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A Comparative Study of Traditional Estimation Methods and Maximum Product Spacings Method in Generalized Inverted Exponential Distribution

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Abstract: In this paper, we propose the method of Maximum product of spacings for point estimation of parameter of generalized inverted exponential distribution (GIED). The aim of this paper is to analyse the small sample behaviour of proposed estimators. Further, we have also proposed asymptotic confidence intervals of the parameters and the estimates of reliability and hazard function using Maximum Product Spacings (MPS) method and compared with corresponding asymptotic confidence intervals and the estimates of reliability and hazard function of Maximum Likelihood estimation (MLEs). A comparative study among the method of MLE, method of least square (LSE) and the method of maximum product of spacings (MPS) is performed on the basis of simulated sample of GIED. The MPS method outperforms the method of MLE and the method of LSE. Furthermore, comparison of different estimation method have been proposed on the basis of K-S distance and AIC. For numerical illustration one real data set has been considered.

Keywords: GIED, Reliability characteristic, method of Maximum Product Spacings, method of Maximum Likelihood Estimation, method of Least Squares Estimates and Interval Estimation.

1 Introduction

In statistical inference problem, we are given a set of observations $x_1, x_2, \cdots, x_n$. These are the values taken by some random phenomena about whose distribution we have some knowledge. For parameter estimation, various estimation methods are widely discussed in literature. One often uses traditional estimation methods such as the method of moments, method of least square, method of weighted least square and maximum likelihood estimation (MLE). Each of them having their own advantages and limitations but among these methods the most popular method of estimation is maximum likelihood estimation method. Which can be justified on the ground of its various useful properties like consistency, sufficiency, invariance and asymptotic efficiency and its easy computations. The MLE method works efficiently if each contribution to the likelihood function is bounded above. It is the situation with all discrete distributions. However, having such nice properties and better applicability it also has some weakness as mentioned by various authors in different context. Its greatest weakness is that it can not work for ‘heavy tailed’ continuous distribution with unknown location and scale parameters (Pitman, 1979 , p . 70). It also creates problem in situations where there is only mixture of continuous distribution and then MLE method can break down. It was established by some authors that MLE does not always provide precise estimates for certain distributions such as gamma, Weibull, and log normal distributions. In all these cases the critical difficulty is that there are paths in parameter space with location parameter tends to smallest observation along which the likelihood becomes infinite. Unfortunately in such situations estimates of other parameters becomes inconsistent. Harter and Moore [5] suggests a alternative way to use local maxima as an alternative of global maxima, this can be effective but not full proof there being some weakness as pointed out by Cheng for this see [1].

In the context of Harter and Moore, Huzurbazar [19] has shown that no stationary point (and hence no local maximum) can provide a consistent estimator, when the concern distribution is J-shaped, for example in the case of

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Weibull and gamma distributions when the shape parameter is less than unity. Thus whether we consider a global or a local maximum, Maximum Likelihood estimation is bound to fail. The practical problem is that even if the distribution is not J-shaped, so that parameters can in principle be consistently estimated by local Maximum Likelihood estimation as the sample size tends to infinity, it can happen that, with fixed sample size, a particular random sample gives rise to a likelihood function with no local maximum at all (Griffiths, [20]), this occur mainly when shape parameter equal to unity.

Several authors has suggested alternative methods to MLE, either involving modification to MLE method or method of moments or percentiles. Despite of the above problems when MLE is applied it outperforms the alternative methods.

In order to overcome these shortcomings and having better applicability in such types of situations which possess properties similar to MLE, Cheng and Amin [1] introduced the Maximum Product of Spacings (MPS) method as an alternative to MLE for the estimation of parameters of continuous univariate distributions. Cheng and Amin proposed to replace the likelihood function by an product of spacings and conjectured that it retains most of the properties of the method of maximum likelihood. Ranneby [3] independently developed the same method as an approximation to the Kullback-Leibler measure of information. The approach of Cheng and Amin is more intuitively attractive and can, to some extent, be regarded as a practical solution to the problems linked with likelihood (Titterington, [15]), but that of Ranneby is more powerful theoretically and allows the derivation of the properties of MPS estimators. It may be noted that MPS method is especially suited to the cases where one of the parameter has an unknown shifted origin, as it is the case in three parameter lognormal, gamma and Weibull distributions or to the distributions having J-shape..

In order to make a general idea of advantages of MPS estimation over MLE, we first list some good properties of MPS estimation, which were showed by Cheng and Amin [1], including sufficiency, consistency and asymptotic efficiency. In certain cases, it is possible to obtain the distributional behaviour of an MPS estimator for all sample sizes n. Thus, for the uniform distribution with unknown endpoints, the MPS estimators are precisely the MVU estimators and so their distribution is known exactly solved by Cheng and Amin [1].

The consistency of MPS estimators have been discussed in detail by Cheng and Amin [16]. In brief, asymptotically MPS are at least as efficient as MLE estimators when they exit. For distribution where the end points are unknown and the density is J-shaped then MLE is bound to fail, but MPS gives asymptotically efficient estimators. MPS estimators will not necessarily be function of sufficient statistics in general. However, for the case when the support of density functions are known, MPS estimator will show the same asymptotic properties as ML estimators including the one of asymptotic sufficiency.

1.1 The Model

The random variable X has a generalized inverted exponential distribution with two parameter α and λ if it has a probability density function of the form:

$$f(x) = \left(\frac{\alpha\lambda}{x^2}\right)\left(\exp -\frac{\lambda}{x}\right)\left[1-\exp\left(-\frac{\lambda}{x}\right)\right]^{(\alpha-1)}, \quad x \geq 0, \quad \lambda,\alpha > 0 \tag{1}$$

Where α is shape parameter and λ is scale parameter, and its CDF is given by

$$F(x) = 1-\left[1-\exp\left(-\frac{\lambda}{x}\right)\right]^{\alpha}, \quad \alpha,\lambda > 0 \tag{2}$$

The model can be considered as another useful two-parameter generalization of the Inverted exponential distribution (IED). This lifetime distribution can model various shapes of failure rates and hence various shapes of ageing criteria. It is noted that the GIED is reduced to the IED for $\alpha = 1$. In literature, estimation of parameters in the two parameter GIED is discussed extensively, but no one has performed comparison of MLE and MPS. Readers are referred to the following references: Abouammah and Alshingiti [12], Gupta and Kundu [14], Gupta and Kundu [13]. Various properties of the GIED like reliability and hazard function, mean and mode is discussed extensively by Abouammah and Alshingiti [12].

In this paper, the method of product of spacings is applied for estimating the parameters in a two parameter GIED. The purpose here is to examine MPS estimates of the parameters of the GIED and we also construct 95% confidence interval using MLE and MPS. The method of product of spacings is compared with the method of Least squares estimates (LSE) and the method of MLE using simulation. MSE and K-S distance are calculated and on the basis of K-S distance through maximum product of spacings method is better fitted than MLE to the considered real data. AIC is calculated for MPS and MLE and both are compared.

The main objective of this paper is to analyse the small sample behaviour of MPS. As we all know that it is impossible to analyse the whole data set due various reasons like cost factor, time factor etc.

The organisation of the paper is as follows:
In section 2, Different estimation procedures are mentioned and estimates of reliability and hazard functions using MPS method is proposed and compared with MLE. In section 3, asymptotic confidence intervals of the parameters using MPS method is proposed and compared with MLE. In section 4, real data illustration and its application is discussed, and comparison of estimation procedure based on K-S statistics is proposed. In section 5, a comparison is conducted using simulation study. Finally concluding remarks are presented in section 6.

2 Parameter estimation

For the considered distribution, we use two very known and popular method namely least squares method and the maximum likelihood estimation method and one which is not very common i.e MPS method for estimating the parameters α and λ.

2.1 Least square estimation

Let $x_1 < x_2 < \cdots < x_n$ be n ordered random sample of any distribution with CDF F(x), we get

$$E(F(x_i)) = i/(n+1) \tag{3}$$

The least squares estimates are obtained by minimizing

$$P(\alpha,\lambda) = \sum_{i=1}^{n} (F(x_i) - i/(n+1))^2 \tag{4}$$

Putting the cdf of GIED in equation (4) we get

$$P(\alpha,\lambda) = \sum_{i=1}^{n} \left(1 - \left(1 - exp\left(-\frac{\lambda}{x_i}\right)\right)^{\alpha} - i/(n+1)\right)^2 \tag{5}$$

In order to minimize Equation (5), we have to differentiate it with respect to λ and α , which gives the following equation:

$$\sum_{i_1}^{n} \frac{(\alpha)\exp\left(-\frac{\lambda}{x_i}\right)\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)^{\alpha-1}\left(1-\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)^{\alpha} - i/(n+1)\right)}{x_i} = 0 \tag{6}$$

$$\sum_{i_1}^{n} \left[\left(1-\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)^{\alpha} - i/(n+1)\right)\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)^{\alpha} \ln\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)\right] = 0 \tag{7}$$

The above Likelihood equation cannot be solved analytically therefore we can use any iterative procedure such as Newton-Rapson method, to get the solution.

2.2 Maximum likelihood estimators

The likelihood function for a sample of size n from GIED (1) is given by:

$$L(\theta) = (\alpha^n\lambda^n)\exp\left(-\lambda\sum_{i=1}^{n}(1/x_i)\right)\prod_{i=1}^{n}(1/x_i^2)\left[\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)\right]^{\alpha-1}, t \geq 0, \quad \alpha,\lambda > 0 \tag{8}$$

and the log likelihood function is given as

$$M = \ln L(\theta) = n\ln\lambda + n\ln\alpha + \sum_{i=1}^{n}\ln\left(1/x_i^2\right) - \lambda\sum_{i=1}^{n}\ln(1/x_i) + (\alpha-1)\sum_{i=1}^{n}\ln\left[(1-\exp\left(-\frac{\lambda}{x_i}\right))\right], \tag{9}$$

After differentiating the above equation with respect to parameter α and λ and then equating them to zero we got the normal equation as follows:

$$(n/\alpha)+\sum_{i=1}^{n}\ln\left[1-\exp\left(-\frac{\lambda}{x_i}\right)\right]=0 \tag{10}$$

$$(n/\lambda)-\sum_{i=1}^{n}(1/x)+(\alpha-1)\sum_{i=1}^{n}\frac{(1/x_i)\exp(-\lambda/x_i)}{[1-\exp(-\lambda/x_i)]}=0 \tag{11}$$

The above normal equations are not in nice closed form, therefore we can use any iterative procedure such as Newton-Rapson method, to get the solution.

2.3 Maximum Product of spacings estimators

Here, the method of maximum product of spacings is described briefly as follows:

Considering a univariate distribution F($x|\theta$) with density f($x|\theta$) where it is required to estimate θ. The density is assumed to be strictly positive in an interval (α,β) and zero elsewhere, α and β may also be elements of θ, α= -∞ and β=∞ are included. That is F($x|\theta$)=0 and f($x|\theta$)=0 for $x<\alpha$:, F($x|\theta$)=1. and f($x|\theta$)=0 for $x>\beta$. Let $x_1<x_2....<x_n$ be a complete ordered sample, further define $x_0=\alpha$, $x_{n+1}=\beta$.

The spacings are defined as follows:
$D_1=F(x_{1:n},\theta)$, $D_{n+1}=1-F(x_{n:n},\theta)$, $D_i=F(x_{i:n},\theta)-F(x_{i-1:n},\theta), i=2,3,\cdots,n$ as the spacings of the sample. Clearly the spacings sum to unity i.e $\sum D_i=1$. The MPS method is to choose θ which maximizes the geometric mean of the spacings i.e G = $(\prod_{i=1}^{n+1}D_i)^{1/n+1}$,
or equivalently , its logarithm S = log G. The main aim for maximizing G (or S) is that the maximum , which is bounded above because of the condition $\sum D_i=1$, is found only when all $D_i's$ are equal. Cheng and Amin [1] showed that maximizing S as a method of parameter estimation is as efficient as ML estimation. Additionally, they showed that ties present in data would not be a matter of concern in parameter estimation.
The CDF of the GIED is given by the equation (2) and the spacings are defined as follows:

$$D_1=F(x_1)=1-\left[(1-\exp(-\lambda/x_1))^{\alpha}\right] \tag{12}$$

$$D_{(n+1)}=1-F(x_n)=\left[(1-\exp(-\lambda/x_n))^{\alpha}\right] \tag{13}$$

And the general term of spacings is given by,

$$D_i=F(x_i)-F(x_{(i-1)})=\left[\left(1-\exp^{-\lambda/x_{(i-1)}}\right)^{\alpha}\right]-\left[\left(1-\exp^{-\lambda/x_i}\right)^{\alpha}\right] \tag{14}$$

Such that $\sum D_i=1$,
MPS method choose θ which maximizes the product of spacings or in other words to maximize the geometric mean of the spacings i.e

$$G=\left(\prod_{i=1}^{n+1}D_i\right)^{1/n+1} \tag{15}$$

Taking the logarithm of G we get,

$$S=1/(n+1)\sum_{i=1}^{n+1}\ln D_i \tag{16}$$

Or we may write S as

$$\begin{aligned}S&=\frac{1}{(n+1)}\left\{\ln D_1+\sum_{i=2}^{n}\ln D_i+\ln D_{n+1}\right\}\\&=\frac{1}{(n+1)}\left\{\ln\left[1-(1-e^{-\lambda/x_1})^{\alpha}\right]+\sum_{i=2}^{n}\ln\left[(1-e^{-\lambda/x_{i-1}})^{\alpha}-(1-e^{-\lambda/x_i})^{\alpha}\right]\right\}\\&+\frac{1}{(n+1)}\left\{\ln\left[(1-e^{-\frac{\lambda}{x_n}})^{\alpha}\right]\right\}\end{aligned} \tag{17}$$

After differentiating the above equation with respect to parameters and then equating them to zero we get the normal equation as follows:

$$S'_{\alpha}=\frac{1}{n+1}\left[-\frac{(1-e^{-\lambda/x_1})^{\alpha}\ln(1-e^{-\lambda/x_1})}{1-(1-e^{-\lambda/x_1})^{\alpha}}-\ln(1-e^{-\lambda/x_n}))\right]+ \\ \frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{(1-e^{\frac{-\lambda}{x_{i-1}}})^{\alpha}\ln(1-e^{\frac{-\lambda}{x_{i-1}}})-(1-e^{\frac{-\lambda}{x_i}})^{\alpha}\ln((1-e^{\frac{-\lambda}{x_i}}))}{(1-e^{\frac{-\lambda}{x_{i-1}}})^{\alpha}-(1-e^{\frac{-\lambda}{x_i}})^{\alpha}}\right]=0 \tag{18}$$

$$S'_{\lambda}=\frac{1}{n+1}\left[\frac{-(\alpha/x_1)((1-e^{\frac{-\lambda}{x_1}})^{\alpha-1})(e^{\frac{-\lambda}{x_1}})}{1-(1-e^{\frac{-\lambda}{x_1}})^{\alpha}}\right] \\ +\frac{1}{n+1}\sum_{i=2}^{n}\left[\frac{-(\alpha/x_{i-1})((1-e^{\frac{-\lambda}{x_{i-1}}})^{\alpha-1})(e^{\frac{-\lambda}{x_{i-1}}})-(\alpha/x_i)((1-e^{\frac{-\lambda}{x_i}})^{\alpha-1})(e^{\frac{-\lambda}{x_{i-1}}})}{(1-e^{\frac{-\lambda}{x_{i-1}}})^{\alpha}-(1-e^{\frac{-\lambda}{x_i}})^{\alpha}}\right] \\ +\frac{1}{n+1}\left[\frac{-(\alpha/x_n)((1-e^{\frac{-\lambda}{x_n}})^{\alpha-1})(e^{\frac{-\lambda}{x_n}})}{(1-e^{\frac{-\lambda}{x_n}})^{\alpha}}\right]=0 \tag{19}$$

The above normal equations cannot be solved analytically therefore we can use Newton-Rapson method, in order to get the solution.

2.4 Reliability and hazard function

In this section, we propose the estimation of reliability and hazard function using MPS for specified value of time say (t=4). Cheng and Amin [1] and Coolen and Newby [17] had mentioned in their paper that MPS also shows the invariance property just like MLE. So on this basis using the invariance property we estimate the reliability and hazard function.

The MPS estimates of the reliability and hazard function is given as:

$$\hat{R}_{MPS}(t)=\left(1-e^{-\frac{\hat{\lambda}}{t}}\right)^{\hat{\alpha}},\ \alpha,\lambda,t>0 \tag{20}$$

$$\hat{H}_{MPS}(t)=\frac{\frac{\hat{\alpha}\hat{\lambda}}{t^2}\left(e^{-\frac{\hat{\lambda}}{t}}\right)}{\left(1-e^{-\frac{\hat{\lambda}}{t}}\right)},\ \alpha,\lambda,t>0 \tag{21}$$

respectively, where $\hat{\alpha}=\hat{\alpha}_{mp}$ and $\hat{\lambda}=\hat{\lambda}_{mp}$ are the MPS estimates of the parameter α and λ respectively. In equation (20) and (21) putting the estimates of MLE, we can get the expression for the reliability and hazard function using MLE.

3 Asymptotic confidence intervals

In this section, we propose the asymptotic confidence intervals using MPS, as it was mentioned by Cheng and Amin [1] and Stanislav Anatolyevin and Grigory Kosenok [18] in their papers that the MPS method also shows asymptotic properties like the Maximum likelihood estimator. Keeping this in mind, we may propose the asymptotic confidence intervals using MPS. The exact distribution of the MPS cannot be obtained explicitly. Therefore, the asymptotic properties of MPS can be used to construct the confidence intervals for the parameters. $I(\hat{\alpha},\hat{\lambda})$ is the observed Fishers information matrix and is define as:

$$I(\hat{\alpha},\hat{\lambda})=\begin{bmatrix}-S''_{\alpha\alpha} & -S''_{\alpha\lambda}\\ -S''_{\lambda\alpha} & -S''_{\lambda\lambda}\end{bmatrix}_{(\alpha=\hat{\alpha}_{mp},\lambda=\hat{\lambda}_{mp})} \tag{22}$$

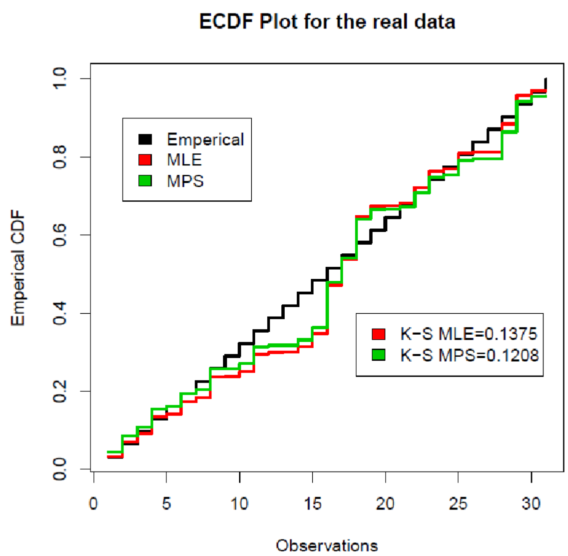


Fig. 1

Variation of sample size with respect to MSE for different choices of α and for fixed value of $\lambda = 1$

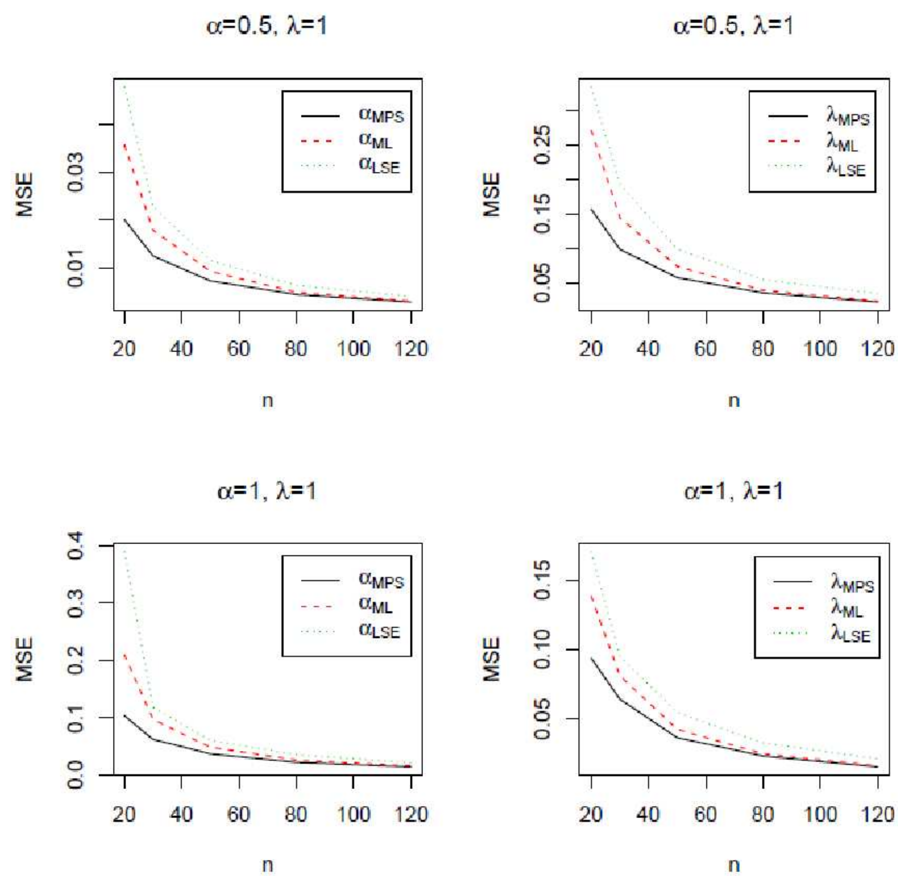


Fig. 2: Mean Square Error of the estimates for $\alpha = 0.5, 1$ and $\lambda = 1$ with variation of sample size (n)

The first derivatives of the product of spacings i.e the function S with respect to parameter α and λ are given by Equations (18) and (19) and hence the second derivatives are calculated as follows:
The second derivative of the function S with respect to α is given as:

$$
\begin{aligned}
S''_{\alpha\alpha} &= \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\lambda)F''_{\alpha\alpha}(x_1,\alpha,\lambda)-F'_{\alpha}(x_1,\alpha,\lambda)^2}{F(x_1,\alpha,\lambda)^2}\right] \\
&+\frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}\left\{F''_{\alpha\alpha}(x_i,\alpha,\lambda)-F''_{\alpha\alpha}(x_{i-1},\alpha,\lambda)\right\}}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\left\{F'_{\alpha}(x_i,\alpha,\lambda)-F'_{\alpha}(x_{i-1},\alpha,\lambda)\right\}^2}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\{1-F(x_n,\alpha,\lambda)\}F''_{\alpha\alpha}(x_n,\alpha,\lambda)+\left\{F'_{\alpha}(x_n,\alpha,\lambda)\right\}^2}{\{1-F(x_n,\alpha,\lambda)\}^2}\right]
\end{aligned} \tag{23}
$$

The second derivative of the function S with respect to λ is given as,

$$
\begin{aligned}
S''_{\lambda\lambda} &= \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\lambda)F''_{\lambda\lambda}(x_1,\alpha,\lambda)-F'_{\lambda}(x_1,\alpha,\lambda)^2}{F(x_1,\alpha,\lambda)^2}\right] \\
&+\frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}\left\{F''_{\lambda\lambda}(x_i,\alpha,\lambda)-F''_{\lambda\lambda}(x_{i-1},\alpha,\lambda)\right\}}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\left\{F'_{\lambda}(x_i,\alpha,\lambda)-F'_{\lambda}(x_{i-1},\alpha,\lambda)\right\}^2}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\{1-F(x_n,\alpha,\lambda)\}F''_{\lambda\lambda}(x_n,\alpha,\lambda)+\left\{F'_{\lambda}(x_n,\alpha,\lambda)\right\}^2}{\{1-F(x_n,\alpha,\lambda)\}^2}\right]
\end{aligned} \tag{24}
$$

and the second derivative of the function S with respect to α,λ is given as:

$$
\begin{aligned}
S''_{\alpha\lambda} = S''_{\lambda\alpha} &= \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\lambda)F''_{\alpha\lambda}(x_1,\alpha,\lambda)-F'_{\alpha}(x_1,\alpha,\lambda)F'_{\lambda}(x_1,\alpha,\lambda)}{F(x_1,\alpha,\lambda)^2}\right] \\
&+\frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}\left\{F''_{\alpha\lambda}(x_i,\alpha,\lambda)-F''_{\alpha\lambda}(x_{i-1},\alpha,\lambda)\right\}}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\left\{F'_{\alpha}(x_i,\alpha,\lambda)-F'_{\alpha}(x_{i-1},\alpha,\lambda)\right\}\left\{F'_{\lambda}(x_i,\alpha,\lambda)-F'_{\lambda}(x_{i-1},\alpha,\lambda)\right\}}{\{F(x_i,\alpha,\lambda)-F(x_{i-1},\alpha,\lambda)\}^2}\right] \\
&-\frac{1}{n+1}\left[\frac{\{1-F(x_n,\alpha,\lambda)\}F''_{\alpha\lambda}(x_n,\alpha,\lambda)+\left\{F'_{\alpha}(x_n,\alpha,\lambda)\right\}\left\{F'_{\lambda}(x_n,\alpha,\lambda)\right\}}{\{1-F(x_n,\alpha,\lambda)\}^2}\right]
\end{aligned} \tag{25}
$$

Where

$$F'_{\alpha}(x,\alpha,\lambda) = -(1-e^{-\lambda/x})^{\alpha}\ln(1-e^{-\lambda/x})$$

$$F''_{\alpha\alpha}(x,\alpha,\lambda) = (1-e^{-\lambda/x})^{\alpha}(\ln(1-e^{-\lambda/x}))^2$$
$$F'_{\lambda}(x,\alpha,\lambda) = -\frac{\alpha}{x}(1-e^{-\lambda/x})^{\alpha-1}e^{-\lambda/x}$$
$$F''_{\lambda\lambda}(x,\alpha,\lambda) = -\frac{\alpha}{x}\left\{\frac{\alpha-1}{x}(1-e^{-\lambda/x})^{\alpha-2}(e^{-\lambda/x})^2 - \frac{1}{x}(1-e^{-\lambda/x})^{\alpha-1}e^{-\lambda/x}\right\}$$

$$F''_{\lambda\alpha}(x,\alpha,\lambda) = -\frac{e^{-\lambda/x}}{x}(1-e^{-\lambda/x})^{\alpha-1}\left\{\alpha(\alpha-1)\ln(1-e^{-\lambda/x})+1\right\}$$

The asymptotic confidence intervals of the parameters of GIED using MLE is already calculated by A. M.Abouammoh and Arwa M. Alshingiti [12]. the first derivatives of the log likelihood function of GIED using MLE with respect to parameters are given by equation (10) and (11), and the second derivatives are as follows:

$$(\ln L)''_{\alpha\alpha} = -\frac{n}{\alpha^2} \tag{26}$$

$$(\ln L)''_{\lambda\lambda} = (\alpha-1)\sum_{i=1}^{n}\left[\frac{e^{-\frac{\lambda}{x_i}}}{x_i^2(1-e^{-\frac{\lambda}{x_i}})}\right] - \frac{n}{\lambda^2} \tag{27}$$

and the second derivative with respect to α,λ is given as:

$$(\ln L)''_{\alpha\lambda} = \sum_{i=1}^{n}\left[\frac{e^{-\frac{\lambda}{x_i}}}{x_i e^{-\frac{\lambda}{x_i}}}\right] \tag{28}$$

So on the basis of these derivatives, we obtain the information matrix I(α,λ). The approximate $(1-\beta)100\%$ confidence intervals for the parameters α and λ is given as, $\hat{\alpha} \pm \gamma_{\frac{\beta}{2}}\sqrt{V(\hat{\alpha})}$ and $\hat{\lambda} \pm \gamma_{\frac{\beta}{2}}\sqrt{V(\hat{\lambda})}$ respectively, where $\gamma_{\frac{\beta}{2}}$ is the upper $(\frac{\beta}{2})$ percentile of standard normal distribution, $\hat{\alpha} = \hat{\alpha}_{mp}$ and $\hat{\lambda} = \hat{\lambda}_{mp}$ are the MPS estimates of the parameter α and λ and $V(\hat{\alpha})$ and $V(\hat{\lambda})$ are elements of $I^{-1}(\hat{\alpha},\hat{\lambda})$.

4 Real data illustration

The data set considered in this section for illustration, contains strengths of glass polished aeroplane window. The use of this data set is described by Fuller et al. (1994) to predict the lifetime for a glass aeroplane window. Since the data were used in a study to predict failure times, such type of study is a form of reliability analysis. The data are as follows:

18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

The above data set is already fitted to GIED and the K-S statistics, between the fitted and the empirical distribution is also calculated and estimates of the parameter using MLE method is calculated by Abouammoh and Alshingiti [12]. MLE of the parameters of the GIED $\hat{\alpha}$ and $\hat{\lambda}$ are 90.855 and 148.412 respectively and for the same data set we have calculated the estimates of the parameter through MPS method and the estimates are $\hat{\alpha}_{mp}$ = 60.642 and $\hat{\lambda}_{mp}$ = 135.714 respectively, and corresponding K-S distance calculated using estimates of MLE and MPS respectively, and it comes out 0.137462 and 0.1207925 respectively. So on the basis of estimates and K-S statistics, for the considered data set MPS fits better as compared to MLE. The result of these distances shows that MPS serve better than MLE in this data set.

As K-S statistics, is extensively used for different model comparison and is considered one of the best way of model comparison, but no one had paid attention on comparison of different estimation procedure based on K-S statistics. In literature several authors have discussed K-S methodology for different model comparison in terms of distances for a given data set. Considering the similar approach on the basis of the K-S statistics, here, we propose comparison of estimation procedure or method as least K-S distance provides the better method of estimation for given data set. For the above data set we notice that K-S distance through MPS is smaller than K-S distance through MLE. In the support of the above proposition empirical cumulative distribution function (ECDF) plot has been given below for MPS and MLE. AIC

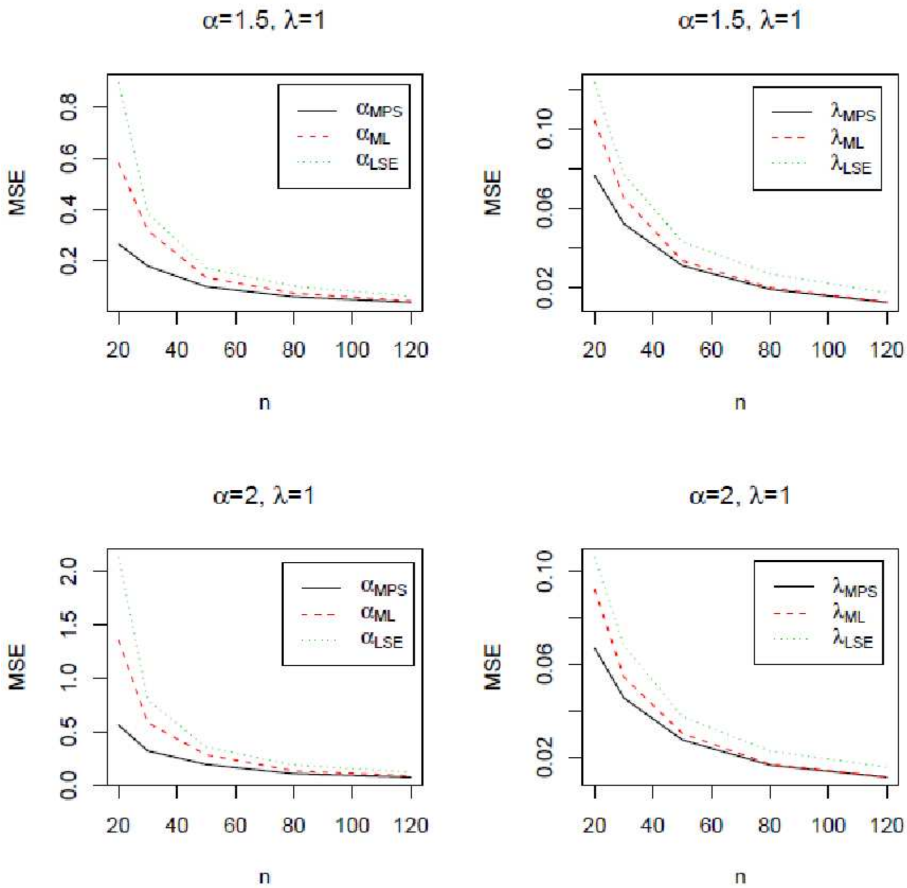


Fig. 3: Figure 2: Mean Square Error of the estimates for $\alpha = 1.5, 2$ and $\lambda = 1$ with variation of sample size (n)

is also calculated using both MLE and MPS for this real data set and it comes out smaller when estimates were provided from MPS as compare to MLE. Since MPS provides lesser K-S distance and AIC, we may say that MPS serve better than MLE for the considered data set. Thus we propose to estimation method comparison on the basis of K-S statistic.

$$AIC_{MLE} = 9.234804$$
$$AIC_{MPS} = 9.138706$$

5 Simulation studies

In order to compare the MPS, the MLE and the LSE methods in parameter estimation, a set of simulation was done based on GIED. We have generated five thousand samples from GIED for different parameters settings. It may be mentioned here that the exact expression of MSE can not be obtained because estimates are not found in nice close forms. It may be also noted here that MSE will depend on sample size n, scale parameter λ and shape parameter α respectively. In this study different variation of sample size(n) say n(=20,40,60,80,100,120), shape parameter α say $\alpha(=1,2,3,4,5,6,7)$ and scale parameter λ say λ(=1,2,3,4,5,6,7) have been considered. To study the effect of variation of the sample size n we have considered the simulated MSE for α and λ. Firstly, sample size n is varied i.e different values of n are taken for fixed value of scale parameter λ taken as 1 and 2, for different choices of shape parameter α as α(= 0.5,1,1.5,2,3,4). Secondly, we varied shape parameter α i.e different values of α for two different choices of sample sizes (n=30 and n=50) for fixed values of scale parameter λ as $\lambda(= 1,2)$, thirdly, we varied the scale parameter λ i.e different values of λ is taken for two different choices of sample sizes (n=30 and n=50) for fixed values of shape parameter α as (α = 2,3), corresponding graphs are attached. For all the above considered choices graph of MSE is plotted and attached.

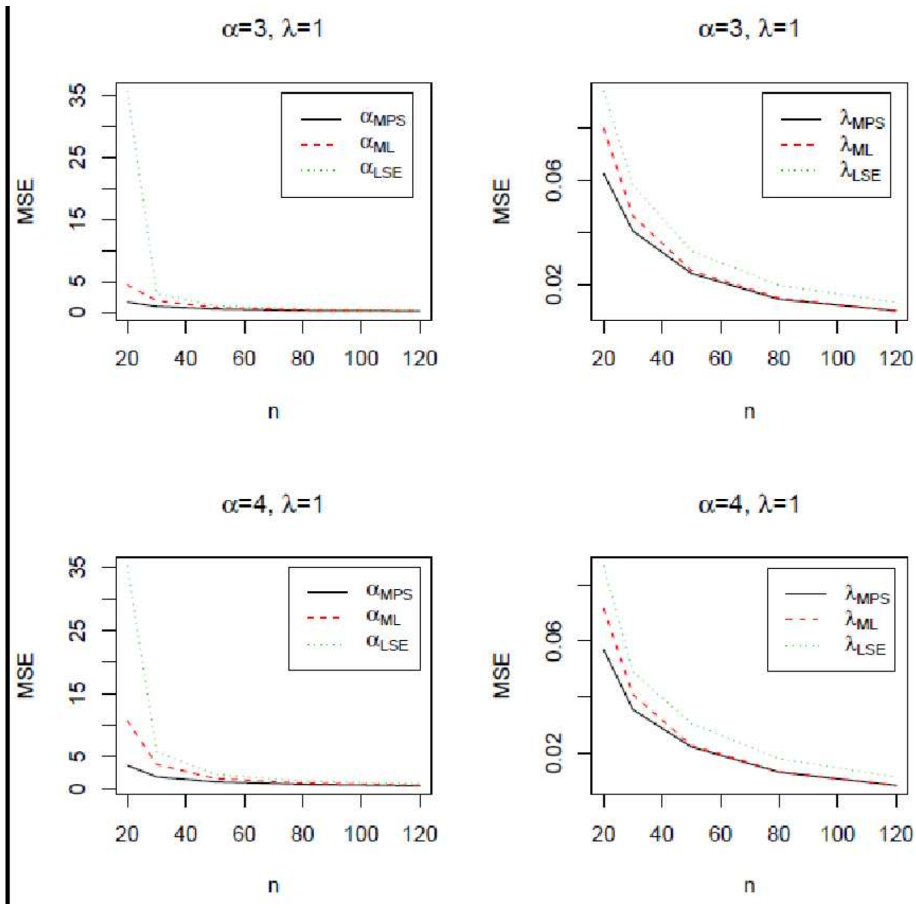
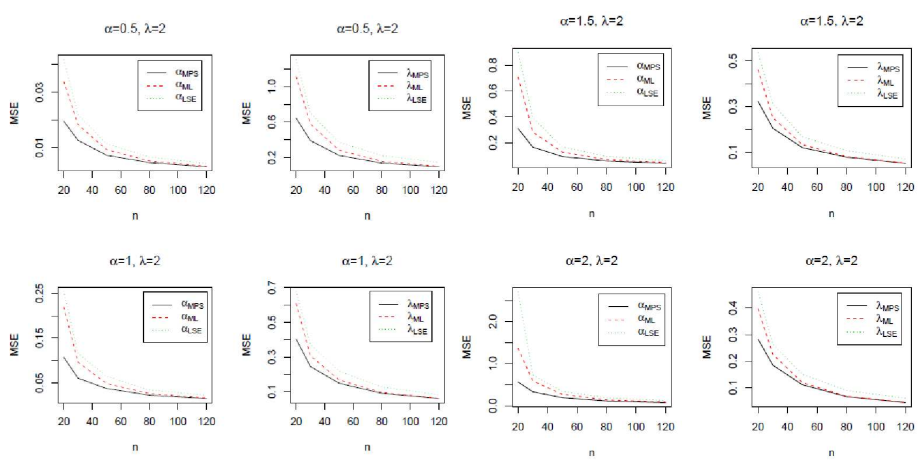


Fig. 4: Mean Square Error of the estimates for $\alpha = 3, 4$ and $\lambda = 1$ with variation of sample size (n)



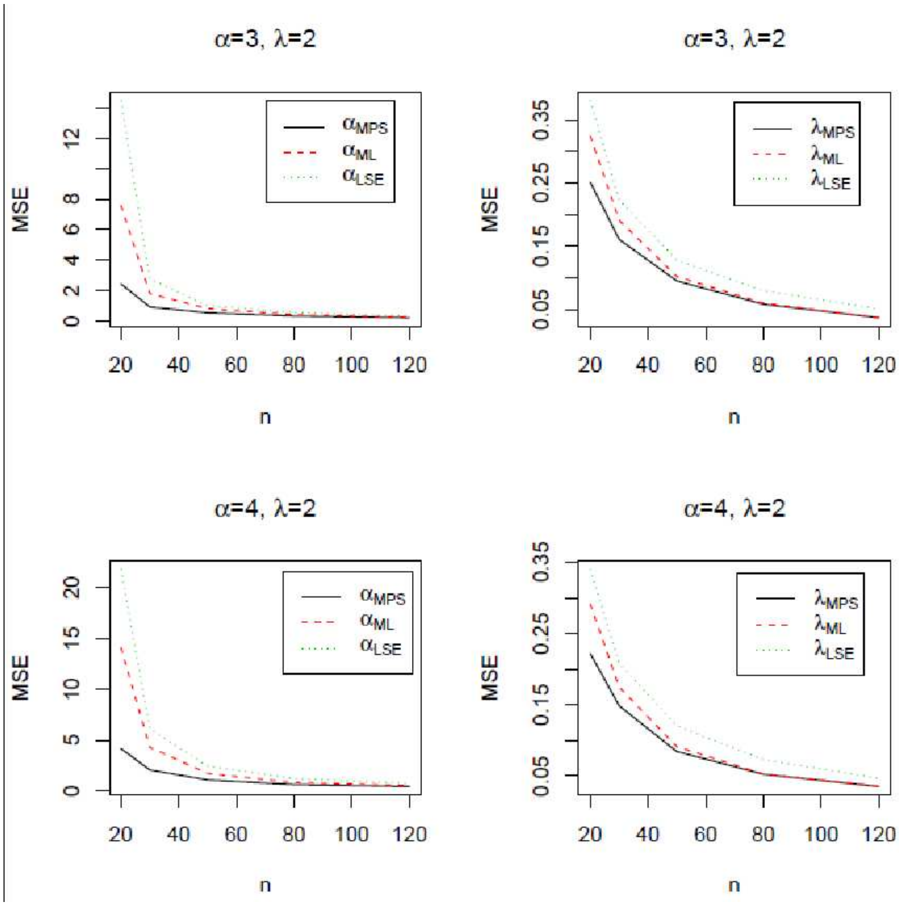


Fig. 5: Variation of sample size with respect to MSE for different choices of α and $\lambda = 2$

We have chosen sample size $n = 30$ in second and third case because at $n = 30$ changes are more visible in terms of MSE. We have also estimates the reliability and hazard functions for α and λ = 1,2 and 3, for sample size $n = 20, 30, 50, 80$ considering MLE and MPS both, for this see Table 1, 2. Further we have constructed the confidence interval and coverage probability for different choices of α and λ = 1,2 and 3 and for sample size $n = 20, 30, 50, 80$ for this see table 2,3,4 and 5. We have also compared the average length of confidence intervals of MPS with corresponding MLEs. R software is used in all computations.

On the basis of the results summarized in graphs and table, some conclusions can be drawn which are stated as follows:

1: It is observed that as sample size increases for fixed values of α and λ the mean square error of the estimates decreases in all the three considered methods and for large size of n i.e after $n = 80$ all of them are nearly equivalent but MPS performs better than other two considered method, see Figures 1 and 2. It is also observed that as n increases MSE of reliability and hazard function follows the similar trend as stated above and here also MPS perform better than MLE (see Table 1). Furthermore, it is noticed that the average length of confidence interval decreases as sample size n increases in both the considered case of MLE as well as MPS but the average length is smaller in case of MPS as compared to MLE. The coverage probability obtained here fairly attains the prescribed confidence interval. This is also true for small sample size, see Tables 2-5.

2: From graph and table, it is observed that as shape parameter α increases for fixed sample size n and λ the MSE of the estimates of shape parameter α increases for all the three considered cases but MSE of MPS is smaller than LSE as well as MLE (see figure 3). It is noticed that for smaller values of shape parameter all of them are equally good but for larger value of shape parameter MPS is better than other two. Generally, shape parameters are difficult to estimate in most of the cases but here in case of GIED, one should use MPS for all choices of λ and n. It is also observed that for fixed value of λ as the shape parameter α increases MSE of reliability decreases but reverse trend is obtained in the case

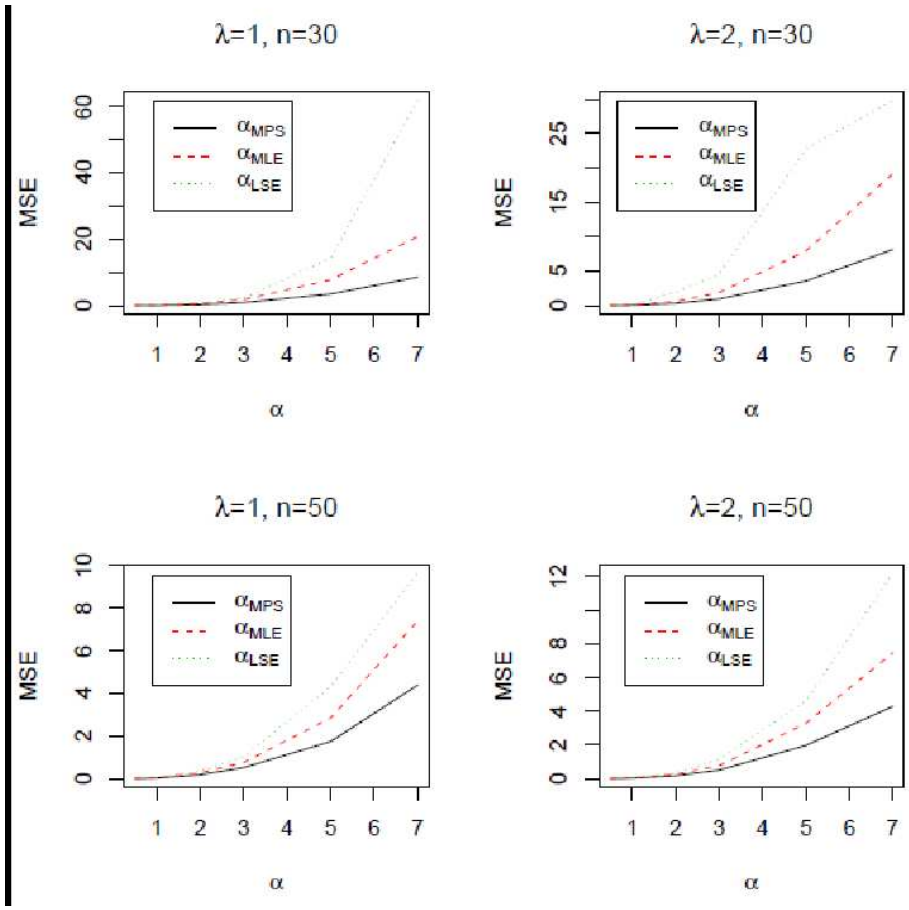


Fig. 6: Variation of α for different choices of sample size (n=30,50) and $\lambda = 1,2$

of hazard, it is true for MPS and MLE both but here also MPS serve better than MLE in terms of MSE. The average length of confidence interval increases as we increase the shape parameter α in both the case of MPS and MLE . Further more, it is also observed that the coverage probability obtained here fairly attains the prescribed confidence interval.

3: It is observed that as scale parameter λ increases for fixed sample size n and shape parameter α the MSE of the estimates of scale parameter λ increases and MPS performs better in terms of MSE in comparison to other two methods. It is also observed that for smaller value of λ all of them are equivalent but for larger value of λ MPS is far better than LSE as well as MLE see figure 4. It is also observed that for fixed value of α as the scale parameter λ increases MSE of reliability decreases but reverse trend is obtained in the case of hazard in case of MLE and MPS both see Table 1. The average length of confidence interval increases as we increase the scale parameter λ for fixed value of shape parameter α similar trend has been observed in both the case of MPS and MLE (see Table 2-5). Further more, it is also observed that the coverage probability obtained here fairly attains the prescribed confidence interval and no any specific trend has been observed.

4: From the plot of ECDF and the value of AIC, it is observed that estimates of MPS fits better than the estimates of MLE.

6 Conclusion

This paper involved the comparison of estimates obtained by MPS, MLE, and LSE method using GIED. For smaller sample size it is advised to use MPS as it perform better than LSE as well as MLE. In this paper we have considered the problem of point estimation, confidence interval and it also introduces comparison of different estimation procedure on

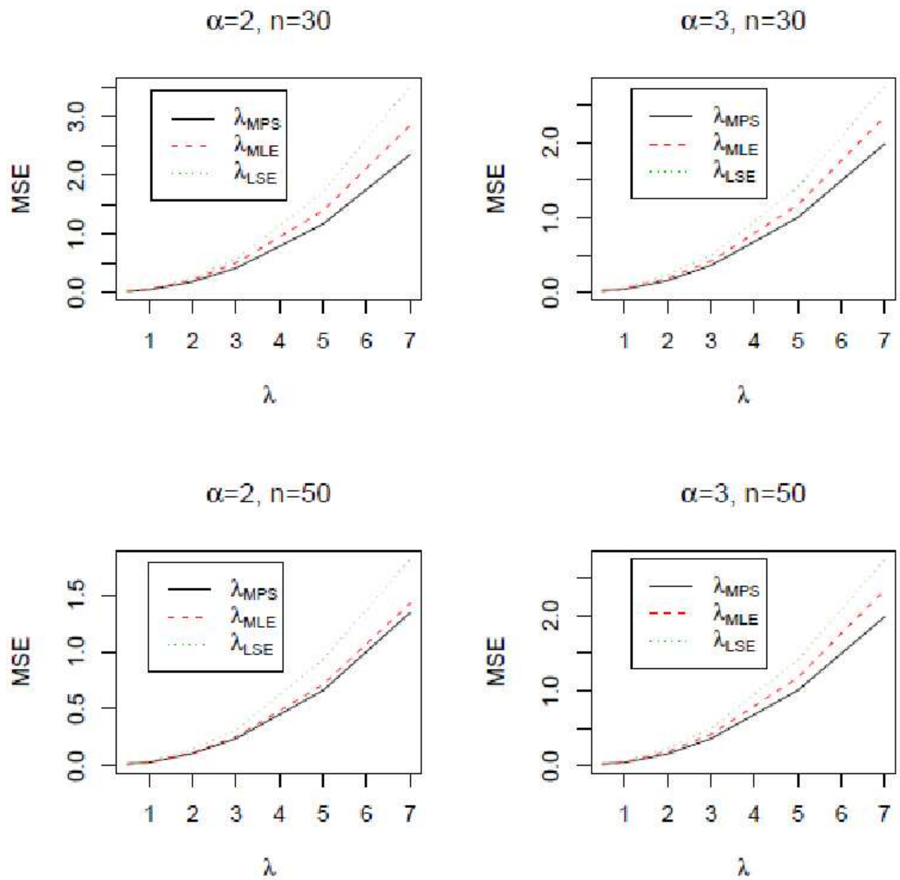


Fig. 7: Variation of λ for different choices of sample size (n=30,50) and $\alpha = 2,3$

the basis of K-S statistic, for this a real data has been incorporated and ECDF plot is also given. From the graphs, it is observed that all the estimates appears to be consistent. The average length of confidence interval using MPS is smaller than that of MLE. We have found that MPS method outperforms the other two method with smaller mean square error. The findings of this paper will be very useful to researchers, statistician and engineers where such types of things were required and also in cases where we have small sample size to analyse and exclusively where GIED is used.

Table 1: Average estimates and corresponding MSEs of the reliability and hazard function using MLE and MPS respectively at specified time say(t=4) for different parameter settings and sample size *n*.

	n=20				n=30				n=50				n=80			
Para	rt		ht		rt		ht		rt		ht		rt		ht	
Reliability and hazard using MLE,t=4																
1,1	0.2134	0.0058	0.2499	0.0076	0.2161	0.0038	0.2392	0.0038	0.2180	0.0023	0.2306	0.0019	0.2194	0.0014	0.2261	0.0010
1,2	0.3914	0.0080	0.2118	0.0042	0.3898	0.0051	0.2057	0.0024	0.3923	0.0030	0.1997	0.0011	0.3928	0.0018	0.1969	0.0006
1,3	0.5296	0.0081	0.1800	0.0024	0.5294	0.0050	0.1750	0.0013	0.5283	0.0030	0.1724	0.0007	0.5291	0.0018	0.1705	0.0004
2,1	0.0487	0.0012	0.5252	0.0536	0.0476	0.0008	0.4950	0.0245	0.0487	0.0005	0.4677	0.0110	0.0491	0.0003	0.4561	0.0060
2,2	0.1495	0.0042	0.4447	0.0314	0.1490	0.0029	0.4265	0.0154	0.1518	0.0017	0.4063	0.0068	0.1529	0.0011	0.3985	0.0039
2,3	0.2715	0.0066	0.3762	0.0159	0.2743	0.0044	0.3619	0.0085	0.2761	0.0026	0.3509	0.0042	0.2774	0.0016	0.3449	0.0023
3,1	0.0124	0.0002	0.8143	0.1636	0.0119	0.0001	0.7557	0.0740	0.0117	0.0001	0.7097	0.0316	0.0112	0.0000	0.6950	0.0178
3,2	0.0594	0.0016	0.6956	0.0999	0.0589	0.0010	0.6533	0.0481	0.0606	0.0006	0.6147	0.0198	0.0601	0.0004	0.6023	0.0105
3,3	0.1418	0.0042	0.5877	0.0573	0.1436	0.0027	0.5527	0.0256	0.1441	0.0016	0.5334	0.0123	0.1458	0.0010	0.5205	0.0068
MAXIMUM PRODUCT SPACINGS R(t) H(t) (MPS)																
1,1	0.2425	0.0057	0.1991	0.0034	0.2383	0.0038	0.2040	0.0022	0.2333	0.0023	0.2066	0.0013	0.2301	0.0014	0.2098	0.0008
1,2	0.4048	0.0065	0.3128	0.0180	0.4002	0.0044	0.3187	0.0182	0.3993	0.0027	0.3255	0.0190	0.3977	0.0017	0.3301	0.0198
1,3	0.5263	0.0064	0.3700	0.0429	0.5270	0.0042	0.3767	0.0450	0.5265	0.0026	0.3843	0.0477	0.5278	0.0017	0.3898	0.0497
2,1	0.0717	0.0023	0.2275	0.0497	0.0648	0.0013	0.2308	0.0464	0.0606	0.0007	0.2326	0.0446	0.0574	0.0005	0.2361	0.0425
2,2	0.1801	0.0048	0.3545	0.0072	0.1721	0.0032	0.3612	0.0044	0.1679	0.0019	0.3655	0.0025	0.1641	0.0012	0.3710	0.0015
2,3	0.2952	0.0058	0.4174	0.0107	0.2921	0.0041	0.4254	0.0106	0.2883	0.0025	0.4325	0.0108	0.2860	0.0016	0.4376	0.0113
3,1	0.0239	0.0006	0.2560	0.1711	0.0201	0.0003	0.2580	0.1662	0.0172	0.0002	0.2599	0.1626	0.0148	0.0001	0.2653	0.1574
3,2	0.0850	0.0027	0.4009	0.0432	0.0780	0.0016	0.4046	0.0372	0.0739	0.0009	0.4081	0.0326	0.0694	0.0005	0.4142	0.0291
3,3	0.1723	0.0048	0.4706	0.0106	0.1667	0.0030	0.4747	0.0061	0.1601	0.0018	0.4825	0.0035	0.1570	0.0011	0.4873	0.0021

Table 2: Average estimates, coverage probability (in the brackets) and corresponding confidence intervals of the parameters α and λ using MPS for different choices of parameter and for sample sine n=20 and 30.

CI AND COVERAGE PROBABILITY MPS				
	n=20		n=30	
Para	α(CP)	λ(CP)	α(CP)	λ(CP)
1,1	0.9228 (0.9654)	0.8999 (0.964)	0.9298 (0.9636)	0.9174 (0.9548)
	(0.2395,1.4505)	(0.1284,1.2788)	(0.8975,1.8539)	(0.5718,1.4983)
1,2	0.9294 (0.966)	1.8253 (0.9628)	0.9355 (0.966)	1.8516 (0.953)
	(0.1571,1.4442)	(0.1775,2.5652)	(0.6109,1.5836)	(1.8005,3.6929)
1,3	0.9234 (0.9656)	2.7147 (0.9618)	0.9352 (0.965)	2.7648 (0.956)
	(0.4953,1.7380)	(2.0397,5.5381)	(0.0923,1.0677)	(0.9450,3.81775)
2,1	1.8310 (0.9642)	0.9134 (0.9482)	1.8581 (0.9634)	0.93018 (0.9528)
	(0,2.8277)	(0.1480,1.1370)	(0.2349,2.4415)	(0.5779,1.37016)
2,2	1.8480 (0.9714)	1.8184 (0.9566)	1.8453 (0.9648)	1.8577 (0.95)
	(0,3.0538)	(0.6993,2.6476)	(0.7629,3.0160)	(1.0253,2.6279)
2,3	1.8430 (0.966)	2.7441 (0.9596)	1.8522 (0.9666)	2.7820 (0.9484)
	(1.1893,4.1223)	(1.6403,4.6007)	(0.5983,2.8156)	(0.5318,2.8675)
3,1	2.7630 (0.9668)	0.9145 (0.9596)	2.7674 (0.9682)	0.9317 (0.95)
	(0.3334,5.4452)	(0.7535,1.6584)	(2.2181,5.9873)	(0.6909,1.4533)
3,2	2.7461 (0.963)	1.8299 (0.953)	2.7542 (0.967)	1.8553 (0.9474)
	(0,3.9963)	(0.5667,2.3851)	(0,3.6252)	(0.9020,2.3533)
3,3	2.7456 (0.963)	2.7479 (0.953)	2.7661 (0.9698)	2.7930 (0.9456)
	(0,4.9593)	(1.7279,4.5179)	(0.7889,4.5274)	(1.5604,3.7566)

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Table 3: Average estimates, coverage probability (in the brackets) and corresponding confidence intervals of the parameters α and λ using MPS for different choices of parameter and for sample sine n=50 and 80.

CI AND COVERAGE PROBABILITY MPS				
	n=50		n=80	
Para	α(CP)	λ(CP)	α(CP)	λ(CP)
1,1	0.9422 (0.9572)	0.9334 (0.94940)	0.9584 (0.9518)	0.9554 (0.9450)
	(0.5423,1.2488)	(0.3020,1.0117)	(0.5543,1.1285)	(0.7195,1.2810)
1,2	0.9428 (0.9608)	1.8729 (0.9530)	0.9575 (0.9544)	1.9056 (0.9468)
	(0.7531,1.4718)	(1.2761,2.7254)	(0.4942,1.0670)	(1.0908,2.2410)
1,3	0.9415 (0.9624)	2.8127 (0.9470)	0.9565 (0.9534)	2.8486 (0.9446)
	(0.6232,1.3199)	(2.1661,4.2704)	(0.5486,1.1047)	(1.8874,3.5803)
2,1	1.8836 (0.9632)	0.9477 (0.9480)	1.8898 (0.9550)	0.9556 (0.9438)
	(0.9364,2.6279)	(0.6742,1.2819)	(0.7514,2.0445)	(0.5774,1.0650)
2,2	1.8699 (0.9616)	1.8802 (0.9434)	1.8967 (0.9498)	1.9134 (0.9440)
	(1.8147,3.4827)	(1.9248,3.1392)	(1.5378,2.8039)	(1.6764,2.6076)
2,3	1.8765 (0.9604)	2.8379 (0.9468)	(1.9003,0.9538)	(2.8732,0.9392)
	(1.0153,2.7006)	(1.2994,3.1588)	(1.4479,2.7118)	(2.4640,3.8683)
3,1	2.7939 (0.9636)	0.9454 (0.9434)	2.8391 (0.9540)	0.9625 (0.9426)
	(1.3852,4.0453)	(0.5759,1.1409)	(1.1708,3.2430)	(0.5245,0.9668)
3,2	2.7949 (0.9606)	1.9041 (0.9448)	2.8427 (0.9600)	1.9209 (0.9460)
	(0.8577,3.4972)	(1.1978,2.3377)	(1.8999,4.0433)	(1.5154,2.4169)
3,3	2.7865 (0.9628)	2.8346 (0.9382)	2.8471 (0.9556)	2.8852 (0.9428)
	(1.9603,4.6485)	(2.0974,3.7866)	(1.4215,3.5387)	(2.1025,3.4448)

Table 4: Average estimates, coverage probability (in the brackets) and corresponding confidence intervals of the parameters α and λ using MLE for different choices of parameter and for sample size *n*=20 and 30.

CI AND COVERAGE PROBABILITY MLE				
	n=20		n=30	
Para	α(CP)	λ(CP)	α(CP)	λ(CP)
1,1	1.1618 (0.9706)	1.1246 (0.9538)	1.1001 (0.9568)	1.0827 (0.9502)
	(0.4667,1.8570)	(0.5008,1.7484)	(0.5712,1.6290)	(0.5865,1.5788)
1,2	1.1708 (0.9724)	2.2800 (0.9476)	1.1076 (0.9622)	2.1853 (0.947)
	(0.4680,1.8740)	(1.0174,3.5427)	(0.5745,1.6408)	(1.1859,3.1846)
1,3	1.1631 (0.9642)	3.3926 (0.9556)	1.1071 (0.9598)	3.2634 (0.9396)
	(0.4661,1.8591)	(1.5103,5.2748)	(0.5741,1.6402)	(1.7709,4.7559)
2,1	2.4310 (0.965)	1.1055 (0.9418)	2.2820 (0.9666)	1.0713 (0.9484)
	(0.7223,4.1396)	(0.5803,1.6307)	(1.0060,3.5580)	(0.6529,1.4897)
2,2	2.4551 (0.9636)	2.2002 (0.9516)	2.2667 (0.9666)	2.1409 (0.943)
	(0.7196,4.1907)	(1.1558,3.2442)	(1.0009,3.5325)	(1.3039,2.9780)
2,3	2.4471 (0.9622)	3.3202 (0.9486)	2.2749 (0.9686)	3.2053 (0.9506)
	(0.7244,4.1698)	(1.7454,4.8951)	(1.0042,3.5456)	(1.9530,4.4575)
3,1	3.8064 (0.9678)	1.0929 (0.9526)	3.4870 (0.9616)	1.0633 (0.94)
	(0.8246,6.7882)	(0.6092,1.5765)	(1.3339,5.6402)	(0.6762,1.4504)
3,2	3.7745 (0.9658)	2.1865 (0.9486)	3.4666 (0.9654)	2.1173 (0.9526)
	(0.8276,6.7213)	(1.2178,3.1552)	(1.3316,5.6016)	(1.3458,2.8888)
3,3	3.7787 (0.964)	3.2833 (0.942)	3.4836 (0.962)	3.1872 (0.9498)
	(0.8252,6.7323)	(1.8293,4.7372)	(1.3351,5.6322)	(2.0271,4.3473)

Table 5: Average estimates, coverage probability (in the brackets) and corresponding confidence intervals of the parameters α and λ using MLE for different choices of parameter and for sample size *n*=50 and 80.

CI AND COVERAGE PROBABILITY MLE				
	n=50		n=80	
Para	α(CP)	λ(CP)	α(CP)	λ(CP)
1,1	1.0553 (0.9606)	1.0452 (0.9522)	1.0369 (0.9532)	1.0335 (0.9522)
	(0.6670,1.4436)	(0.6708,1.4196)	(0.7368,1.3371)	(0.7397,1.3274)
1,2	1.0562 (0.9602)	2.0975 (0.9482)	1.0358 (0.9524)	2.0609 (0.9458)
	(0.6675,1.4450)	(1.3465,2.8486)	(0.7356,1.3355)	(1.4749,2.6470)
1,3	1.0548 (0.9668)	3.1500 (0.9556)	1.0350 (0.9568)	3.0826 (0.9486)
	(0.6668,1.4429)	(2.0217,4.2784)	(0.7355,1.3346)	(2.2057,3.9595)
2,1	2.1621 (0.9636)	1.0437 (0.9494)	2.0786 (0.9534)	1.0220 (0.9486)
	(1.2431,3.0811)	(0.7256,1.3617)	(1.3882,2.7690)	(0.7743,1.2698)
2,2	2.1464 (0.9532)	2.0709 (0.9478)	2.0859 (0.9556)	2.0463 (0.957)
	(1.2354,3.0575)	(1.439,2.7029)	(1.3928,2.7792)	(1.5506,2.5420)
2,3	2.1545 (0.9582)	3.1260 (0.9446)	2.0904 (0.958)	3.0729 (0.9538)
	(1.2390,3.0698)	(2.1725,4.0796)	(1.3953,2.7856)	(2.3286,3.8173)
3,1	3.2590 (0.9626)	1.0344 (0.9486)	3.1591 (0.964)	1.0249 (0.953)
	(1.7397,4.7783)	(0.7407,1.3280)	(2.0087,4.3095)	(0.7942,1.2555)
3,2	3.2611 (0.9666)	2.0836 (0.9482)	3.1631 (0.9566)	2.0452 (0.9492)
	(1.7410,4.7812)	(1.4922,2.6750)	(2.0102,4.3160)	(1.5849,2.5055)
3,3	3.2516 (0.9602)	3.1019 (0.9462)	3.1685 (0.956)	3.0720 (0.9514)
	(1.7368,4.7663)	(2.2213,3.9826)	(2.0133,4.3237)	(2.3809,3.7631)

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