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Thermodynamics with an Action Principle

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Attempts to understand Black Holes lead to interesting problems in thermodynamics, and to a very surprising conclusion.

1 Introduction

Black holes have a meaning for mathematicians, and a different meaning for astrophysicists. To a mathematician, the most important black hole is an exact solution to Einstein's field equations for the metric of empty space. This solution, found by Schwarzschild, is a fair approximation to the field on the outside of a spherical concentration of total mass M, say, but there is an anomaly at a certain value of the radial coordinate, at

$$r = 2M,$$

the 'Schwarzschild radius'. The explicit form of the metric is

$$ds^2 = g_{00}\, dt^2 - g_{rr} dr^2 - r^2 d\Omega^2 \tag{1.1}$$

with

$$g_{00} = \mathrm{e}^{\nu} = \mathrm{c}^2 - 2\mathrm{M/r}, \;\; \mathrm{g_{rr}} = \mathrm{e}^{\lambda} = \frac{\mathrm{c}^2}{\mathrm{c}^2 - 2\mathrm{M/r}}. \tag{1.2}$$

A space traveller approaching the star reaches regions of diminishing light velocity. He does not reach the critical radius in finite time.

In his own time he does reach the Schwarzschild radius, and if he goes on he will penetrate to the inside, but he will be unable to turn around, and no signal sent by him will tell his story to an observer on the outside. This has led to a great deal of fancyful speculation. My opinion is that any region about which it is impossible to get information is outside the domain of science.

The black holes of astrophysics are much more interesting, especially so if it can be shown that they have properties resembling the mathematical black holes. The idea is more or less as follows.

It is believed that old stars tend to diminish in size as they burn up their fuel, and that the mass collapses inwards to a very small size. Far away from the center the density of matter is very small, perhaps zero, and Schwarzschild's solution is assumed to hold to very high accuracy outside some radius R, say. The mass remains finite and appears as a parameter in this solution, as in Equations (1.1) and (1.2). Astrophysicists believe that the radius R eventually become less than $2M/c^2$, all the mass is then inside the Schwarzschild radius, and the anomalies of the mathematical black hole become physical reality.

What they actually see, of course, is nothing, directly, but the presence of the star is revealed by the attraction that it excerts on neighbouring, visible objects. A more aft name for this is "invisible stellar objects", rather than the highly suggestive "black hole".

To justify this prediction it is necessary to know more than Einstein's equations for empty space. One needs to understand matter as well, and the interaction of matter with the metric. Here one makes use of a theory developed by Tolman in 1935.

Formally, Einstein's theory of General Relativity already describes everything, for the right hand side of Einstein's equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

is the energy momentum tensor of matter. It does not specify any particular type of matter and one has to make an educated guess about what kind of matter makes up the collapsing star. In principle, a model of matter is needed, a theory of matter and metric in mutual interaction, but Tolman circumvented this by postulating a special form for the energy momentum tensor. In analogy with classical hydro-thermo-dynamics, he postulated that the main dynamical variables include a scalar field ρ (a density), another scalar field (pressure) and a 4-vector field U that is related to the velocity of flow by

$$v_i = U_i/U_0, \;\; i = 1, 2, 3.$$

Since this leaves the fourth component unidentified he postulates the normalization

$$g_{\mu\nu} U^\mu U^\nu = 1.$$

Tolman's formula is

$$T_{\mu\nu} = \rho U_\mu U_\nu - p\, g_{\mu\nu}.$$

This is an extremely simple model; it does not take any account of the pressure of radiation as an independent dynamical variable. And there are other curious features.

For example, although notions of classical theory are freely used, the cornerstone of classical hydrodynamics is abandoned, for there is no continuity equation, no conserved vector current. The need for a conserved current is certainly felt, but it has to be introduced by an independent dynamical postulate involving additional dynamical variables unrelated to the scalar field ρ.

The absence of a vector conservation law, together with the fact that the equations, in the case that the metric takes the form (1.1), do not involve the function ν but only its derivatives, is an obstruction to any real understanding of the boundary conditions. In classical hydrodynamics it is the conservation of mass that characterizes the boundary conditions, and these are the boundary conditions that are used to demonstrate stability of the stationary solutions.

Most other branches of theoretical physics, in their mature form, are dominated by action principles. This is not true of classical thermodynamics, a fact that is difficult to explain, for it does not seem to be difficult to set up a variational formulation. It is a main purpose of this talk to show that it can be done, and to give some hints of its possible uses.

2 The Intuitive Picture of a Collapsing Star

There are reasons why one can be a little skeptical of the notion that stars collapse to become black holes. Let us look at this type of scenario in some detail.

Suppose that the mass distribution is a spherical distribution of something that resembles dust. To avoid having to solve Einstein's equations in the presence of matter, let us suppose that this dust is concentrated in shells $R_{2i} < r < R_{2i+1}, i = 1, 2, \ldots, N$, alternating with shells $R_{2i-1} < r < R_{2i}$ that are empty. Then in each empty shell Einstein's field equations for empty space must hold, and the metric in each of these shells is of the form similar to the Schwarzschild metric, Eq. (1.1) with

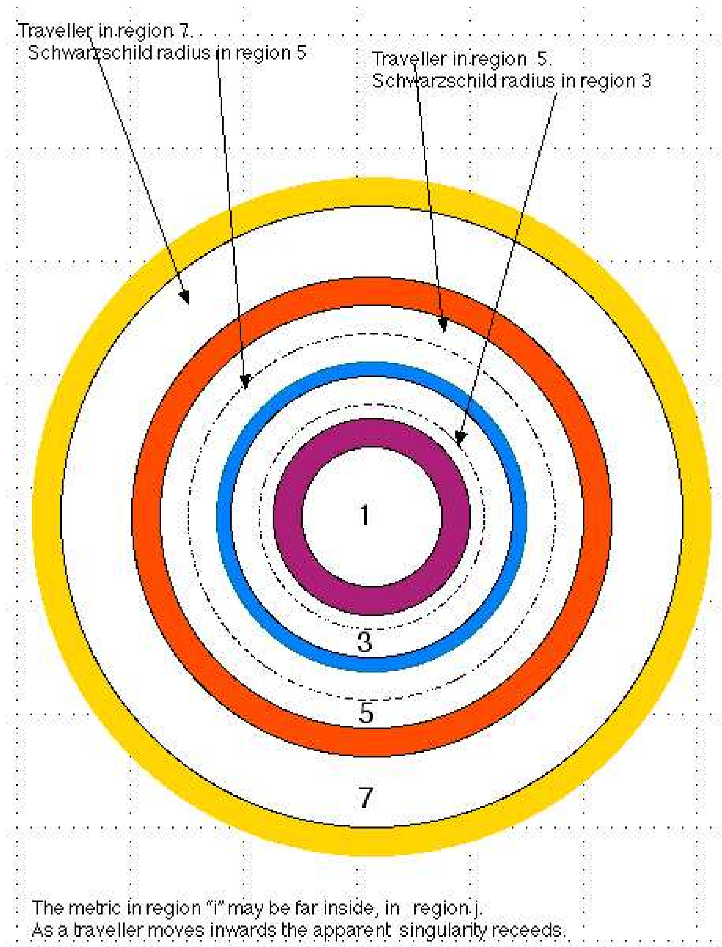
$$g_{00}^{(i)} = c_i^2 - \frac{2M_i}{r}, \ \ i = 1, 2, \ldots \ .$$

Here $c_1, c_2, \ldots$ are constants, undetermined except that the outermost value is the velocity of light at infinity. (To determine these constants we need to solve Einstein's equations in the regions that are not empty.)

Let me emphasize that the metric $g^{(i)}$ is valid in the region $R_{2i-1} < r < R_{2i}$ only, and that this region does not extend to the point $r = 2M_i/c_i^2$; there is no physical singularity of the metric in this region. Nevertheless, we may refer to $2M_i/c_i^2$ as the Schwarzschild radius of this metric.

See the figure.

As a traveler approaches the star, he penetrates into successive, empty regions, and by observing the local metric he can determine both c_i and M_i. Evidently, the mass M_i is diminished by each passing of a layer. It is conceivable that the velocity c_i also decreases so that, eventually, there is a region in which $c_i^2 - 2M_i/r$ vanishes for $R_{2i} < r < R_{2i+1}$.



Let us suppose that, at early times, $g_{00} > 0$ everywhere. The innermost massive shell bounds a region that is empty, and the metric of that region is flat, altough with a velocity of light that may be smaller than c. There is no Schwarzschild horizon for this innermost layer to fall into, the Schwarzshild radius of that region is zero, the first layer is definitely "outside", and it will remain outside at all times. But then the other layers must be outside as well. In this case, as the traveller moves inwards, he passes from one Schwarzschild metric to another, but *the Schwarzschild radius recedes and he never reaches a singularity.*

Even if we assume that, at some later time, there is a region that contains it's own Schwartzschild singularity, then material particles that find themselves on the outside of that singularity will remain on the outside at all times

This picture of an actual collapse suggests that no singularity ever materializes.

Therefore, to make a convincing argument for the idea that black holes are created by collapsing matter it will be necessary to show that the scenario that we have examined is untypical and that matter behaves in another manner altogether. For this, one has to have a model. Models exist, but they do not satisfy requirements that we should like to impose for reasons of coherence. The first of such requirements is a formulation in terms of an action principle.

3 Simplest Variational Model

Consider the total action that is the sum of the Einstein-Hilbert action and the following matter action,

$$A_{Matter} = \int d^x \sqrt{-g}\Big(\rho`(g^{\mu\nu}(\psi_{,\mu}\psi_{,\nu} - c^2) - V[\rho]\Big),$$

where ρ is a density and ψ is a velocity potential related to Tolmans vector potential U. In this model there is no turbulence. The equations that govern stationary solutions are essentially the same as in Tolman's theory. The main difference is that this model comes with a conserved current,

$$J^\mu = \rho g^{\mu\nu}\psi_{,\nu}.$$

This has the effect of making the choice of boundary conditions more transparent. In other respects the model is in full agreement with Tolman's theory.

Some people have pointed out that this idea, of formulating a variational principle for this problem is an old one. This is true except that those attempts did not go far enough. I do not have time to talk about this.

Other people thought that this model is inconsistent with thermodynamics. To dispell this notion it was necessary to examine the nonrelativistic limit (adequate for most astrophysical applications) and the interpretation in terms of thermodynamics.

The nonrelativistic limit of the proposed lagrangian is found by setting

$$\psi = c^2 t + \Phi$$

and interpreting the field Φ as the non-relativistic velocity potential. The action becomes

$$\int dt \int d^3x\mathcal{L},$$

with the lagrangian density

$$L = \rho(\Phi - \vec{v}^2/2 - \phi) - V[\rho],$$

where ϕ is the gravitational potential. This leads to the conventional theory of non-relativistic stellar or earthly atmospheres when one takes for $V[\rho]$ the form

$$V[\rho] = a\rho^\gamma$$

that we shall examine in a moment. As I said, this reduces to to conventional theory in the non relativistic domain, with the interpretation of ρ as the mass density. Nevertheless, by deviating from Tolman's scheme I ran into much opposition from critics who felt that the thermodynamic interpretation was tenuous. For this reason I decided to make a careful study of thermodynamics, and it is to this that I should now like to address myself.

4 Thermodynamics with an Action Principle

This story begins with hydrodynamics, in the simplest setting in which the dynamical variables are the mass density ρ , the temperature T, and the flow velocity $\vec{v}$, with the asumption of irrotational flow, so that

$$\vec{v} = -\text{grad}\,\Phi.$$

The equations that lie at the foundation of this theory are the equation of continuity

$$\dot{\rho} + \vec{\bigtriangledown}\cdot(\rho\vec{v}) = 0,$$

and Newton's equation

$$\rho\frac{D\vec{v}}{Dt} + \rho\vec{\bigtriangledown}\phi = -\vec{\bigtriangledown}p,$$

where ϕ is the gravitational potential. To complete the theory one needs to specify the pressure, as a functional $p[\rho, T]$.

I must emphasize that the last two equations are used witout change in every known approach to hydro-thermo-dynamics, at any rate to every approach to be discussed in this talk. And in every theory an additional equation is needed.

It is well known that this theory can be formulated in terms of an action principle, precisely with the lagrangian density that is shown above. In that case one needs to specify the functional V; that is, the hamiltonian or, what is the same, the internal energy. Then the pressure turns out to be

$$p = \rho\frac{\partial V}{\partial\rho} - V.$$

Therefore, in this approach the hamiltonian defines the theory and the pressure is not an independent variable.

What about the temperature? It is often taken to be determined by the ideal gas law,

$$p = \mathcal{R}T\rho;$$

then T is not an independent variable. As an example of this theory consider the case that

$$V[\rho] = \frac{a}{n}\rho^\gamma,\ \ \gamma = 1 + \frac{1}{n},$$

where γ is a constant that is called the polytropic index and a is another constant. The number n is called the adiabatic index of the gas; it is 3/2 for an ideal, monatomic gas. This is a very successful model for stellar and earthly atmospheres.

But the temperature is actually an independent dynamical variable, and that is recognized in different ways. The standard approach is to put the variational formulation aside and develop the theory in terms of its conservation laws. The equation of continuity is related to the conservation of mass and Newton's equation to the conservation of momentum. It remains to formulate the conservation of energy, and that is expressed by what is called the energy equation,

$$\frac{De}{Dt} + p\frac{d(1/\rho)}{Dt} = 0.$$

In this equation e is the specific energy per unit of mass and p is the pressure; both are functions of ρ and T.

The alternative that I find more promising develops the action principle, promoting the temperature to the role of an independent dynamical variable. Both approaches lead to identical results for an ideal gas.

The crux of the matter is that the hamiltonian of the theory must be related to the internal energy. Knowing that the internal energy density of an ideal gas with adiabatic index n is

$$u = n\mathcal{R}T\rho,$$

we are led to the following, unique lagrangian density

$$\mathcal{L}[\Phi, \rho, T] = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho\log(k/k_0),$$

with

$$k := \rho/T^n, \;\; \vec{v} = -\vec{\bigtriangledown}\Phi.$$

The constant λ is a Lagrange multiplier is used to constrain the mass. The equations of motion are as follows:

(1) Variation of the velocity potential leads to the equation of continuity

$$\dot{\rho} + \vec{\bigtriangledown}\cdot(\rho\vec{v}) = 0.$$

(2) Variation of ρ gives the Newton-Bernoulli equation

$$\rho\frac{D\vec{v}}{Dt} + \rho\vec{\bigtriangledown}\phi = -\vec{\bigtriangledown}p.$$

(3) Variation of the temperature gives

$$\mathcal{R}\Big(n - \log k\Big)\rho = 0,$$

This last equation is the adiabatic relation ρ/T^n = constant.

The theory is in complete agreement with the laws of thermodynamics. The hamiltonian density is

$$h = \rho(\vec{v}^2 + \phi) + \mathcal{R}T\rho \log k = \rho(\vec{v}^2/2 + \phi) + \rho e,$$

and the specific internal energy density, evaluated on shell, is

$$e = n\mathcal{R}T,$$

as it should be for an ideal gas. The pressure is

$$p = \mathcal{R}T\rho$$

and the crucial thermodynamical relation is satisfied,

$$de + pdV = 0.$$

This result is in full agreement with the standard approaches to these problems.

5 The Scandal

The theory that I have presented is very successful when applied to atmospheres. It is my intention to promote it to a relativistic theory but that has not yet been done. In the meantime a very curious circumstance confronts us.

This work was started to try to understand atmospheres, especially the question of the extent to which atmospheres are controlled by radiation. Yet radiation has not been considered up to this point. Somehow, the theory is successful in describing an actual atmosphere that appears to be under the strong influence of radiation, although we have done nothing to deserve this unexpected success. The only input is the known expression for the internal energy of an ideal gas, and the ideal gas law.

Radiation is subject to being turned off, so that any theory that describes radiated atmospheres ought to have a parameter that corresponds to the intensity of radiation. Present theories do not have such a parameter. Furthermore, we do not have a theory of an isolated atmosphere that is consistent with our prejudices.

How do we describe an <u>isolated</u> atmosphere?

The answer is that we just did. For the equation

$$\frac{De}{Dt} + \frac{D(1/\rho)}{Dt} = 0;$$

that is, the energy conservation equation of radiation hydrodynamics, is the statement that the changes induced by the passage of time are adiabatic, hence changes of an isolated system. The so called polytropic atmosphere is an atmosphere that is isolated except for the imposition of boundary conditions. The more precise statement is that the only effect of the incoming radiation is to compensate for the loss due to black body radiation from the gas. There is no change in entropy. But there is a problem with this conclusion.

There is a very strong prejudice among physicists that any isolated body of gas must have uniform temperature at equilibrium, *regardless of the presence of gravitation.* In fact, it is easy to see that the existence of a temperature gradient in an isolated atmosphere contradicts this statement of Clausius:

"Heat cannot flow by itself from a colder to a hotter body"

It is often claimed (first by Clausius) that this statement is equivalent to the second law of thermodynamcs. Since we believe in the second law of thermodynamics we would have to conclude that the isolated gas in equilibrium must have a uniform temperature, even in the presence of gravitation. But a theory of such an equilibrium state does not seem to exist!

Let us compare the results of the action principle with with modern thermodynamics, or more precisely with radiation hydrodynamics. The equations that define radiation hydrodynamics are the equation of continuity, the Bernoulli equation and one additional equation that expresses the principle of energy conservation. This equation is actually an expression of the first law of thermodynamics, in the form

$$\frac{D}{Dt} + p\frac{D(1/\rho)}{Dt} = q.$$

The right hand side stands for a local heat source and is probably to be taken to be zero under the circumstances that we are discussing. In this case this equation is the same as the one we derive from our action by variation of T, with the difference that the above equation has an extra solution with $\partial T/\partial t = 0$. Our action principle is therefore in virtually complete agreement with traditional theory. We have seen that it leads directly to the correct expressions for the specific internal energy $e = n\mathcal{R}T$, and pressure $p = \mathcal{R}T\rho$, of an ideal gas.

But neither approach allows for an isothermal equilibrium in the presence of gravity. There is a theory of isothermal equilibrium, but it describes a substance with specific internal energy $e = \mathcal{R}T\log\rho$.

I shall illustrate the position taken by the textbooks with the following example. In a well known treatise that treats of sound waves and shock waves and especially explosions, in the atmosphere, the theory of such waves is developed on the basis of the two hydrodynamical equations alone. The "energy equation" is not mentioned, though it is a pillar of the earlier chapters, but time development is assumed to be adiabatic and that comes to the same if the heat generation term q is zero. At the end of a discussion of the equilibrium state of the gas the author seems to have some misgivings, for he says:

"This equibrium state should, however, go into a state of thermal equilibrium, owing to the equalization of the temperature in the column of gas."

Note that the propagation of a disturbance in a gas that is initially in thermal equilibrium is not discussed! There is a good reason for this: such a state (of an ideal gas) is not envisaged by the theory, and it may not exist.

In view of the controversial nature of this (tentative) conclusion let us go over the salient point once more. If we wish to describe an ideal gas in a gravitational field then we must fix the specific internal energy density and the pressure according to the ideal gas law. In this case the energy equation, whether it is gotten by an axiom of thermodynamics or from the action principle, leads to the polytropic relation $\rho \propto T^n$, and the temperature cannot be uniform.

There is an easy way out of the dilemma. We just have to accept that, while the entropy of an actual atmosphere happens to be zero, it is not zero for a state of isothermal equilibrium. But accepting this interpretation of our failure is costly, for the theory loses all predictive power.

6 Conclusions

(1) An action principle for thermodynamics exists that incorporates all features of older approaches to thermodynamics. It will be shown to afford a new approach to astrophysics and it will be used to explore the formation of "black holes".

(2) We are far from understanding the interaction between atmospheres and radiation. The temperature profile of the atmosphere seems not to be imposed by radiation but is natural to the gas in the gravitational field. The role of radiation is to maintain the temperature by replacing what is lost by the emission of heat.

(3) The validity of Clausius' statement of the second law of thermodynamics, in the presence of gravitation, is difficult to reconcile with the theory. This goes against the authority of Maxwell and Boltzmann and virtually every other important figure in the history of thermodynamics.

I should add that, because of the equivalence principle, a powerful centrifuge provides a nice laboratory for an experimental verification of the temperature gradient of an isolated atmosphere. A careful experiment carried out with a modern centrifugeI could be a rich source of information. Results of a more difficult experiment, using an isolated, stationary column of water, has been reported by Graeff.

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