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Bayes Prediction Bounds for Right Ordered Pareto Type - II Data

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Abstract: The Pareto Type-II model is considered in present article as the underlying model from which observables are to be predicted under Bayesian approach. The predictions are to be made on the ordered failure items of the remaining businesses by using conditional probability function. A right item failure - censoring criterion has been used for prediction.

Keywords: Pareto Distribution, Right Item Censoring, Predictive distribution, Central Coverage Bayes Prediction Bound Length.

1 Introduction

Pareto distribution has played a major role in the investigations of previous phenomena providing a satisfactory model at the extremities. Pareto distribution and their close relatives provide a very flexible family of fat - tailed distributions, which may be used as a model for income distribution of higher income group.

Freiling (1966) applied the Pareto law to study the distributions of nuclear particles. Harris (1968) used this distribution in determining times of maintenance service while Davis & Feldstein (1979) employed it in studying time to failure of equipment components. This distribution has established its important role in variety of other problems such as size of cities and firms (Steindle (1965)), business mortality (Lomax (1954)).

It is often used as a model for analyzing areas including city population distribution, stock price fluctuation, oil field locations and military areas. It has also been found to be suitable for approximating the right tails of distribution with positive skewness. It has a decreasing failure rate, so it is also useful for modeling survival after some medical procedures (the ability to survive for a longer time appears to increase, the longer one survives after certain medical procedures).

Two-parameter Pareto distribution is transformational equivalent to the two-parameter exponential distribution (Dyer (1981)), thus one could analyze Pareto data using known techniques for exponential distributions. Arnold (1983) gave an extensive historical survey of its use in the context of income distribution. Arnold & Press (1989), Ouyang & Wu (1994), Ali-Mousa (2001), Soliman (2001), Wu et al. (2004) and others those who have studied predictive inference for the future observations under the Pareto model. Recently, Prakash & Singh (2013) present some Bayes prediction length of interval for Pareto model.

Al-Hussaini et al. (2001) were obtaining Bayes prediction bounds for Type - I censored data from a finite mixture of Lomax (Pareto Type - II) components. Some Bayes prediction bounds based on one - sample technique for Pareto model have obtained by Nigm et al. (2003).

The objective of the present paper is to obtain the central coverage Bayes prediction length of bounds for the future observation from Pareto Type - II distribution. We present the Bayesian statistical analysis to predict the future statistic of the considered model based on the right censored item failure data. Based on the first k ordered failure items in a sample of size n from Pareto Type - II distribution, Bayesian prediction bounds for the remaining $(n-k)$ items are derived in two cases, the first is when the scale parameter is known (Section 3) and the second is when both parameters are considered as the random variables (Section 4). A numerical study has been carried out for the illustration of the procedures in next section.

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2 The Considered Model

Probability density function of the considered Pareto model is

$$f(x;\theta,\sigma) = \theta\,\sigma\,(1+\sigma x)^{-\theta-1}\,;\; x>0,\, \theta>0, \sigma>0. \tag{1}$$

Here, θ is the shape parameter and σ is the scale parameter. The Pareto distribution (1), is the result of the mixture of Exponential distribution with the parameter α, and the exponential scale parameter α is distributed as Gamma with parameters θ and σ.

In life testing, fatigue failures and other kinds of destructive test situations, the observations usually occurred in ordered manner such a way that weakest items failed first and then the second one and so on. Let us suppose that, there are $n(>0)$ items are put to test under the considered model without replacement and only $k(>0)$ items are fully measured, while the remaining $(n-k)$ items are censored. These $(n-k)$ censored lifetimes will be ordered separately. This is known as the Right Item Failure - Censoring scheme. The objective of the present article is the predictions are to be made on the ordered failure remaining $(n-k)$ items, by using conditional probability function.

The likelihood function for random sample $\underline{x}\left(=x_{(1)},x_{(2)},\ldots,x_{(k)}\right)$ under the above censoring criterion is define as

$$L(\underline{x}|\theta,\sigma) \propto \left(\prod_{i=1}^{k} f\left(x_{(i)};\theta,\sigma\right)\right).\left(\prod_{i=k+1}^{n}\left(1-\int_0^{x_{(i)}} f\left(x_{(i)};\theta,\sigma\right)\,dx_{(i)}\right)\right)$$

$$\Rightarrow L(\underline{x}|\theta,\sigma) \propto \sigma^k\,\theta^k\,exp\left(-\theta T_1 - T_0\right), \tag{2}$$

where $T_1=\sum_{i=1}^{n} log\left(1+\sigma x_{(i)}\right)$ and $T_0=\sum_{i=1}^{k} log\left(1+\sigma x_{(i)}\right)$.

3 Central Coverage Bayes Prediction Bounds (Known Scale Parameter)

From a Bayesian view point; there is clearly no way in which one can say that one prior is better than other. It is more frequently the case that, we select to restrict attention to a given flexible family of priors, and we choose one from that family, which seems to match best with our personal beliefs. A natural family of conjugate prior for the shape parameter θ is consider here as a Gamma (when scale parameter is known), having probability density function

$$g_1(\theta|\sigma) \propto \theta^{c-1}e^{-d\,\theta}\,;\; \theta>0, d>0, c>0. \tag{3}$$

Posterior density function of the parameter θ is obtained by using (2) and (3) as

$$\pi(\theta|\underline{x}) = \frac{L(\underline{x}|\theta,\sigma)\,.g_1(\theta|\sigma)}{\int_\theta L(\underline{x}|\theta,\sigma)\,.g_1(\theta|\sigma)\,d\theta}$$

$$\Rightarrow (\theta|\underline{x}) = \frac{T_k^{k+c}}{\Gamma(k+c)}\,\theta^{k+c-1}\,e^{-\theta T_k}\,;\; T_k = T_1 + d. \tag{4}$$

Nigm & Handy (1987) consider the problem of predicting $T_{(r+s)}; s=1,2,\ldots,n$ based on the order statistics $t_{(1)} < t_{(2)} < \ldots < t_{(r)}$ from a sample of size n. In present article, the predictions are to be made on the ordered failure remaining $(n-k)$ items, using the conditional probability function for the Right Item failure - censored data.

The conditional predictive density of $Y_{(s)} = X_{k+s}; s=1,2,\ldots,n-k$ at given θ is defined as

$$h_1\left(y_{(s)}|\theta,\underline{x}\right) = \left[\psi\left(x_{(k)}\right)-\psi\left(y_{(s)}\right)\right]^{s-1}\left[\psi\left(y_{(s)}\right)\right]^{n-k-s}\left[\psi\left(x_{(k)}\right)\right]^{-n+r}\left[f\left(y_{(s)}\right)\right], \tag{5}$$

where $\psi(.)$ is the survival function.

Solving (5), we have

$$h_1\left(y_{(s)}|\theta,\underline{x}\right) = \theta\,\sigma\sum_{i=0}^{s-1}\frac{\Delta_i}{B_{(s)}}\left(\frac{B_{(s)}}{B_{(k)}}\right)^{-n_i\theta}\,; \tag{6}$$

where $B_{(s)} = \left(1+\sigma y_{(s)}\right), B_{(k)} = \left(1+\sigma x_{(k)}\right), n_i = n-k-s+i+1$ and $\Delta_i = (-1)^i \left({}^{s-1}C_i\right)$.

The joint probability density function of $Y_{(s)}$ given the data is defined as

$$h'\left(y_{(s)}, \theta | \underline{x}\right) = h_1\left(y_s | \theta, \underline{x}\right) . \pi\left(\theta | \underline{x}\right)$$

$$\Rightarrow h'\left(y_{(s)}, \theta | \underline{x}\right) = \frac{T_k^{k+c}}{\Gamma(k+c)} \sigma\, \theta^{k+c} e^{-\theta T_k} \sum_{i=0}^{s-1} \frac{\Delta_i}{B_{(s)}} \left(\frac{B_{(s)}}{B_{(k)}}\right)^{-n_i\theta}. \tag{7}$$

Therefore the predictive density of $Y_{(s)}$ is thus obtained as

$$h^*\left(y_{(s)} | \underline{x}\right) = \int_\theta h'\left(y_{(s)}, \theta | \underline{x}\right) d\theta. \tag{8}$$

Using (7) in (8) for $y_{(s)} > x_{(r)}$, we have

$$h^*\left(y_{(s)} | \underline{x}\right) = (k+c)\, \sigma\, T_k^{(k+c)} \sum_{i=0}^{s-1} \frac{\Delta_i}{B_{(s)}} \left(T_k + n_i \log\left(\frac{B_{(s)}}{B_{(k)}}\right)\right)^{-(k+c+1)}. \tag{9}$$

Now, the predictive survival function is defined as

$$P\left[Y_{(s)} > y | \underline{x}\right] = \frac{\int_y^\infty h^*\left(Y_{(s)} | \underline{x}\right) dy_s}{\int_{x_k}^\infty h^*\left(Y_{(s)} | \underline{x}\right) dy_s}$$

$$= \frac{(k+c)\, \sigma\, T_k^{(k+c)} \sum_{i=0}^{s-1} \Delta_i \int_y^\infty \frac{1}{B_{(s)}} \left(T_k + n_i \log\left(\frac{B_{(s)}}{B_{(k)}}\right)\right)^{-(k+c+1)} dy_s}{(k+c)\, \sigma\, T_k^{(k+c)} \sum_{i=0}^{s-1} \Delta_i \int_{x_k}^\infty \frac{1}{B_{(s)}} \left(T_k + n_i \log\left(\frac{B_{(s)}}{B_{(k)}}\right)\right)^{-(k+c+1)} dy_s}$$

$$P\left[Y_{(s)} > y | \underline{x}\right] = \frac{\sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \left(T_k + n_i \log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}}{\sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \left(T_k\right)^{-(k+c)}} ; B_{(0)} = (1+\sigma y). \tag{10}$$

We know that

$$\sum_{i=0}^{a} (-1)^i \left({}^aC_i\right) (i+b)^{-1} = \left\{(a+1)\left({}^{a+b}C_{a+1}\right)\right\}^{-1}. \tag{11}$$

Hence, the denominator of equation (10) are solving by using (11) as

$$\sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} (T_k)^{-(k+c)} = \sum_{i=0}^{s-1} (-1)^i \left({}^{s-1}C_i\right) \left(i + (n-k-s+1)\right)^{-1} (T_k)^{-(k+c)}$$

$$= \left(s\left({}^{n-k}C_s\right)\right)^{-1} (T_k)^{-(k+c)}. \tag{12}$$

Using (12) and (10), the predictive survival function is thus written as

$$P\left[Y_{(s)} > y | \underline{x}\right] = s\left({}^{n-k}C_s\right) (T_k)^{k+c} \sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \left(T_k + n_i \log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}. \tag{13}$$

In the context of Bayes prediction, we say that (l_1, l_2) is a $100(1-\varepsilon)\%$ prediction limits for the future observation $Y_{(s)}$, if they satisfies

$$Pr\left(l_1 < Y_{(s)} < l_2\right) = 1 - \varepsilon. \tag{14}$$

Here l_1 and l_2 are said to be lower and upper Bayes prediction bounds for the random variable $Y_{(s)}$, and $1-\varepsilon$ is called the confidence prediction coefficient. The central coverage Bayes prediction lower and upper bounds are obtain by solving following equality

$$Pr\left(Y_{(s)} \leq l_1\right) = \frac{1-\varepsilon}{2} = Pr\left(Y_{(s)} \geq l_2\right). \tag{15}$$

Using (13) and (15), the lower and upper central coverage Bayes prediction bounds l_1 and l_2 are obtained by solving following equalities as

$$\frac{1+\varepsilon}{2} = s\left({}^{n-k}C_s\right)(T_k)^{k+c}\sum_{i=0}^{s-1}\frac{\Delta_i}{n_i}\left(T_k + n_i\, log\left(\frac{B_{(l_1)}}{B_{(k)}}\right)\right)^{-(k+c)}$$

and

$$\frac{1-\varepsilon}{2} = s\left({}^{n-k}C_s\right)(T_k)^{k+c}\sum_{i=0}^{s-1}\frac{\Delta_i}{n_i}\left(T_k + n_i\, log\left(\frac{B_{(l_2)}}{B_{(k)}}\right)\right)^{-(k+c)}, \tag{16}$$

where $B_{(l_1)} = (1+l_1\sigma)$ and $B_{(l_2)} = (1+l_2\sigma)$. Further, simplification of the above equations (16), do not possible. A numerical technique is applied here for obtaining the values of l_1 and l_2 for some ε.

For a particular case, substituting $s=1$ in the predictive survival function (13), for predicting the item $Y_{(1)} = X_{(k+1)}$, of the next item to fail, and is obtain as

$$P\left[Y_{(1)} > y|\underline{x}\right] = \left(1 + \frac{(n-k)}{T_k}\, log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}. \tag{17}$$

Similarly, for $s = n-k$, corresponding to predicting the item $Y_{(n-k)} = X_{(n)}$, of the last item to fail, the predictive survival function (13) is written as

$$P\left[Y_{(n-k)} > y|\underline{x}\right] = \sum_{i=1}^{n-k}(-1)^{i-1}\left({}^{n-k}C_i\right)\left(1 + \frac{i}{T_k} log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}. \tag{18}$$

Now, the central coverage Bayes prediction lower and upper bounds for Y_1 are now obtain as

$$l_1 = \sigma^{-1}\left\{B_k\, exp\left(\frac{T_k}{n-k}\left(\left(\frac{1+\varepsilon}{2}\right)^{-1/(k+c))} - 1\right)\right) - 1\right\}$$

and

$$l_2 = \sigma^{-1}\left\{B_k\, exp\left(\frac{T_k}{n-k}\left(\left(\frac{1-\varepsilon}{2}\right)^{-1/(k+c))} - 1\right)\right) - 1\right\}. \tag{19}$$

The central coverage Bayes prediction length of bounds for $Y_{(1)}$ is obtain as

$$L = l_2 - l_1. \tag{20}$$

The Central coverage Bayes prediction lower and upper bounds do not exist for the item $Y_{(n-k)}$. However, one may obtain lower and upper bounds for $Y_{(n-k)}$ by equating following equality.

$$\frac{1+\varepsilon}{2} = \sum_{i=1}^{n-k}(-1)^{i-1}\left({}^{n-k}C_i\right)\left(1 + \frac{i}{T_k}\, log\left(\frac{B_{(l_1)}}{B_{(k)}}\right)\right)^{-(k+c)}$$

and

$$\frac{1-\varepsilon}{2} = \sum_{i=1}^{n-k}(-1)^{i-1}\left({}^{n-k}C_i\right)\left(1 + \frac{i}{T_k}\, log\left(\frac{B_{(l_2)}}{B_{(k)}}\right)\right)^{-(k+c)}. \tag{21}$$

4 Central Coverage Bayes Prediction Bounds (Both Parameters Unknown)

When, both scale and shape parameters are considered to be random variable, the likelihood function of the considered model is obtain as

$$L(\underline{x}|\theta,\sigma) \propto \theta^k e^{-\theta T_1} \sigma^k e^{-T_0} \tag{22}$$

where $T_1 = \sum_{i=1}^{n} log\left(1+\sigma x_{(i)}\right)$ and $T_0 = \sum_{i=1}^{k} log\left(1+\sigma x_{(i)}\right)$.

It is clear from equation (22) that, both the functions T_0 and T_1 are depends only upon scale parameter σ, and thus both scale and shape parameters are independently distributed. Therefore, in present case when both shape and scale parameters are consider being random variables for the underlying model, the joint prior density for the parameter θ and σ is considered as

$$g(\theta,\sigma) = g_1(\theta|\sigma)\,.\,g_2(\sigma)\,; \tag{23}$$

where $g_1(\theta|\sigma)$ and $g_2(\sigma)$ are the gamma densities and defined as

$$g_1(\theta|\sigma) = \frac{\sigma^c}{\Gamma(c)}\,\theta^{c-1} e^{-\sigma\theta}\,;\,\theta > 0, \sigma > 0, c > 0 \tag{24}$$

and

$$g_2(\sigma) = \frac{b^a}{\Gamma(a)}\,\sigma^{a-1} e^{-b\sigma}\,;\,\sigma > 0, a > 0, b > 0. \tag{25}$$

Now, the joint prior density is

$$g(\theta,\sigma) \propto \theta^{c-1} \sigma^{a+c-1} e^{-\sigma\theta} e^{-b\sigma}. \tag{26}$$

The joint posterior density is now defined as

$$\pi^{**}(\theta,\sigma|\underline{x}) = \frac{L(\underline{x}|\theta,\sigma)\,.\,g(\theta,\sigma)}{\int_\sigma \int_\theta L(\underline{x}|\theta,\sigma)\,.\,g(\theta,\sigma)\,d\theta d\sigma}.$$

Using (22) and (26) we have,

$$\Rightarrow \pi^{**}(\theta,\sigma|\underline{x}) = \bar{\sigma}\,\theta^{k+c-1} e^{-\theta(T_1+\sigma)} \sigma^{a+c+k-1} e^{-T_0-b\sigma}\,; \tag{27}$$

where $\bar{\sigma} = \left(\Gamma(k+c)\int_\sigma (T_1+\sigma)^{-(k+c)}\,\sigma^{a+c+k-1}\,e^{-T_0-b\sigma}\right)^{-1}$.

The conditional predictive density of $Y_{(s)} = X_{r+s}\,;\,s = 1,2,...,n-k$ at given θ and σ is obtain similarly as

$$h_2\left(y_{(s)}|\theta,\sigma,\underline{x}\right) = \theta\,\sigma \sum_{i=0}^{s-1} \frac{\Delta_i}{B_{(s)}}\,exp\left(-n_i\,\theta\,log\left(\frac{B_{(s)}}{B_{(k)}}\right)\right). \tag{28}$$

The joint probability density function of $Y_{(s)}$ given the data is now obtain as

$$h''\left(y_{(s)},\theta,\sigma|\underline{x}\right) = h_2\left(y_{(s)}|\theta,\sigma,\underline{x}\right)\,.\,\pi^{**}(\theta,\sigma|\underline{x})$$

$$\Rightarrow h''\left(y_{(s)},\theta,\sigma|\underline{x}\right) = \bar{\sigma} \sum_{i=0}^{s-1} \frac{\Delta_i}{B_{(s)}}\,\theta^{k+c} e^{-\theta T_k^*}\,\frac{\sigma^{a+c+k}}{e^{T_0+b\sigma}}; \tag{29}$$

where $T_k^* = \left(T_1+\sigma+n_i\,log\left(\frac{B_{(s)}}{B_{(k)}}\right)\right)$.

Hence, the predictive density of $Y_{(s)}$ given on the data, for $Y_{(s)} > x_{(r)}$ is thus obtained as

$$h^{**}\left(y_{(s)}|\underline{x}\right) = \int_\sigma \int_\theta h''\left(y_{(s)},\theta,\sigma|\underline{x}\right)\,d\theta\,d\sigma.$$

$$\Rightarrow h^{**}\left(y_{(s)}|\underline{x}\right) = \bar{\bar{\sigma}} \sum_{i=0}^{s-1} \Delta_i \int_\sigma \frac{\sigma^{a+c+k}}{B_{(s)}}\,e^{-T_0-b\sigma}\,\left(T_k^*\right)^{-(k+c+1)}\,d\sigma\,; \tag{30}$$

where $\bar{\bar{\sigma}} = \bar{\sigma}\Gamma(k+c+1)$.

The predictive survival function is obtain in present case as

$$P\left[Y_{(s)} > y|\underline{x}\right] = \frac{\int_y^\infty h^{**}\left(Y_{(s)}|\underline{x}\right) dy_s}{\int_{x_k}^\infty h^{**}\left(Y_{(s)}|\underline{x}\right) dy_s}$$

$$\Rightarrow P\left[Y_{(s)} > y|\underline{x}\right] = \frac{\sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \int_\sigma T_k^{**} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}{\sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}; \tag{31}$$

where $T_k^{**} = \left(T_1 + \sigma + n_i \log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}$, $T_k^{***} = (T_1+\sigma)^{-(k+c)}$ and $B_{(0)} = (1+\sigma y)$.

The Central coverage Bayes prediction lower and upper bounds for the random variable $Y_{(s)}$ when both parameters are considered to be unknown, are obtained by equating following equality

$$\frac{1+\varepsilon}{2} = P\left[Y_{(s)} > l_1|\underline{x}\right] = \frac{\Omega_1}{\Omega_*}$$

and

$$\frac{1-\varepsilon}{2} = P\left[Y_{(s)} > l_2|\underline{x}\right] = \frac{\Omega_2}{\Omega_*}; \tag{32}$$

where $\Omega_j = \sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \int_\sigma T_{kj}^{**} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma$, $B_{(l_j)} = (1+l_j\sigma)$, $j=1,2$, $T_{kj}^{**} = \left(T_1 + \sigma + n_i \log\left(\frac{B_{(l_j)}}{B_{(k)}}\right)\right)^{-(k+c)}$ and $\Omega_* = \sum_{i=0}^{s-1} \frac{\Delta_i}{n_i} \int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma$.

Again, closed form of the above equations (32) does not exist. A numerical technique is applied here for obtaining the values of central coverage Bayes prediction lower and upper bounds for some ε.

Putting $s=1$, for predicting the item $Y_{(1)} = X_{(k+1)}$, of the next item to fail, the predictive survival function is

$$P\left[Y_{(s)} > y|\underline{x}\right] = \frac{\int_\sigma T_{k3}^{**} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}{\int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}, \tag{33}$$

where $T_{k3}^{**} = \left(T_1 + \sigma + (n-k) \log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}$.

Similarly, for predicting the last item to fail $\left(Y_{(n-k)} = x_{(n)}\right)$, putting $s = n-k$, the predictive survival function is rewritten as

$$P\left[Y_{(n-k)} > y|\underline{x}\right] = \frac{\Omega_3}{\Omega_4}, \tag{34}$$

where $\Omega_3 = \sum_{i=1}^{n-k} (-1)^{i-1} \left({}^{n-k}C_i\right) \int_\sigma T_{k4}^{**} \sigma^{a+c+k-1} e^{-T_0-b\sigma} d\sigma$,
$\Omega_4 = \sum_{i=1}^{n-k} (-1)^{i-1} \left({}^{n-k}C_i\right) \int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0-b\sigma} d\sigma$ and $T_{k4}^{**} = \left(T_1 + \sigma + i \log\left(\frac{B_{(0)}}{B_{(k)}}\right)\right)^{-(k+c)}$.

The central coverage Bayes prediction lower and upper bounds for $Y_{(1)}$ and $Y_{(n-k)}$ do not exist in closed form. However, one may obtain prediction bounds for $Y_{(1)}$ by equating following equalities.

$$\frac{1+\varepsilon}{2} = P\left[Y_{(s)} > l_1|\underline{x}\right] = \frac{\int_\sigma T_{k31}^{**} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}{\int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}$$

and

$$\frac{1-\varepsilon}{2} = P\left[Y_{(s)} > l_2|\underline{x}\right] = \frac{\int_\sigma T_{k32}^{**} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}{\int_\sigma T_k^{***} \sigma^{a+c+k-1} e^{-T_0} e^{-b\sigma} d\sigma}, \tag{35}$$

where $T_{k3j}^{**} = \left(T_1 + \sigma + (n-k) \log\left(\frac{B_{(l_j)}}{B_{(k)}}\right)\right)^{-(k+c)}$; $j=1,2$.

Table 1: Bayes Prediction Length of Bounds for X_{16} (When σ is Known).

$\sigma\downarrow\varepsilon\rightarrow$	$(c,d)\downarrow$	99%	95%	90%
2.00	0.25, 0.50	2.2975	1.6849	1.4879
	04, 02	2.1501	1.6160	1.3089
	09, 03	1.9072	1.4071	1.2519
1.00	0.25, 0.50	2.2436	1.6454	1.4530
	04, 02	2.0997	1.5781	1.2782
	09, 03	1.8624	1.3741	1.2226
0.50	0.25, 0.50	2.1910	1.6068	1.4189
	04, 02	2.0504	1.5411	1.2482
	09, 03	1.8188	1.3418	1.1939

Similarly, the central coverage Bayes prediction lower and upper bounds for $Y_{(n-k)}$ (the last item to fail) are obtained by equating following equalities.

$$\frac{1+\varepsilon}{2} = P\left[Y_{(n-k)} > l_1|\underline{x}\right] = \frac{\Omega_{31}}{\Omega_4}$$

and

$$\frac{1-\varepsilon}{2} = P\left[Y_{(n-k)} > l_2|\underline{x}\right] = \frac{\Omega_{32}}{\Omega_4}, \tag{36}$$

where $T_{k4j}^{**} = \left(T_1+\sigma+i\,log\left(\frac{B_{(lj)}}{B_{(k)}}\right)\right)^{-(k+c)}$ $\Omega_{3j} = \sum_{i=1}^{n-k}(-1)^{i-1}\left({}^{n-k}C_i\right)\int_\sigma T_{k4j}^{**}\ \sigma^{a+c+k-1}e^{-T_0-b\sigma}\,d\sigma$ and $j=1,2$.

5 Numerical Analysis

5.1 When Scale Parameter is Known

In this section, we are presenting the findings of the central coverage Bayes prediction lower and upper bounds for a particular set of parametric values.

We assume here that the failure time follows the considered Pareto Type - II model. The numerical values of the prior parameters c and d are selected as $(c,d) = (0.25, 0.50), (04, 02)$ and $(09, 03)$ with scale parameter $\sigma = 0.50, 1.00, 2.00$. The selections of prior parametric values meet the criterion that the prior variance should be unity.

Using above considered parametric values, generate $10,000$ random samples. A sample of $20\,(=n)$ items is tested and their ordered failure times are observed.

Suppose that this test is terminated when first $15\,(=k)$ of the ordered lifetimes are available. For selected level of significance $\varepsilon = 99\%, 95\%, 90\%$; the central coverage Bayes prediction lower and upper bounds for X_{16}, (the next failure item) are calculated. The lengths of central coverage Bayes prediction bounds for X_{16} are presented in Table 1.

We observe from the table that the length of the interval tend to be wider as the value of scale parameter σ increases when other parametric values are consider to be fixed. Opposite trend has been seen when prior parameter c increases. It is also noted further that when confidence level ε or prior parameter d decreases the length of intervals tends to be closer.

For the similar set of considered data, the length of central coverage Bayes prediction bounds for X_{20}, (the last failure item) are obtained and presented in Table 2.

It observes from Table 2 that the magnitude of the length of the interval tends to be wider as compare to failure item X_{16}. However, the gain in the interval is robust. This is a natural, as the prediction of the future order statistic that is far away from the last observation, the value has less accuracy than that of other future order statistic. Other properties are seemed to be similar.

Table 2: Bayes Prediction Length of Bounds for X_{20} (When σ is Known).

$\sigma \downarrow \varepsilon \rightarrow$	$(c,d) \downarrow$	99%	95%	90%
2.00	0.25, 0.50	3.5438	2.5989	2.2951
	04, 02	3.3165	2.4927	2.0190
	09, 03	2.9417	2.1704	1.9311
1.00	0.25, 0.50	3.4607	2.5379	2.2413
	04, 02	3.2387	2.4342	1.9716
	09, 03	2.8727	2.1195	1.8858
0.50	0.25, 0.50	3.3795	2.4784	2.1886
	04, 02	3.1627	2.3771	1.9254
	09, 03	2.8054	2.0697	1.8415

Table 3: Bayes Prediction Length of Bounds for X_{16} (Both Parameters Unknown).

$c \downarrow \varepsilon \rightarrow$	$(a,b) \downarrow$	99%	95%	90%
2.00	0.25, 0.50	2.7521	2.0183	1.7823
	04, 02	2.5755	1.9358	1.5679
	09, 03	2.2846	1.6855	1.4996
1.00	0.25, 0.50	2.6875	1.9710	1.7405
	04, 02	2.5152	1.8904	1.5311
	09, 03	2.2309	1.6460	1.4645
0.50	0.25, 0.50	2.6245	1.9247	1.6997
	04, 02	2.4561	1.8460	1.4952
	09, 03	2.1787	1.6073	1.4301

Table 4: Bayes Prediction Length of Bounds for X_{20} (Both Parameters Unknown).

$c \downarrow \varepsilon \rightarrow$	$(a,b) \downarrow$	99%	95%	90%
2.00	0.25, 0.50	4.4538	3.2663	2.8845
	04, 02	4.1681	3.1328	2.5375
	09, 03	3.6971	2.7277	2.4270
1.00	0.25, 0.50	4.3494	3.1896	2.8168
	04, 02	4.0704	3.0593	2.4779
	09, 03	3.6104	2.6638	2.3701
0.50	0.25, 0.50	4.2473	3.1148	2.7506
	04, 02	3.9748	2.9875	2.4198
	09, 03	3.5258	2.6012	2.3144

5.2 When Both Parameters are Unknown

When both parameters are treated as the random variable, study also has been carried out for studying the properties of the length of the central coverage Bayes predictive bounds.

We generate the values of the scale parameter σ from (25) by using selected values of prior parameters $(a,b) = (0.25, 0.50), (04, 02)$ and $(09, 03)$. The selections of prior parametric values meet the criterion that the prior variance should be unity.

With the help of the generated values of σ and $c\,(= 0.50, 1.00, 2.00)$, we generated the shape parameter θ. With the help of above generated values, $10,000$ random samples have been generated from the considered model, and a sample of $20\,(= n)$ items is tested and their ordered failure times are observed.

The length of central coverage Bayes prediction bounds are obtained for X_{16}, (the next failure item, since test is terminated when the first $15\,(= k)$ of the ordered lifetimes are available) and X_{20}, and presented respectively in the Tables 3 and 4.

Similar properties have been seen for length of the central coverage Bayes prediction bounds in both cases. It is also noted that the magnitude of the length of bounds are wider (when both parameter are unknown) than compare to length of bounds when one parameter is unknown. The gain in magnitude in the length of the Bayes prediction bounds is nominal.

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