&= G(T(x,x),T(x_1,x_1),T(x_1,x_1))\\
&\leq \frac{a_1}{2}[G(g(x),g(x_1),g(x_1))+G(g(x),g(x_1),g(x_1))]\\
&+\frac{a_2}{2}[G(g(x),T(x,x),T(x,x))+G(g(x),T(x_1,x_1),T(x_1,x_1))\\
&+G(g(x),g(x_1),g(x_1))]\\
&+\frac{a_3}{2}[G(g(x),T(x_1,x_1),T(x_1,x_1))\\
&+G(g(x_1),T(x,x),T(x,x))\\
&+G(g(x),g(x_1),g(x_1))]\\
&=(a_1+a_2+a_3)G(g(x),g(x_1),g(x_1))\\
&+\frac{a_3}{2}G(g(x_1),g(x),g(x)).
\end{aligned}$$

Similarly, we may show that

$$\begin{aligned}
&G(g(x_1),g(x),g(x)) \leq (a_1+a_2+a_3)G(g(x_1),g(x),g(x))\\
&+\frac{a_3}{2}G(g(x),g(x_1),g(x_1)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&G(g(x),g(x_1),g(x_1))+G(g(x_1),g(x),g(x))\\
&\leq \frac{2a_1+2a_2+3a_3}{2}[G(g(x),g(x_1),g(x_1))\\
&+G(g(x_1),g(x),g(x))].
\end{aligned}$$

It implies that $G(g(x),g(x_1),g(x_1)) = G(g(x_1),g(x),g(x)) = 0$ and so $g(x) = g(x_1)$. Hence, $(g(x),g(x))$ is a unique coupled point of coincidence of mappings T and g. Now, we show that T and g have common coupled fixed point. For this, let $u = g(x)$. Then, we have $u = g(x) = T(x,x)$. By w^*-compatibility of T and g, we have

$$g(u) = g(g(x)) = g(T(x,x)) = T(g(x),g(x)) = T(u,u).$$

Then, $(g(u),g(u))$ is a coupled point of coincidence of mappings T and g. By the uniqueness of coupled point of coincidence, we have $g(x) = g(u)$. Therefore, (u,u) is the common coupled fixed point of T and g. To prove the uniqueness, let $v \in X$ with $v \neq u$ such that (v,v) is the common coupled fixed point of T and g.
Then, using (1),

$$\begin{aligned}
&G(u,v,v) = G(T(u,u),T(v,v),T(v,v))\\
&\leq \frac{a_1}{2}[G(gu,gv,gv)+G(gu,gv,gv)]\\
&+\frac{a_2}{2}[G(gu,T(u,u),T(u,u))+G(gv,T(v,v),T(v,v))\\
&+G(gu,gv,gv)]\\
&+\frac{a_3}{2}[G(gu,T(v,v),T(v,v))+G(gv,T(u,u),T(u,u))\\
&+G(gu,gv,gv)]\\
&=(a_1+\frac{a_2}{2}+a_3)G(u,v,v)+\frac{a_3}{2}G(v,u,u).
\end{aligned}$$

Similarly, we may show that

$$G(v,u,u) \leq (a_1+\frac{a_2}{2}+a_3)G(v,u,u)+\frac{a_3}{2}G(u,v,v).$$

Hence,

$$G(u,v,v)+G(v,u,u) \leq \frac{2a_1+a_2+3a_3}{2}[G(u,v,v)+G(v,u,u)].$$

Since $\frac{2a_1+a_2+3a_3}{2} < 1$, so that $G(u,v,v) = G(v,u,u) = 0$ and $u = v$. Thus T and g have a unique common coupled fixed point. In Theorem 2.1, take $w = u$ and $z = v$, to obtain the following corollary.

Corollary 22*Let (X,G) be a $G-$metric space. Set $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. Assume there exist $a_1,a_2,a_3 \geq 0$ with $2a_1+3a_2+3a_3 < 2$ such that*

$$\begin{aligned}
&G(T(x,y),T(u,v),T(u,v))\\
&\leq \frac{a_1}{2}[G(gx,gu,gu)+G(gy,gv,gv)]\\
&+\frac{a_2}{2}[G(gx,T(x,y),T(x,y))+G(gu,T(u,v),T(u,v))\\
&+G(gy,gv,gv)]\\
&+\frac{a_3}{2}[G(gx,T(u,v),T(u,v))+G(gu,T(x,y),T(x,y))\\
&+G(gy,gv,gv)],
\end{aligned} \tag{6}$$

for all $x,y,u,v,w,z \in X$. If $T(X\times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Now, putting $g = I_X$ (the identity map of X)in the Theorem 2.1, we obtain

Corollary 23*Let (X,G) be a complete $G-$metric space. Assume $T : X\times X \longrightarrow X$ be a function satisfying (1)(with $g = I_X$)for all $x,y,u,v,w,z \in X$. Then T has a unique fixed point.*

By choosing a_1,a_2 and a_3 suitably, one can deduce some corollaries from Theorem 2.1.
For example, if $a_1 = 2k$ and $a_2 = a_3 = 0$ in Theorem 2.1, then the following corollary is obtained which extends and generalizes the comparable results of[10].

Corollary 24*Let (X,G) be a $G-$metric space. Set $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. Assume there exist $k \in [0,\frac{1}{2})$ such that*

$$\begin{aligned}
&G(T(x,y),T(u,v),T(w,z))\\
&\leq k[G(gx,gu,gw)+G(gy,gv,gz)],
\end{aligned} \tag{7}$$

for all $x,y,u,v,w,z \in X$. If $T(X\times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Theorem 25*Let (X,G) be a complete $G-$metric space. Assume that $T : X\times X \longrightarrow X$ and $g : X \longrightarrow X$ are continuous and there exists $k \in [0,\frac{2}{3})$ and $h \in M(x,y,u,v,w,z)$ satisfying*

$$G(T(x,y),T(u,v),T(w,z)) \leq kh, \tag{8}$$

for all $x,y,u,v,w,z \in X$ where

$$M(x,y,u,v,w,z) = \{\frac{G(gx,gu,gw)+G(gy,gv,gz)}{2}, \frac{1}{2}G(gx,T(x,y),T(x,y))+\frac{1}{2}G(gu,T(u,v),T(u,v)) + \frac{1}{2}G(gy,gv,gz), \frac{1}{2}G(gx,T(u,v),T(u,v))+\frac{1}{2}G(gu,T(x,y),T(x,y)) + \frac{1}{2}G(gy,gv,gz)\}.$$

If $T(X\times X)\subseteq g(X)$ and g commutes with T, then T and g have a coupled coincidence point.

Proof. Let $gx_{n+1} = T(x_n,y_n)$ and $gy_{n+1} = T(y_n,x_n)$ for all $n \geq 0$. For each $n \in \mathbb{N}$, there exists

$$h_n \in \{\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}, \frac{1}{2}G(gx_{n-1},T(x_{n-1},y_{n-1}),T(x_{n-1},y_{n-1})) + \frac{1}{2}G(gx_n,T(x_n,y_n),T(x_n,y_n)) + \frac{1}{2}G(gy_{n-1},gy_n,gy_n), \frac{1}{2}G(gx_{n-1},T(x_n,y_n),T(x_n,y_n)) + \frac{1}{2}G(gx_n,T(x_{n-1},y_{n-1}),T(x_{n-1},y_{n-1})) + \frac{1}{2}G(gy_{n-1},gy_n,gy_n)\}$$
$$= \{\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}, \frac{1}{2}G(gx_{n-1},gx_n,gx_n)+\frac{1}{2}G(gx_n,gx_{n+1},gx_{n+1}) + \frac{1}{2}G(gy_{n-1},gy_n,gy_n), \frac{1}{2}G(gx_{n-1},gx_{n+1},gx_{n+1})+\frac{1}{2}G(gx_n,gx_n,gx_n) + \frac{1}{2}G(gy_{n-1},gy_n,gy_n)\},$$

such that

$$G(gx_n,gx_{n+1},gx_{n+1}) = G(T(x_{n-1},y_{n-1}),T(x_n,y_n),T(x_n,y_n)) \leq kh_n.$$

Now, we consider three cases:

1.If $h_n = \frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}$, then

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq k\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2} \leq \frac{2k}{2-k}\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2};$$

2.$h_n = \frac{G(gx_{n-1},gx_n,gx_n)+G(gx_n,gx_{n+1},gx_{n+1})+G(gy_{n-1},gy_n,gy_n)}{2}$, then

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq \frac{k}{2}G(gx_n,gx_{n+1},gx_{n+1}) + k\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2},$$

which gives that

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq \frac{2k}{2-k}\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2};$$

3.$h_n = \frac{G(gx_{n-1},gx_{n+1},gx_{n+1})+G(gx_n,gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}$, then by case 2, we also have

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq \frac{2k}{2-k}\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2};$$

since

$$G(gx_{n-1},gx_{n+1},gx_{n+1}) \leq G(gx_{n-1},gx_n,gx_n)+G(gx_n,gx_{n+1},gx_{n+1}).$$

Thus, we have

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq \frac{2k}{2-k}\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}, \quad \forall n\in\mathbb{N}.$$

By a similar proof, one can also show that

$$G(gy_n,gy_{n+1},gy_{n+1}) \leq \frac{2k}{2-k}\frac{G(gx_{n-1},gx_n,gx_n)+G(gy_{n-1},gy_n,gy_n)}{2}, \quad \forall n\in\mathbb{N}.$$

Then, we conclude that

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq (\frac{2k}{2-k})^n\frac{G(gx_0,gx_1,gx_1)+G(gy_0,gy_1,gy_1)}{2}, \quad \forall n\in\mathbb{N},$$

and

$$G(gy_n,gy_{n+1},gy_{n+1}) \leq (\frac{2k}{2-k})^n\frac{G(gx_0,gx_1,gx_1)+G(gy_0,gy_1,gy_1)}{2}, \quad \forall n\in\mathbb{N}.$$

It follows from $0 \leq k < \frac{2}{3}$ that $0 \leq \frac{2k}{2-k} < \frac{2}{3}$. Then, analogously to the corresponding proof of Theorem 2.1, $\{gx_n\}$ and $\{gy_n\}$ are $G-$Cauchy sequences in the $G-$metric space (X,G),which is complete. Then, there are $x,y \in X$ such that $\{gx_n\}$ and $\{gy_n\}$ are respectively $G-$convergent to x and y. From Proposition 1.1, we have

$$\lim_{n\longrightarrow\infty} G(gx_n,gx_n,x) = \lim_{n\longrightarrow\infty} G(gx_n,x,x) = 0, \tag{9}$$

$$\lim_{n\longrightarrow\infty} G(gy_n,gy_n,y) = \lim_{n\longrightarrow\infty} G(gy_n,y,y) = 0. \tag{10}$$

From (9), (10) and the continuity of g, we have

$$\lim_{n\longrightarrow\infty} G(g(gx_n),g(gx_n),g(x))$$
$$= \lim_{n\longrightarrow\infty} G(g(gx_n),g(x),g(x)) = 0,$$
$$\lim_{n\longrightarrow\infty} G(g(gy_n),g(gy_n),g(y))$$
$$= \lim_{n\longrightarrow\infty} G(g(gy_n),g(y),g(y)) = 0.$$

Since $gx_{n+1} = T(x_n,y_n)$ and $gy_{n+1} = T(y_n,x_n)$, the commutativity of T and g yields that

$$g(gx_{n+1}) = g(T(x_n,y_n)) = T(gx_n,gy_n) \tag{11}$$

$$g(gy_{n+1}) = g(T(y_n,x_n)) = T(gy_n,gx_n). \tag{12}$$

By using (11), (12) and the continuity of T, we get $\{g(gx_{n+1})\}$ is $G-$convergent to $T(x,y)$ and $\{g(gy_{n+1})\}$ is $G-$convergent to $T(y,x)$. By uniqueness of the limit,we have $gx = T(x,y)$ and $gy = T(y,x)$, and this ends the proof.
Now, putting $g = I_X$ (the identity map of X)in the previous result, we obtain

Corollary 26*Let* (X,G) *be a complete* $G-$*metric space. Assume that* $T : X\times X \longrightarrow X$ *is continuous and there exists* $k \in [0,\frac{2}{3})$ *and* $h \in M(x,y,u,v,w,z)$ *satisfying*

$$G(T(x,y),T(u,v),T(w,z)) \le kh,$$

for all $x,y,u,v,w,z \in X$ *where*

$$M(x,y,u,v,w,z) = \{\frac{G(x,u,w)+G(y,v,z)}{2},$$
$$\frac{1}{2}G(x,T(x,y),T(x,y))+\frac{1}{2}G(u,T(u,v),T(u,v))$$
$$+\frac{1}{2}G(y,v,z),$$
$$\frac{1}{2}G(x,T(u,v),T(u,v))+\frac{1}{2}G(u,T(x,y),T(x,y))$$
$$+\frac{1}{2}G(y,v,z)\}.$$

Then T *has a coupled fixed point.*

Now, we introduce an example to support the usability of our results. **Example.**Let $X = [0,1]$. Define $T : X\times X \longrightarrow X$ by $T(x,y) = \frac{1}{16}x^2y^2$ and define $g : X \longrightarrow X$ by $g(x) = \frac{1}{2}x^2$. Define a $G-$metric on X by $G(x,y,z) = |x-y|+|x-z|+|y-z|$ for all $x,y,z \in X$.
By routine calculations, the reader can easily verify that the following assumptions hold:

(1)$T(X\times X) \subseteq g(X)$;
(2)$g(X)$ is a $G-$complete subset of X;
(3)T is w^*-compatible with g.

Here, we show only that T and g are condition (1) in Theorem 2.1 is satisfied for all real numbers a_1,a_2,a_3 with $0 \le 2a_1 + 3a_2 + 3a_3 < 2$. Since $|xy-uv| \le |x-u|+|y-v|$ holds for all $x,y,u,v \in X$, we have

$$G(T(x,y),T(u,v),T(w,z))$$
$$= G(\frac{1}{16}x^2y^2,\frac{1}{16}u^2v^2,\frac{1}{16}w^2z^2)$$
$$= \frac{1}{16}|x^2y^2-u^2v^2|+\frac{1}{16}|x^2y^2-w^2z^2|$$
$$+\frac{1}{16}|u^2v^2-w^2z^2|$$
$$\le \frac{1}{16}[|x^2-u^2|+|y^2-v^2|+|x^2-w^2|+|y^2-z^2|+|u^2-w^2|$$
$$+|v^2-z^2|]$$
$$\le \frac{1}{16}[|x^2-u^2|+|y^2-v^2|+|x^2-w^2|+|y^2-z^2|+|u^2-w^2|$$
$$+|v^2-z^2|$$
$$+|x^2-\frac{1}{8}x^2y^2|+|x^2-\frac{1}{8}u^2v^2|+|u^2-\frac{1}{8}u^2v^2|+|u^2-\frac{1}{8}x^2y^2|]$$
$$\le \frac{1}{8}[|\frac{1}{2}x^2-\frac{1}{2}u^2|+|\frac{1}{2}x^2-\frac{1}{2}w^2|+|\frac{1}{2}u^2-\frac{1}{2}w^2|$$
$$+|\frac{1}{2}y^2-\frac{1}{2}v^2|+|\frac{1}{2}y^2-\frac{1}{2}z^2|+|\frac{1}{2}v^2-\frac{1}{2}z^2|]$$
$$+\frac{1}{16}[|\frac{1}{2}x^2-\frac{1}{16}x^2y^2|+|\frac{1}{2}u^2-\frac{1}{16}u^2v^2|+|\frac{1}{2}y^2-\frac{1}{2}v^2|$$
$$+|\frac{1}{2}y^2-\frac{1}{2}z^2|+|\frac{1}{2}v^2-\frac{1}{2}z^2|]+\frac{1}{16}[|\frac{1}{2}x^2-\frac{1}{16}u^2v^2|$$
$$+|\frac{1}{2}u^2-\frac{1}{16}x^2y^2|+|\frac{1}{2}y^2-\frac{1}{2}v^2|+|\frac{1}{2}y^2-\frac{1}{2}z^2|$$
$$+|\frac{1}{2}v^2-\frac{1}{2}z^2|]$$
$$\le \frac{\frac{1}{4}}{2}[G(gx,gu,gw)+G(gy,gv,gz)]$$
$$+\frac{\frac{1}{8}}{2}[G(gx,T(x,y),T(x,y))$$
$$+G(gu,T(u,v),T(u,v))+G(gy,gv,gz)]$$
$$+\frac{\frac{1}{8}}{2}[G(gx,T(u,v),T(u,v))$$
$$+G(gu,T(x,y),T(x,y))+G(gy,gv,gz)].$$

Thus, (1) is satisfied with $a_1 = \frac{1}{4}$ and $a_2 = a_3 = \frac{1}{8}$ where $2a_1+3a_2+3a_3 < 2$. Hence, all the conditions of Theorem 2.1 are satisfied. Moreover, $(0,0)$ is the unique common coupled fixed point of T and g.

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