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Concomitants of Case-II of Generalized Order Statistics from Farlie-Gumbel-Morgenstern Distributions

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Abstract: In this paper, the concomitants of case-II of generalized order statistics from Farlie-Gumbel-Morgenstern (FGM) distributions and recurrence relations between their moments are studied. Further, we provide the minimum variance linear unbiased estimator of the location and scale parameters of the concomitants of ordinary order statistics from some distributions.

Keywords: Generalized order statistics; Ordinary order statistics; Concomitants; Recurrence relation; Minimum variance linear unbiased estimator; Farlie-Gumbel-Morgenstern distributions.

1 Introduction

Kamps [6] has introduced case-I of generalized order statistics, Beg and Ahsanullah [1] have studied the concomitants of case-I of generalized order statistics from Farlie-Gumbel-Morgenstern (FGM) distributions. Kamps and Cramer [7] have introduced case-II of generalized order statistics. The main aim of this paper is to study the concomitants of case-II of generalized order statistics from Farlie-Gumbel-Morgenstern (FGM) distributions and obtain recurrence relations between moments.

Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \dots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k+n-r+M_r \geq 1$ for all $r \in \{1, 2, \dots, n-1\}$, and let $\tilde{m} = (m_1, \dots, m_{n-1}) \in \Re^{n-1}$. Let F be a distribution function, the random variables $X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$, are called generalized order statistics (based on F). The joint density function of the generalized order statistics $X(1, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$ is given by (see Kamps [6]):

$$f_{X_{(1,n,\tilde{m},k)},\dots,X_{(n,n,\tilde{m},k)}}(x_1, \dots, x_n) = k(\prod_{j=1}^{n-1} \gamma_j)(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i} f(x_i))(1-F(x_n))^{k-1} f(x_n),$$

$x_1 \leq x_2 \leq \dots \leq x_n$.

Kamps and Cramer [7] have introduced some lemmas about the one and two dimensional marginal density functions of X_s and X_r, $1 \leq r < s \leq n$, of case-II of generalized order statistics as follow:

$$f_{X_{(r,n,\tilde{m},k)}}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i - 1} f(x), \tag{1}$$

$$f_{X_{(r,n,\tilde{m},k)},X_{(s,n,\tilde{m},k)}}(x,y) = c_{s-1} \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{1-F(y)}{1-F(x)} \right)^{\gamma_i} \right) \left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} \right) \left(\frac{f(x)}{1-F(x)} \right) \times \left(\frac{f(y)}{1-F(y)} \right), \tag{2}$$

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where

$$a_i(r)=\prod_{j=1,i\neq j}^{r}\frac{1}{\gamma_j-\gamma_i},\gamma_j\neq\gamma_i,1\leq i\leq r\leq n,$$

$$a_i^{(r)}(s)=\prod_{j=r+1,i\neq j}^{s}\frac{1}{\gamma_j-\gamma_i},\gamma_j\neq\gamma_i,r+1\leq i\leq s\leq n,$$

$$a_i(s)=a_i^{(0)}(s),c_{r-1}=\prod_{j=1}^{r}\gamma_j,1\leq r\leq n.$$

Remark 1.1. Case-I of generalized order statistics can be considered as a special case of case-II of generalized order statistics as follow: For $m_i=m_j=m$, we have:

$$a_i(r)=\frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!},$$

$$a_i^{(r)}(s)=\frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}.$$

Thus, case-II reduces to case-I of generalized order statistics.

Morgenstern [10] has introduced FGM distributions, Gumbel [5] has studied FGM for exponential distribution. Farlie [4] has considered this family in the general form. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of the random variables X and Y, respectively, then the probability density function (pdf) of the bivariate FGM distributions is given by:

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)],\quad -1\leq\alpha\leq 1. \tag{3}$$

Here, $f_X(x)$ and $f_Y(y)$ are the marginal $pdf's$ of X and Y respectively. The parameter α is known as the dependence parameter of the random variables X and Y. If α is zero, then X and Y are independent. The conditional probability density function (pdf) of Y given X is given by:

$$f_{Y|X}(y\mid x)=f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)],\quad -1\leq\alpha\leq 1. \tag{4}$$

The general theory of concomitants of order statistics has originally studied by David, O'Connell and Yang [3]. Let (X_i,Y_i), $i=1,2,...,n$, be n pairs of independent random variables from some bivariate population with distribution function (df) $F(x,y)$. Let $X_{(r,n,\tilde{m},k)}$ be the $r-th$ generalized order statistics of case-II, then the Y value associated with $X_{(r,n,\tilde{m},k)}$ is called the concomitant of the $r-th$ generalized order statistics of case-II and is denoted by $Y_{[r,n,\tilde{m},k]}$. The pdf of $Y_{[r,n,\tilde{m},k]}$ is given BY:

$$g_{[r,n,\tilde{m},k]}(y)=g_{Y_{[r,n,\tilde{m},k]}}(y)=\int_{-\infty}^{\infty}f_{Y|X}(y\mid x)f_{(r,n,\tilde{m},k)}(x)dx, \tag{5}$$

where $f_{(r,n,\tilde{m},k)}(x)$ is the pdf of $X_{(r,n,\tilde{m},k)}$ defined in (1). Moreover, the joint pdf of the concomitants $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ of the $r-th$ and $s-th$ generalized order statistics of case-II has the form:

$$g_{[r,s,n,\tilde{m},k]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f_{Y|X}(y_1\mid x_1)f_{Y|X}(y_2\mid x_2)f_{(r,s,n,\tilde{m},k)}(x_1,x_2)dx_1dx_2, \tag{6}$$

where $f_{(r,s,n,\tilde{m},k)}(x_1,x_2)$ is the joint pdf of $X_{[r,n,\tilde{m},k]}$ and $X_{[s,n,\tilde{m},k]}$ defined in (2), $1\leq r<s\leq n$.

2 Concomitants of Case-II of Generalized Order Statistics

For the FGM distributions with pdf given by (3), utilizing (1) and (4) in (5), we obtain the pdf of $Y_{[r,n,\tilde{m},k]}$ as follow:

$$\begin{aligned}g_{[r,n,\tilde{m},k]}(y)&=\int_{-\infty}^{\infty}f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)]\times c_{r-1}\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i-1}f(x)dx\\&=f_Y(y)-\alpha c_{r-1}f_Y(y)(2F_Y(y)-1)\int_{-\infty}^{\infty}(1-2F_X(x))\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i-1}f(x)dx\\&=f_Y(y)-\alpha c_{r-1}f_Y(y)(2F_Y(y)-1)\int_{-\infty}^{\infty}(2(1-F_X(x))-1)\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i-1}f(x)dx\\&=f_Y(y)+f_Y(y)(2F_Y(y)-1)D_{[r,n,\tilde{m},k]},\end{aligned} \tag{7}$$

where $D_{[r,n,\tilde{m},k]} = \alpha[1 - 2c_{r-1}\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}]$, $\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i} = \frac{1}{c_{r-1}}$.

2.1 Recurrence Relation between Moments of Concomitants

We will study the recurrence relation between moments of concomitants of case-II of generalized order statistics and the ordinary order statistics as follows:
Equation (7) can be written as:

$$g_{[r,n,\tilde{m},k]}(y) = f_{Y_{1:1}}(y) + D_{[r,n,\tilde{m},k]}[f_{Y_{2:2}}(y) - f_{Y_{1:1}}(y)], \tag{8}$$

where $f_{Y_{1:1}}(y) = f_Y(y)$ is the pdf of $Y_{1:1}$, the first ordinary order statistic of a random sample of size one of Y variate, and $f_{Y_{2:2}}(y) = 2F_Y(y)f_Y(y)$ is the pdf of $Y_{2:2}$, the second ordinary order statistic of a random sample of size two of Y variate. Using this result, we derive the moments and the moment generating function of $Y_{[r,n,\tilde{m},k]}$ as follows: From (8), the $p-th$ moment of $Y_{[r,n,\tilde{m},k]}$ is

$$\begin{aligned} \mu^p_{[r,n,\tilde{m},k]} &= E[Y^p_{[r,n,\tilde{m},k]}] = \int_{-\infty}^{\infty} y^p g_{[r,n,\tilde{m},k]}(y)dy \\ &= \mu^p_{1:1} + D_{[r,n,\tilde{m},k]}[\mu^p_{2:2} - \mu^p_{1:1}], \end{aligned} \tag{9}$$

where $\mu^p_{1:1} = E[Y^p_{1:1}]$ and $\mu^p_{2:2} = E[Y^p_{2:2}]$. Also, the moment generating function of $Y_{[r,n,\tilde{m},k]}$ is given by:

$$M_{[r,n,\tilde{m},k]}(t) = M_{1:1}(t) + D_{[r,n,\tilde{m},k]}[M_{2:2}(t) - M_{1:1}(t)], \tag{10}$$

where $M_{1:1}(t) = E[\exp(tY_{1:1})]$ and $M_{2:2}(t) = E[\exp(tY_{2:2})]$.

Theorem 2.1. Let $1 \le i_1 \le i_2 \le n-r$ and $1 \le j_1 \le j_2 \le r-1$. For a bivariate random variable (X,Y) having pdf in (3), the following recurrence relations between moments of concomitants are valid:

$$\mu^p_{[r-j_1,n-i_1,\tilde{m},k]} - \mu^p_{[r-j_2,n-i_2,\tilde{m},k]} = [D_{[r-j_1,n-i_1,\tilde{m},k]} - D_{[r-j_2,n-i_2,\tilde{m},k]}][\mu^p_{2:2} - \mu^p_{1:1}], \tag{11}$$

and the relation between the moment generating functions of concomitants are:

$$M_{[r-j_1,n-i_1,\tilde{m},k]}(t) - M_{[r-j_2,n-i_2,\tilde{m},k]}(t) = [D_{[r-j_1,n-i_1,\tilde{m},k]} - D_{[r-j_2,n-i_2,\tilde{m},k]}][M_{2:2}(t) - M_{1:1}(t)]. \tag{12}$$

3 Joint Distribution of Two Concomitants of Case-II of Generalized Order Statistics

In this section, we derive the joint distribution of concomitants of two generalized order statistics of case-II. From (6) we have:

$$g_{[r,s,n,\tilde{m},k]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{Y|X}(y_1 \mid x_1) f_{Y|X}(y_2 \mid x_2) f_{(r,s,n,\tilde{m},k)}(x_1,x_2)dx_1dx_2$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f_Y(y_1)[1+\alpha(2F_X(x_1)-1)(2F_Y(y_1)-1)] \\
&\quad \times f_Y(y_2)[1+\alpha(2F_X(x_2)-1)(2F_Y(y_2)-1)] \\
&\quad \times c_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)(1-F_X(x_1))^{\gamma_i}\right]\left(\frac{f_X(x_1)}{1-F_X(x_1)}\right) \\
&\quad \times \left(\frac{f_X(x_2)}{1-F_X(x_2)}\right) dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f_Y(y_1)[1+\alpha(2F_X(x_1)-1)(2F_Y(y_1)-1)] \\
&\quad \times f_Y(y_2)[1+\alpha(2F_X(x_2)-1)(2F_Y(y_2)-1)] \\
&\quad \times c_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)(1-F_X(x_1))^{\gamma_i}\right] \\
&\quad \times \left(\frac{f_X(x_1)}{1-F_X(x_1)}\right)\left(\frac{f_X(x_2)}{1-F_X(x_1)}\right)\left(\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{-1} dx_1 dx_2 \\
&= c_{s-1}\int_{-\infty}^{\infty} f_Y(y_1)[1+\alpha(2F_X(x_1)-1)(2F_Y(y_1)-1)] \\
&\quad \times \left[\sum_{i=1}^{r} a_i(r)(1-F_X(x_1))^{\gamma_i-1} f_X(x_1)\right] \\
&\quad \times \left\{\int_{x_1}^{\infty} f_Y(y_2)[1+\alpha(2F_X(x_2)-1)(2F_Y(y_2)-1)]\right. \\
&\quad \times \left.\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{\gamma_i-1}\left(\frac{f_X(x_2)}{1-F_X(x_1)}\right)\right] dx_2\right\} dx_1 \\
&= c_{s-1}\int_{-\infty}^{\infty} f_Y(y_1)[1+\alpha(2F_X(x_1)-1)(2F_Y(y_1)-1)] \\
&\quad \times \left[\sum_{i=1}^{r} a_i(r)(1-F_X(x_1))^{\gamma_i-1} f_X(x_1)\right] \\
&\quad \times \left\{\int_{x_1}^{\infty} f_Y(y_2)[1-\alpha(2(1-F_X(x_2))-1)(2F_Y(y_2)-1)]\right. \\
&\quad \times \left.\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{\gamma_i-1}\left(\frac{f_X(x_2)}{1-F_X(x_1)}\right)\right] dx_2\right\} dx_1 \\
&= \left[f_Y(y_2)(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}) + \alpha f_Y(y_2)(2F_Y(y_2)-1)(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i})\right] \\
&\quad \times \left[c_{s-1} f_Y(y_1)(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i}) + \alpha c_{s-1} f_Y(y_1)(2F_Y(y_1)-1)\left\{(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i}) - 2(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1})\right\}\right] \\
&\quad - \left[2\alpha f_Y(y_2)(2F_Y(y_2)-1)\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i+1}\right] \\
&\quad \times \left[c_{s-1} f_Y(y_1)(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}) + \alpha c_{s-1} f_Y(y_1)(2F_Y(y_1)-1)\left\{(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}) - 2(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+2})\right\}\right],
\end{aligned} \tag{13}
$$

since

$$
(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i})(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i}) = \frac{1}{c_{s-1}},\ \sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i} = \frac{c_{r-1}}{c_{s-1}},\ \sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i} = \frac{1}{c_{r-1}}. \tag{14}
$$

From (13) in (14) we have:

$$g_{[r,s,n,\tilde{m},k]}(y_1,y_2) = f_Y(y_1)f_Y(y_2)[1+(2F_Y(y_1)-1)D_1+(2F_Y(y_2)-1)D_2 + (2F_Y(y_1)-1)(2F_Y(y_2)-1)D_3], \tag{15}$$

where

$$\begin{aligned} D_1 &= \alpha\left(1-2c_{r-1}\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}\right), \\ D_2 &= \alpha\left(1-2c_{s-1}(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i+1})(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1})\right), \\ D_3 &= \alpha^2\left(1-2c_{r-1}\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}-2(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i+1})(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+1}-2\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i+2})\right). \end{aligned} \tag{16}$$

3.1 Recurrence Relation and Product Moments of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$

With the joint density $g_{[r,s,n,\tilde{m},k]}(y_1,y_2)$ of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$, the product moments $E[Y^p_{[r,n,\tilde{m},k]}Y^q_{[s,n,\tilde{m},k]}]$, denoted by $\mu^{(p,q)}_{[r,s,n,\tilde{m},k]}$, $p,\, q>0$, are given by:

$$\begin{aligned} \mu^{(p,q)}_{[r,s,n,\tilde{m},k]} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1^p y_2^q g_{[r,s,n,\tilde{m},k]}(y_1,y_2)dy_1dy_2 \\ &= \mu^p_{1:1}\mu^q_{1:1}+[\mu^q_{1:1}\mu^p_{2:2}-\mu^p_{1:1}\mu^q_{1:1}]D_1+[\mu^p_{1:1}\mu^q_{2:2}-\mu^p_{1:1}\mu^q_{1:1}]D_2 \\ &\quad +[\mu^p_{2:2}-\mu^p_{1:1}][\mu^q_{2:2}-\mu^q_{1:1}]D_3, \end{aligned} \tag{17}$$

where $\mu^p_{1:1}=\int_{-\infty}^{\infty}y_1^p f_{Y_{1:1}}(y_1)dy_1$, $\mu^q_{1:1}=\int_{-\infty}^{\infty}y_2^q f_{Y_{1:1}}(y_2)dy_2$, $\mu^p_{2:2}=\int_{-\infty}^{\infty}y_1^p f_{Y_{2:2}}(y_1)dy_1$ and $\mu^q_{2:2}=\int_{-\infty}^{\infty}y_2^q f_{Y_{2:2}}(y_2)dy_2$. Furthermore, using (9) and (17), the covariance of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ is given by:

$$Cov(Y_{[r,n,\tilde{m},k]},Y_{[s,n,\tilde{m},k]}) = \mu_{[r,s,n,\tilde{m},k]}-\mu_{[r,n,\tilde{m},k]}\mu_{[s,n,\tilde{m},k]},\ r\neq s. \tag{18}$$

The joint moment generating function of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ is given by:

$$\begin{aligned} M_{[r,s,n,\tilde{m},k]}(t_1,t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{t_1y_1+t_2y_2\}g_{[r,s,n,\tilde{m},k]}(y_1,y_2)dy_1dy_2 \\ &= M_{1:1}(t_1)M_{1:1}(t_2)+[M_{1:1}(t_2)M_{2:2}(t_1)-M_{1:1}(t_1)M_{1:1}(t_2)]D_1 \\ &\quad +[M_{1:1}(t_1)M_{2:2}(t_2)-M_{1:1}(t_1)M_{1:1}(t_2)]D_2 \\ &\quad +[M_{2:2}(t_1)-M_{1:1}(t_1)][M_{2:2}(t_2)-M_{1:1}(t_2)]D_3. \end{aligned} \tag{19}$$

Differentiating (19) with respect to t_1 and t_2, p times and q times, respectively, and putting $t_1=t_2=0$, we get (17).

4 Moments of Case-II of Generalized Order Statistics from some Well known Distributions

In this section, we will use the pdf $g_{[r,n,\tilde{m},k]}(y)$ given in (7) and the joint pdf $g_{[r,s,n,\tilde{m},k]}(y_1,y_2)$ given in (15) to study the moments of some distributions such as Weibull, exponential, Pareto and power function distributions.

4.1 Weibull Distribution

The pdf and cdf for Weibull distribution are given by, respectively:

$$f(y) = \frac{a}{\lambda^a}y^{a-1}e^{-\frac{y^a}{\lambda^a}},\ 0\leq y<\infty, \tag{20}$$

$$F(y) = 1 - e^{-\frac{y^a}{\lambda^a}},\ a,\ \lambda > 0, \tag{21}$$

from (7) the pdf of the concomitant of $X_{[r,n,\tilde{m},k]}$ is

$$g_{[r,n,\tilde{m},k]}(y) = \frac{a}{\lambda^a} y^{a-1} e^{-\frac{y^a}{\lambda^a}} + D_{[r,n,\tilde{m},k]} \frac{a}{\lambda^a} y^{a-1} e^{-\frac{y^a}{\lambda^a}} [1 - 2e^{-\frac{y^a}{\lambda^a}}]. \tag{22}$$

Theorem 4.1. Let $\mu^p_{[r,n,\tilde{m},k]}$ be the $p-th$ moment of $Y_{[r,n,\tilde{m},k]}$, then:

$$\mu^p_{[r,n,\tilde{m},k]} = \lambda^p \Gamma(\frac{p}{a}+1) \left[1 + D_{[r,n,\tilde{m},k]}[1 - 2^{\frac{-p}{a}}]\right]. \tag{23}$$

From (15) the joint pdf of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ is given by:

$$\begin{aligned} g_{[r,s,n,\tilde{m},k]}(y_1,y_2) = \frac{a_1 a_2}{\lambda_1^{a_1}\lambda_2^{a_2}} y_1^{a_1-1} e^{-\frac{y_1^{a_1}}{\lambda_1^{a_1}}} y_2^{a_2-1} e^{-\frac{y_2^{a_2}}{\lambda_2^{a_2}}} \Bigg[1 + [1 - 2e^{-\frac{y_1^{a_1}}{\lambda_1^{a_1}}}]D_1 \\ + [1 - 2e^{-\frac{y_2^{a_2}}{\lambda_2^{a_2}}}]D_2 + [1 - 2e^{-\frac{y_1^{a_1}}{\lambda_1^{a_1}}}][1 - 2e^{-\frac{y_2^{a_2}}{\lambda_2^{a_2}}}]D_3 \Bigg]. \end{aligned} \tag{24}$$

Theorem 4.2. Let $\mu^{(p,q)}_{[r,s,n,\tilde{m},k]}$ be the $p-th$ and $q-th$ joint moments of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$, then:

$$\mu^{(p,q)}_{[r,s,n,\tilde{m},k]} = \lambda_1^p \lambda_2^q \Gamma(\frac{p}{a_1}+1)\Gamma(\frac{q}{a_2}+1) \left[1 + D_1[1 - 2^{\frac{-p}{a_1}}] + D_2[1 - 2^{\frac{-q}{a_2}}] + D_3[1 - 2^{\frac{-p}{a_1}}][1 - 2^{\frac{-q}{a_2}}]\right]. \tag{25}$$

Remark 4.1. From Weibull distribution we can get the moments of other related distributions like exponential and Rayleigh distributions by changing the parameters.

4.2 Pareto Distribution

The pdf and cdf for Pareto distribution are given by, respectively:

$$f(y) = a y^{-(a+1)},\ y \geq 1, \tag{26}$$

$$F(y) = 1 - y^{-a},\ a > 0, \tag{27}$$

from (7) the pdf of the concomitant of $X_{[r,n,\tilde{m},k]}$ is

$$g_{[r,n,\tilde{m},k]}(y) = a y^{-(a+1)} + D_{[r,n,\tilde{m},k]} a y^{-(a+1)} [1 - 2y^{-a}]. \tag{28}$$

Theorem 4.3. Let $\mu^p_{[r,n,\tilde{m},k]}$ be the $p-th$ moment of $Y_{[r,n,\tilde{m},k]}$, then:

$$\mu^p_{[r,n,\tilde{m},k]} = \frac{a}{a-p}\left[1 + D_{[r,n,\tilde{m},k]} \frac{p}{2a-p}\right]. \tag{29}$$

From (15) the joint pdf of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ is given by:

$$\begin{aligned} g_{[r,s,n,\tilde{m},k]}(y_1,y_2) = a_1 a_2 y_1^{-(a_1+1)} y_2^{-(a_2+1)} \Big[1 + D_1[1 - 2y_1^{-a_1}] + D_2[1 - 2y_2^{-a_2}] \\ + D_3[1 - 2y_1^{-a_1}][1 - 2y_2^{-a_2}] \Big]. \end{aligned} \tag{30}$$

Theorem 4.4. Let $\mu^{(p,q)}_{[r,s,n,\tilde{m},k]}$ be the $p-th$ and $q-th$ joint moments of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$, then:

$$\mu^{(p,q)}_{[r,s,n,\tilde{m},k]} = \frac{a_1 a_2}{(a_1-p)(a_2-q)}\left[1 + D_1 \frac{p}{2a_1-p} + D_2 \frac{q}{2a_2-q} + D_3 \frac{pq}{(2a_1-p)(2a_2-q)}\right]. \tag{31}$$

4.3 Power Distribution Function

The pdf and cdf for Power distribution function are given by, respectively:

$$f(y) = ay^{a-1},\ 0 \le y \le 1, \tag{32}$$

$$F(y) = y^a,\ a > 0, \tag{33}$$

from (7) the pdf of the concomitant of $X_{[r,n,\tilde{m},k]}$ is

$$g_{[r,n,\tilde{m},k]}(y) = ay^{a-1} + D_{[r,n,\tilde{m},k]}ay^{a-1}[2y^a - 1]. \tag{34}$$

Theorem 4.5. Let $\mu^p_{[r,n,\tilde{m},k]}$ be the $p-th$ moment of $Y_{[r,n,\tilde{m},k]}$, then:

$$\mu^p_{[r,n,\tilde{m},k]} = \frac{a}{a+p}\left[1 + D_{[r,n,\tilde{m},k]}\frac{p}{2a+p}\right]. \tag{35}$$

From (15) the joint pdf of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$ is given by:

$$\begin{aligned} g_{[r,s,n,\tilde{m},k]}(y_1,y_2) = a_1a_2y_1^{a_1-1}y_2^{a_2-1}\left[1 + D_1[2y_1^{a_1} - 1] + D_2[2y_2^{a_2} - 1]\right. \\ \left. + D_3[2y_1^{a_1} - 1][2y_2^{a_2} - 1]\right]. \end{aligned} \tag{36}$$

Theorem 4.6. Let $\mu^{(p,q)}_{[r,s,n,\tilde{m},k]}$ be the $p-th$ and $q-th$ joint moments of $Y_{[r,n,\tilde{m},k]}$ and $Y_{[s,n,\tilde{m},k]}$, then:

$$\mu^{(p,q)}_{[r,s,n,\tilde{m},k]} = \frac{a_1a_2}{(a_1+p)(a_2+q)}\left[1 + D_1\frac{p}{2a_1+p} + D_2\frac{q}{2a_2+q} + D_3\frac{pq}{(2a_1+p)(2a_2+q)}\right]. \tag{37}$$

5 Applications

An interesting application of this work is to obtain the minimum variance linear unbiased estimates (MVLUE) of the location and scale parameters. The method of Llyod [8] may be used, however due to the difficulty in obtaining the inverse of the variance-covariance matrix in closed form, numerical methods were used instead to obtain these estimates.

We have used Mathematica program to calculate the estimates of the location and scale parameters of the concomitants of order statistics ($\gamma_i = n-i+1$) when the marginal distributions are Weibull, Pareto and power function (see Tables (5.1), (5.2) and (5.3)).

Suppose the random variables have the location parameter μ and scale parameter σ. Using Lloyd's method, the MVLUE of θ as $\hat{\theta} = (A'V^{-1}A)^{-1}(A'V^{-1}\mathbf{y})$, where $V = (V_{i,j})$ is the variance of the $i-th$ and $j-th$ concomitants, V^{-1} is the inverse of the matrix V, $\mathbf{y}'$ is the observed value of the vector $\mathbf{Y}' = [Y_{[1,n,\tilde{m},k]}, Y_{[2,n,\tilde{m},k]}, \ldots, Y_{[n,n,\tilde{m},k]}]$, $\mu = 0$ and $\sigma = 1$. A and θ are defined as:

$$A' = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \mu_{[1,n,\tilde{m},k]} & \mu_{[2,n,\tilde{m},k]} & \ldots & \mu_{[n,n,\tilde{m},k]}. \end{bmatrix},\ \theta' = [\mu \quad \sigma].$$

Table 5.1: Coefficients of MVLUE of μ and σ for Weibull distribution based on order statistics with $n = 5$, $a = 4$, $a_1 = 4$, $a_2 = 6$, $\lambda = 3$, $\lambda_1 = 3$ and $\lambda_2 = 5$.

α	Estimates	Coefficients				
-0.9	$\hat{\mu}$	1.46353	0.799855	0.0826532	-0.714274	-1.63176
	$\hat{\sigma}$	-3.7743	-1.97731	-0.0303169	2.13907	4.64286
-0.5	$\hat{\mu}$	2.68766	1.42007	0.0919568	-1.32628	-2.87341
	$\hat{\sigma}$	-7.10467	-3.66316	-0.0537653	3.80441	8.01718
-0.1	$\hat{\mu}$	13.773	6.9724	0.101797	-6.87021	-13.9769
	$\hat{\sigma}$	-37.2507	-18.7598	-0.0776814	18.8811	38.2071
0.1	$\hat{\mu}$	-13.9692	-6.8931	0.106959	6.99965	13.7557
	$\hat{\sigma}$	38.1841	18.9442	-0.0899027	-18.833	-37.2054
0.5	$\hat{\mu}$	-2.88577	-1.33844	0.117856	1.45423	2.65213
	$\hat{\sigma}$	8.04145	3.84251	-0.115065	-3.75192	-7.01698
0.9	$\hat{\mu}$	-1.66628	-0.712543	0.129666	0.83872	1.41044
	$\hat{\sigma}$	4.71921	2.14434	-0.141485	-2.07618	-3.64589

Table 5.2: Coefficients of MVLUE of μ and σ for Pareto distribution based on order statistics with $n = 5$, $a = 4$, $a_1 = 4$ and $a_2 = 6$.

α	Estimates	Coefficients				
-0.9	$\hat{\mu}$	2.71376	1.99605	0.851726	-1.04666	-4.51488
	$\hat{\sigma}$	-3.42756	-2.45888	-0.925477	1.60449	6.20742
-0.5	$\hat{\mu}$	5.52065	3.47708	0.815618	-2.64507	-7.16828
	$\hat{\sigma}$	-7.1667	-4.43389	-0.881249	3.73096	9.75087
-0.1	$\hat{\mu}$	30.7135	16.1573	0.777886	-15.3817	-32.267
	$\hat{\sigma}$	-40.7527	-21.3424	-0.835822	20.7097	43.2213
0.1	$\hat{\mu}$	-32.2941	-15.3016	0.758368	16.0646	30.7728
	$\hat{\sigma}$	43.2603	20.6014	-0.812568	-21.2202	-40.829
0.5	$\hat{\mu}$	-7.1163	-2.61602	0.718009	3.37723	5.63708
	$\hat{\sigma}$	9.69665	3.68298	-0.764873	-4.30581	-7.30895
0.9	$\hat{\mu}$	-4.34698	-1.12419	0.675898	1.90871	2.88657
	$\hat{\sigma}$	6.01311	1.68702	-0.715483	-2.34923	-3.63543

6 Conclusion

We derived an analytical expression of one and two concomitants of case-II of generalized order statistics in FGM family. We also provided the recurrence relation between moments. Some application for well known distributions such as Weibull, Pareto and power function distributions to obtain the moments. We use this moments to get the minimum variance linear unbiased estimates of the location and scale parameters of concomitants of order statistics. A good application of this setup is the use of concomitants to infer about the population parameters when the original variables are not observable. Finally, for $m_i = m_j = m$ case-II reduces to case-I of generalized order statistics.

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Table 5.3: Coefficients of MVLUE of μ and σ for power function distribution based on order statistics with $n = 5$, $a = 4$, $a_1 = 4$ and $a_2 = 6$.

α	Estimates	Coefficients				
-0.9	$\hat{\mu}$	8.76695	2.6828	-0.925244	-3.76573	-6.75878
	$\hat{\sigma}$	-6.76128	-1.97885	0.89342	3.19481	5.65189
-0.5	$\hat{\mu}$	14.5378	5.84291	-0.756931	-6.66594	-12.9578
	$\hat{\sigma}$	-11.3971	-4.49274	0.774058	5.51861	10.5972
-0.1	$\hat{\mu}$	68.3431	32.9249	-0.570715	-33.5055	-67.1918
	$\hat{\sigma}$	-54.4666	-26.1439	0.648814	27.0005	53.9612
0.1	$\hat{\mu}$	-66.7509	-34.5547	-0.464951	34.0977	67.6729
	$\hat{\sigma}$	53.5921	27.848	0.580657	-27.0738	-53.947
0.5	$\hat{\mu}$	-12.9499	-7.54418	-0.207585	7.34739	13.3543
	$\hat{\sigma}$	10.5055	6.26126	0.422298	-5.65021	-10.5388
0.9	$\hat{\mu}$	-7.19962	-4.55642	0.161742	4.64426	6.95004
	$\hat{\sigma}$	5.82696	3.90964	0.209273	-3.4553	-5.49057

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