Some linear isometric operators on the Dirichlet space

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Abstract: We study some isometric operators, including composition, Volterra type and Toeplitz operators on several different Dirichlet spaces. Besides, the isometric equivalence of Toeplitz operators on Dirichlet space is also characterized.

Keywords: Dirichlet space; isometry; composition operators; Volterra type operator; Toeplitz operator.

1. Introduction

Let be the unit disk in the complex plane C, and $S()$ be the set of holomorphic self-maps of . The algebra of all holomorphic functions on domain will be denoted by $H()$ and the set of all holomorphic automorphisms on will be denoted by $Aut()$. Let ^{n} be the unit poly-disk in C^n, and $S(^n)$ be the set of all holomorphic self-maps of ^{n}.

At the beginning, we introduce a generalized Dirichlet space with respect to a general positive measure on the open unit disk. If μ is a positive measure on , let the notation $\mathcal{D}_2(\mu)$ denote the space of holomorphic functions on with

$$\|f\|_\mu = \left(|f(0)|^2 + \int_D |f'(z)|^2 \, d\mu(z) \right)^{1/2} < \infty,$$

and we call it a Dirichlet space with respect to μ. It is clear that $\mathcal{D}_2(\mu)$ is a Hilbert space, and its inner product induced by its norm has the following formula:

$$\langle f, g \rangle_\mu = f(0)\overline{g(0)} + \int_D f'(z)\overline{g'(z)} d\mu(z).$$

The Sobolev space of order 1 on the unit poly-disk is denoted by $\mathcal{W}_2^1$, and it is a Hilbert space which is the completed space of smooth functions f on D^n for which

$$\|f\|^2 = \left| \int_{D^n} f dv \right|^2$$
$$+ \sum_{i=1}^n \int_{D^n} \left(\left| \frac{\partial f}{\partial z_i} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}_i} \right|^2 \right) dv < \infty.$$

Here, $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i} \right)$ and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i})$. Let $\langle \cdot, \cdot \rangle$ denote the inner product of $\mathcal{W}_2^1$ induced by its norm as follow

$$\langle f, g \rangle = \int_{D^n} f dv \int_{D^n} \bar{g} dv$$
$$+ \sum_{i=1}^n \int_{D^n} \left(\frac{\partial f}{\partial z_i} \frac{\overline{\partial g}}{\partial z_i} + \frac{\partial f}{\partial \bar{z}_i} \frac{\overline{\partial g}}{\partial \bar{z}_i} \right) dv.$$

Recall that the Dirichlet space on the poly-disk $\mathcal{D}$ is a subspace of $\mathcal{W}_2^1$. Each point evaluation is easily verified to be a bounded linear functional on $\mathcal{D}$.

In this paper, the notation $\mathcal{D}_2$ is applied to denote the Dirichlet space in the unit disk . And the norm on $\mathcal{D}_2$ is as the following:

$$\|f\| = \{ |f(0)|^2 + \int |f'(z)|^2 \, dA(z) \}^{1/2}.$$

Then the inner product on $\mathcal{D}_2$ is defined as:

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int f' \bar{g}' dA(z).$$

For each $z \in$, there exists an unique reproducing kernel $R_z \in \mathcal{D}_2$ which has the reproducing property

$$f(z) = \langle f, R_z \rangle$$

for each $f \in \mathcal{D}_2$. It is well known that the formula of the reproducing kernel R_z is given by

$$R_z(w) = 1 + \log(\frac{1}{1 - \bar{z}w}) \quad (w \in).$$

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Readers interested in the topic of holomorphic function spaces can refer to the books [16] and [17].

Let P be the orthogonal projection from $\mathcal{W}_2^1$ onto $\mathcal{D}$, and

$$\Omega = \left\{ u \in \mathcal{C}^1(D^n) : u, \frac{\partial u}{\partial z_i}, \frac{\partial u}{\partial \bar{z}_i} \in L^\infty(D^n, dv) \right\},$$

where $i = 1, \cdots, n$. Given $u \in \Omega$, the Toeplitz operator T_u with symbol u is a linear operator on $\mathcal{D}$ which is defined by

$$T_u f = P(uf) \tag{1}$$

for functions $f \in \mathcal{D}$. It is easy to show that the Toeplitz operator $T_u : \mathcal{D} \to \mathcal{D}$ is always bounded whenever $u \in \Omega$.

Let $\psi \in H(D)$ and $\varphi \in S(D)$, the composition operator C_φ induced by φ is defined as $C_\varphi f = f \circ \varphi$ for $f \in H(D)$; the multiplication operator induced by ψ is defined by $M_\psi f(z) = \psi(z) f(z)$. The *Volterra operator* J_ψ is defined by

$$J_\psi f(z) = \int_0^z f(\zeta) \psi'(\zeta) d\zeta;$$

and another integral operator, named *co-Volterra operator*, I_ψ is defined by

$$I_\psi f(z) = \int_0^z f'(\zeta) \psi(\zeta) d\zeta$$

where $z \in$ and $f \in H()$. The operator J_ψ is actually a generalization of the integral operator (when $\psi(z) = z$). The operators J_ψ and I_ψ are close companions as a consequence of their relations to the multiplication operator $M_\psi f(z) = \psi(z) f(z)$. To see this, note that integration by parts gives

$$M_\psi f = f(0) \psi(0) + J_\psi f + I_\psi f.$$

A linear operator T on a normed space X is said to be an isometric linear operator if $\|Tf\|_X = \|f\|_X$, for any $f \in X$. Let X and Y be two Banach spaces and T_1 and T_2 are bounded linear operators on X and Y respectively. We say that T_1 and T_2 are isometrically equivalent if there exists surjective isometries U_X and U_Y on X and Y respectively such that $U_X T_1 = T_2 U_Y$. For $Y = X$, two operators T_1 and T_2 are said to be similar if there is a bounded invertible operator S on X such that $S T_2 = T_1 S$. If S could be chosen to be an isometry as well, then T_1 and T_2 are said to be isometrically isomorphic.

The isometric composition operator on analytic functions spaces has been studied by many authors. In [12,7], the authors characterized the isometric composition operator on several classic reflexive spaces just like Hardy space, weighted Bergman space and analytic Besov spaces, respectively. The isometric composition operators on the Bloch spaces both in the unit disk, unit poly-disk and unit ball were discussed in [14,3,1,2,11,4,9,10]. Recently, isometric problems on the spaces of BMOA and Dirichlet spaces are concerned in [8,13].

Isometric equivalence of composition operators on Hardy space and several Banach spaces of analytic functions spaces are investigate in [15,5] and [6], resp..

Base on these foundations, we firstly characterize isometric composition operators on generalized Dirichlet spaces in the unit disk. Secondly, we study isometric composition and Volterra type operators on classic Dirichlet space in the unit disk. Then, we investigate the isometric equivalence of Toeplitz operators on classic Dirichlet space. At the end of the paper, isometric composition operators on Dirichlet space in polydisk are also studied to some degree.

2. Isometry on generalized Dirichlet space

Let ν_0 be a positive Borel measure on the unit interval $[0, 1]$, another positive measure ν on is called a *radial measure* with respect to ν_0 if $d\nu = d\nu_0(r) d\theta / 2\pi$ where $d\theta$ represents the Lebesgue measure of ∂.

Suppose that ν_0 is a positive Borel measure on $[0, 1]$, a real sequence $\{a_n\}$ is called a *moment sequence* of ν_0 if $a_n = \int_0^1 r^n d\nu_0$ for $n = 0, 1, 2, \ldots$.

Let ν_0 be a positive Borel measure on $[0, 1]$ and $\varphi \in S()$ with $\|\varphi\|_\infty = 1$. If positive measure ν is a radial measure with respect to ν_0, and μ is another positive measure on , then the composition operator C_φ is isometric from $\mathcal{D}_2(\nu)$ into $\mathcal{D}_2(\mu)$ if and only if

$$\int_D \phi^n \overline{\phi^m} |\phi'|^2 d\mu = 0 \quad \text{for} \quad \forall n \neq m,$$

and the sequence $\{\int_D |\phi|^n |\phi'|^2 d\mu\}$ is a moment sequence of ν_0.

Proof. Firstly, assume that C_ϕ is an isometric composition operator from $\mathcal{D}_2(\nu)$ into $\mathcal{D}_2(\mu)$, then

$$\langle z^n, z^m \rangle_\nu = \langle C_\phi z^n, C_\phi z^m \rangle_\mu. \tag{2}$$

Putting $n = 1$ and $m = 0$ in equation e1, we have $\phi(0) = 0$. Hence

$$\int_D (z^n)' \overline{(z^m)'} d\nu = \int_D (\phi^n)' \overline{(\phi^m)'} d\mu.$$

By the polarization formula, we have the following two items:

(1) $\int_D |\phi'|^2 d\mu = \int_0^1 d\nu_0(r)$ when $n = m = 1$;

(2)

$$nm \delta_{(n-1)(m-1)} \int_0^1 r^{(n-1)} r^{(m-1)} d\nu_0(r)$$
$$= \int_D (z^n)' \overline{(z^m)'} d\nu = \int_D (\phi^n)' \overline{(\phi^m)'} d\mu$$

when $n > 1$ and $m > 1$.

So we have

$$\int_D |(\phi^n)'|^2 \, d\mu = n^2 \int_0^1 r^{2(n-1)} d\nu_0 \quad (n = 1, 2, 3, \cdots),$$

that is

$$\int_D |\phi^{n-1}|^2 |\phi'|^2 \, d\mu = \int_0^1 r^{2(n-1)} d\nu_0 \quad (n = 1, 2, 3, \cdots).$$

For $0 \leq x \leq 1$, we have

$$x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n$$

and

$$\sum_{n=0}^{\infty} |a_n| (1 - x^2)^n < \infty \quad (0 \leq x \leq 1).$$

Since

$$\left| \sum_{n=0}^{k} a_n (1 - |\phi|^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n|$$

and

$$\left| \sum_{n=0}^{k} a_n (1 - r^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n| < \infty,$$

by the dominated convergence theorem, we have

$$\int_D |\phi| |\phi'|^2 \, d\mu = \int_D \sum_{n=0}^{\infty} a_n (1 - |\phi|^2)^n |\phi'|^2 \, d\mu$$

$$= \sum_{n=0}^{\infty} a_n \int_D (1 - |\phi|^2)^n |\phi'|^2 \, d\mu$$

$$= \sum_{n=0}^{\infty} a_n \int_0^1 (1 - r^2)^n d\nu_0$$

$$= \int_0^1 \sum_{n=0}^{\infty} a_n (1 - r^2)^n d\nu_0 = \int_0^1 r \, d\nu_0.$$

Similarly, because $x^{2l+1} = \sqrt{1 - (1 - x^{4l+2})}$, we can get $\int_D |\phi|^{2n+1} |\phi'|^2 \, d\mu = \int_0^1 r^{2n+1} d\nu_0 \quad (n = 0, 1, 2, \cdots)$. Thus $\{\int_D |\phi|^n |\phi'|^2 \, d\mu\}$ is a moment sequence of ν_0.

Conversely if $\int_D \phi^n \overline{\phi^m} |\phi'|^2 \, d\mu = 0$ for $\forall n \neq m$, and $\{\int_D |\phi|^n |\phi'|^2 \, d\mu\}$ is a moment sequence of ν_0, then

$$\int_D \left| \sum_{n=0}^{k} a_n \phi^n \right|^2 |\phi'|^2 \, d\mu$$

$$= \sum_{n=0}^{k} |a_n|^2 \int_D |\phi^n|^2 |\phi'|^2 \, d\mu$$

$$= \sum_{n=0}^{k} |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_D \left| \sum_{n=0}^{k} a_n z^n \right|^2 d\nu,$$

that is, the operator C_ϕ is an isometric operator.

3. Isometric operators on $\mathcal{D}_2$

In this section, our discussion starts from the isometric property for the product of composition and Volterra type operators.

Let $\varphi \in S()$ and $\psi \in H()$. Then neither the products of composition and Volterra type operators $C_\varphi I_\psi$ and $C_\varphi J_\psi$, nor the products of Volterra type and composition operators $I_\psi C_\varphi$ and $J_\psi C_\varphi$ can be isometric on Dirichlet space $\mathcal{D}_2$.

Proof. We here just prove the front half of the theorem, and it is similar for the back half. Note that the Dirichlet norm of $f \in \mathcal{D}_2$ has the following formula:

$$\|f\|_{\mathcal{D}_2}^2 = |f(0)|^2 + \int_| f'(z)|^2 dA(z). \tag{3}$$

We assume the operator $C_\varphi I_\psi$ to be isometric on $\mathcal{D}_2$. And making $C_\varphi I_\psi$ acting on a holomorphic function $f \in \mathcal{D}_2$, we have

$$C_\varphi I_\psi(f)(z) = \int_0^{\varphi(z)} f'(\zeta)\psi(\zeta)d\zeta.$$

Next we take the Dirichlet norm of the operator $C_\varphi I_\psi(f)$,

$$\|C_\varphi I_\psi f\|_{\mathcal{D}_2}^2 = \left| \int_0^{\varphi(0)} f'(\zeta)\psi(\zeta)d\zeta \right|^2$$

$$+ \int_| |f'(\varphi(z))\psi(\varphi(z))\varphi'(z)|^2 dA(z). \tag{4}$$

The formulas V1 and V2 are equal by the assumption on isometry of $C_\varphi I_\psi$. By putting $f \equiv 1$ in both V1 and V2, we have V1= 1 and V2= 0. That is a contradiction, so the operator $C_\varphi I_\psi$ can not be isometric.

Now assume that $C_\varphi J_\psi$ is isometric on $\mathcal{D}_2$. By making $C_\varphi J_\psi$ acting on $f \in \mathcal{D}_2$, we obtain

$$C_\varphi J_\psi(f)(z) = \int_0^{\varphi(z)} f(\zeta)\psi'(\zeta)d\zeta,$$

and then take its Dirichlet norm,

$$\|C_\varphi J_\psi f\|_{\mathcal{D}_2}^2 = \left| \int_0^{\varphi(0)} f(\zeta)\psi'(\zeta)d\zeta \right|^2$$

$$+ \int_| |f(\varphi(z))\psi'(\varphi(z))\varphi'(z)|^2 dA(z). \tag{5}$$

Similarly, formulas V1 and V3 are equal from the isometric assumption of $C_\varphi J_\psi$. Putting $f \equiv 1$ in both V3 and V1, we have

$$\left| \int_0^{\varphi(0)} \psi'(\zeta)d\zeta \right|^2 + \int_| |\psi'(\varphi(z))\varphi'(z)|^2 dA(z) = 1. \tag{6}$$

Let $f(z) = z^n$ for $n = 1, 2, \ldots$ and $z \in$ in both V3 and V1, then

$$\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|^2$$
$$+ \int_{|\varphi^n(z)\psi'(\varphi(z))\varphi'(z)|^2 dA(z)}$$
$$= 0 + \int_{|nz^{n-1}|^2 dA(z)}$$
$$= n^2 \int_0^{2\pi} d\theta \int_0^1 r^{2(n-1)} \cdot r dr = n\pi \to \infty,$$

as $n \to \infty$. But from the formula of V5, it is obvious that

$$\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|^2$$
$$+ \int_{|\varphi^n(z)\psi'(\varphi(z))\varphi'(z)|^2 dA(z)}$$
$$\leq \left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|^2 + \int_{|\psi'(\varphi(z))\varphi'(z)|^2 dA(z)}$$
$$= \left(\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|^2 - \left| \int_0^{\varphi(0)} \psi'(\zeta) d\zeta \right|^2 \right)$$
$$+ \left| \int_0^{\varphi(0)} \psi'(\zeta) d\zeta \right|^2 + \int_{|\psi'(\varphi(z))\varphi'(z)|^2 dA(z)}$$
$$= \left(\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|^2 - \left| \int_0^{\varphi(0)} \psi'(\zeta) d\zeta \right|^2 \right) + 1$$
$$\to \infty.$$

Hence $\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right| \to \infty$ as $n \to \infty$. Considering that the line integral is independent on the integration path, we assume that the line integral between 0 and $\varphi(0)$ is taken along the straight complex line from 0 to $\varphi(0)$, and here we denote this straight line by $[0, \varphi(0)]$ and the notation $L[0, \varphi(0)]$ as its Lebesgue length. So

$$\left| \int_0^{\varphi(0)} \zeta^n \psi'(\zeta) d\zeta \right|$$
$$\leq \int_0^{\varphi(0)} |\zeta|^n |\psi'(\zeta)| |d\zeta|$$
$$\leq \sup_{z \in D_{|\varphi(0)|}} |\psi'(z)| \cdot L[0, \psi(0)] < \infty,$$

where D_r represents the closed disc with the origin as its center and r as its radius. Thence this contradiction implies that the isometric assumption on $C_\varphi J_\psi$ on $\mathcal{D}_2$ must be false.

The proof of last two negative isometric assertion is left to the interested readers.

In the remaining text of this section, we will look into the properties of quasi-isometric composition operator. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called a quasi-isometry if $T^{*2}T^2 = T^*T$. We will characterize the quasi-isometric composition operator on $\mathcal{D}_2$ in the following theorem. Let $\varphi \in S()$, then C_φ is a quasi-isometry on $\mathcal{D}_2$ if and only if φ is a rotation.

Proof. Since the composition operator induced by a rotation is clearly a quasi-isometry, it suffices to show that if C_φ is a quasi-isometry, then φ is a rotation.

Suppose that the composition operator C_φ is quasi-isometry. By the definition of quasi-isometry, we have

$$\langle C_\varphi^{*2} C_\varphi^2 f, g \rangle_{\mathcal{D}_2} = \langle C_\varphi^* C_\varphi f, g \rangle_{\mathcal{D}_2}$$

for $\forall f, g \in \mathcal{D}_2$. That is

$$\langle C_\varphi^2 f, C_\varphi^2 g \rangle_{\mathcal{D}_2} = \langle C_\varphi f, C_\varphi g \rangle_{\mathcal{D}_2} \tag{7}$$

If we put $f = \mathrm{id}, g \equiv 1$ in 1, then $\varphi(\varphi(0)) = \varphi(0)$, that is $\varphi(0) = 0$.

Put $f(z) = g(z) = z^n$ for $\forall z \in$ in 1, then

$$\langle \varphi^n \circ \varphi, \varphi^n \circ \varphi \rangle_{\mathcal{D}_2} = \langle \varphi^n, \varphi^n \rangle_{\mathcal{D}_2}.$$

From the definition of inner product on $\mathcal{D}_2$, we get

$$\int_D |\varphi^n \circ \varphi|^2 |\varphi'|^2 dA(z) = \int_D |\varphi^n|^2 |\varphi'|^2 dA(z)$$

that is

$$\int_D (|\varphi^n \circ \varphi|^2 - |\varphi^n|^2) |\varphi'|^2 dA(z) = 0.$$

From Schwarz lemma, we obtain that

$$|\varphi^n \circ \varphi|^2 - |\varphi^n|^2 = 0. \tag{8}$$

Noting that $\varphi \in S()$ and $\varphi(0) = 0$, suppose $\sum_{n=1}^{\infty} a_n z^n$ is the Taylor expansion of $\varphi(z)$, and we put it in equation 2, then $a_1 = \lambda$, $|\lambda| = 1$, $a_n = 0$ for $n \geq 2$, that is φ is a rotation.

If φ is a rotation of the open unit disk, then C_φ is an isometry. Hence if C_φ is a quasi-isometry on $\mathcal{D}_2$, then C_φ is an isometry on $\mathcal{D}_2$.

4. Isometric equivalence of Toeplitz operators

In this section, we will study the isometric equivalence of two Toeplitz operators on the Dirichlet space of the disk. Let $f, g \in \mathcal{W}_2^1()$, if two Toeplitz operators T_f and T_g are isometric equivalent on $\mathcal{D}_2$, then there exists a constant γ with modular 1 and an automorphism σ of the unit disk such that

$$f(\sigma(z)) - f(\sigma(0)) = \gamma(g(z) - g(0)).$$

*Proof.*Suppose λ and κ are two complex numbers with modular 1 and σ is an automorphism of the disk, we define an integral operator Q on $h \in \mathcal{W}_2^1()$ as the following:

$$Qh(z) = \lambda h(0) + \kappa \int_0^z h'(\sigma(\xi))\sigma'(\xi)d\xi$$

for any $h \in \mathcal{W}_2^1()$. It is easy to see that Q is a surjective isometry on $\mathcal{W}_2^1()$ and $Q\mathcal{D}_2 = \mathcal{D}_2$. Suppose P is the Bergman projection from $\mathcal{W}_2^1()$ to $\mathcal{D}_2$, then we have

$$h = (h - Ph) + Ph, \text{ and}$$

$$P(h - Ph) = Ph - PPh = Ph - Ph = 0$$

for any $h \in \mathcal{W}_2^1()$. Hence

$$PQh = P[Q(h - Ph) + QPh] = PQPh = QPh.$$

Then it follows from the above equation that $PQ = QP$.

For $h \in \mathcal{W}_2^1$, we have

$$QT_f h = QP(fh) = PQ(fh)$$

$$= P(\lambda f(0)h(0) + \kappa \int_0^z (f'h + h'f)(\sigma(\xi))\sigma'(\xi)d\xi)$$

and

$$T_g Qh = T_g(\lambda h(0) + \kappa \int_0^z h'(\sigma(\xi))\sigma'(\xi)d\xi)$$

$$= P(g\lambda h(0) + \kappa g \int_0^z h'(\sigma(\xi))\sigma'(\xi)d\xi).$$

Suppose two Toeplitz operators T_f and T_g are isometric equivalent on $\mathcal{D}_2$, then we have $QT_f h = T_g Qh$, where Q is the surjective isometry on $\mathcal{D}_2$ as above and h is an arbitrary function in $\mathcal{D}_2$. From the formula of Q, we have the following equation

$$\lambda f(0)h(0) + \kappa \int_0^z (f'h + h'f)(\sigma(\xi))\sigma'(\xi)d\xi \qquad (9)$$

$$= g\lambda h(0) + \kappa g \int_0^z h'(\sigma(\xi))\sigma'(\xi)d\xi \qquad (10)$$

Let $h \equiv 1$ in to, then

$$\lambda f(0) + \kappa \int_0^z f'(\sigma(\xi))\sigma'(\xi)d\xi = \lambda g(z).$$

Taking derivative of both sides, we get $\kappa f'(\sigma(z))\sigma'(z) = \lambda g'$. Hence

$$\kappa(f(\sigma(z)) - f(\sigma(0))) = \lambda(g(z) - g(0)).$$

Since λ and κ are two complex numbers with modular 1, there exists a complex number γ with modular 1 such that $f(\sigma(z)) - f(\sigma(0)) = \gamma(g(z) - g(0))$.

5. Isometric composition operators on $\mathcal{D}$

Recall that a self-map φ of is said to be a univalent full map if it is one-to-one and $A[\backslash\varphi()] = 0$. In the following, we will characterize the isometric composition operator on Dirichlet space on the poly-disk. Let $z = (z_1, z_2, \ldots, z_n) \in^n$ and τ be a permutation of $\{1, \ldots, n\}$. For $j = 1, \ldots, n$, suppose that $\varphi_j \in S()$, $\varphi_j(0) = 0$ and all φ_j's are univalent full maps on . Denote φ by $\varphi(z) = (\varphi_1(z_{\tau(1)}), \ldots, \varphi_n(z_{\tau(n)}))$. Then C_φ is isometric composition operator on $\mathcal{D}$.

*Proof.*For simplicity, we shall assume that $\tau(j) = j$ for each $j = 1, \ldots, n$. Let us set aside the trivial case for each $\varphi_j \in Aut()$. Then

$$f(\varphi) = f(\varphi_1(z_1), \ldots, \varphi_n(z_n))$$

for each $f \in \mathcal{D}$, where $z_i \in D$ with $i = 1, \ldots, n$.

From the definition of the Dirichlet norm and the fact that $\varphi_j(0) = 0$ for $j = 1, \ldots, n$, we have:

$$\|C_\varphi f\|^2 = |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial \varphi_i} \frac{\partial \varphi_i}{\partial z_i} \right|^2 dv$$

$$= |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial w_i} \right|^2 n_{\varphi_i} dv.$$

However φ_j is univalent self map, $n_{\varphi_j} = 1$ for each $j = 1, \ldots, n$, so

$$\|C_\varphi f\|^2 = |f(0)|^2 + \sum_{i=1}^n \int_{D^n} \left| \frac{\partial f}{\partial w_i} \right|^2 dv,$$

that is $\|C_\varphi f\| = \|f\|$, then C_φ is an isometry on $\mathcal{D}$.

From the expression of inner product on $\mathcal{D}$, we can easily get the following proposition: For $\varphi \in S(^n)$, if the composition operator C_φ is isometric on $\mathcal{D}$, then $\varphi(0) = 0$.

The converse of Theorem 5 is still an open question.

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