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Moment Generating Functions of Exponential-Truncated Negative Binomial Distribution based on Ordered Random Variables

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Abstract: In this paper, we established explicit expressions and some recurrence relations for marginal and joint moment generating functions of generalized order statistics from exponential-truncated negative binomial distribution. The results for record values and order statistics are deduced from the result. Further, a characterizing result of this distribution on using the conditional expectation of generalized order statistics is discussed.

Keywords: Generalized order statistics; order statistics; record values; exponential-truncated negative binomial distribution; marginal and joint moment generating functions; recurrence relations and characterization.
AMS Subject Classification: 62G30, 62E10

1 Introduction

A random variable X is said to have exponential-truncated negative binomial distribution if its probability density function (pdf) is of the form

$$f(x) = \frac{(1-\alpha)\lambda\theta\alpha^{\theta}e^{-\lambda x}}{(1-\alpha^{\theta})[1-(1-\alpha)e^{-\lambda x}]^{\theta+1}}, \quad x>0, \quad \alpha,\theta>0 \ and \ \lambda>0 \tag{1.1}$$

and the corresponding survival function is

$$\bar{F}(x) = \frac{\alpha^{\theta}}{(1-\alpha^{\theta})}[\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta}-1], \quad x>0, \quad \alpha,\theta>0 \ and \ \lambda>0. \tag{1.2}$$

Some distributions arise as special cases of the exponential-truncated negative binomial distribution:
i) For $\theta = 1$, we obtain the Marshall-Olkin exponential distribution with scale parameter λ and shape parameter α.
ii) For $\theta = 1$ and $\alpha = 2$, we obtain the half-logistic distribution with scale parameter λ.
iii) When $\alpha \to 0$, exponential-truncated negative binomial distribution reduces to the exponential distribution with scale parameter α.
iv) When $\theta \to 0$ and $0 < \alpha < 2$, exponential-truncated negative binomial distribution reduces to the Exponential-Logarithmic distribution with scale parameter λ and shape parameter $1-\alpha$, see Tahmasbi and Rezaei [16].
For more details and some applications of this distribution one may refer to Nadarajah *et al*. [11].

Kamps [4] introduced and extensively studied the generalized order statistics (gos). The order statistics, sequential order statistics, Stigler's order statistics, record values are special cases of *gos*. Suppose $X(1,n,m,k),\dots,X(n,m,m,k)$ are n *gos* from an absolutely continuous cumulative distribution function (cdf) $F(x)$ with the corresponding pdf $f(x)$. Their joint pdf is

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\Big)[1-F(x_n)]^{k-1}f(x_n) \tag{1.3}$$

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for $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n > F^{-1}(1)$, $m \geq -1$, $\gamma_r = k+(n-r)(m+1) > 0$, $r = 1,2,\ldots,n-1$, $k \geq 1$ and n is a positive integer.
Choosing the parameters appropriately, models such as ordinary order statistics $(\gamma_i = n-i+1;\ i = 1,2,\ldots,n\ \ i.e\ \ m_1 = m_2 = \ldots = m_{n-1} = 0,\ k = 1)$, $k-$ th record values $(\gamma_i = k\ \ i.e\ \ m_1 = m_2 = \ldots = m_{n-1} = -1,\ k \in N)$, sequential order statistics $(\gamma_i = (n-i+1)\alpha;\ \alpha_1,\alpha_2,\ldots,\alpha_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1;\ \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1,\beta_2,\ldots,\beta_n > 0)$ and progressive type II censored order statistics $(m_i \in N,\ k \in N)$ are obtained (Kamps [4], Kamps and Cramer [5]).
For simplicity we shall assume $m_1 = m_2 = \ldots = m_{n-1} = m$.
The marginal pdf of the $r-$th gos, $X(r,n,m,k)$, $1 \leq r \leq n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.4}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y),\ \ x < y, \tag{1.5}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\ ,\ \ \bar{F}(x) = 1 - F(x),\ \ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ \ x \in [0,1).$$

Ahsanullah and Raqab [1], Raqab and Ahsanullah [12], [13] have established recurrence relations for moment generating functions of record values from Pareto and Gumble, power function and extreme value distributions. Recurrence relations for marginal and joint moment generating functions of gos from power function distribution, Gompertz distribution, Erlang-truncated exponential distribution and extended type II generalized logistic distribution are derived by Saran and Pandey [14], Khan *et al*. [6], Kulshrestha *et al*. [7] and Kumar [10], respectively. Saran and Pandey [15] and Kumar [9] have established recurrence relations for marginal and joint moment generating functions of lower generalized order statistics from power function distribution and Marshall-Olkin extended logistic distribution respectively. Al-Hussaini *et al*. [2], [3] have established recurrence relations for conditional and joint moment generating functions of gos based on mixed population. Kumar [8] have established explicit expressions and some recurrence relations for moment generating functions of record values from generalized logistic.

The aim of this note is to gave exact expressions and some recurrence relations for marginal and joint moment generating functions of gos from exponential-truncated negative binomial distribution. Results for order statistics and record values are deduced as special cases and a characterization of this distribution is obtained by using the conditional expectation of function of gos.

2 Relations for marginal moment generating functions

Note that for exponential-truncated negative binomial distribution defined in (1.1)

$$\bar{F}(x) = \left[1 - \frac{1}{\theta}\sum_{u=2}^{\theta+1}(-1)^u \binom{\theta+1}{u}\{(1-\alpha)e^{-\lambda x}\}^{u-1}\right] f(x). \tag{2.1}$$

The relation in (2.1) will be exploited in this paper to derive some recurrence relations for the moment generating functions of gos from the exponential-truncated negative binomial distribution.
Let us denote the marginal moment generating functions of $X(r,n,m,k)$ by $M_{X(r,n,m,k)}(t)$ and its $j-$th derivative by $M^{(j)}_{X(r,n,m,k)}(t)$ and the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ by $M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)$ and

its $(i,j)-$th partial derivatives with respect to t_1 and t_2, respectively by $M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)$.
For the exponential-truncated negative binomial distribution as given in (1.2), the moment generating functions of $X(r,n,m,k)$ is given as

$$M_{X(r,n,m,k)}(t) = \int_{-\infty}^{\infty} e^{tx} f_{X(r,n,m,k)}(x)dx$$

$$= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx. \tag{2.2}$$

Further, on using the binomial expansion, we can rewrite (2.2) as

$$M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}$$

$$\times \int_{-\infty}^{\infty} e^{tx} [\bar{F}(x)]^{\gamma_{r-u}-1} f(x)dx. \tag{2.3}$$

Now letting $z = \bar{F}(x)$ in (2.3), we get

$$M_{X(r,n,m,k)}(t) = \frac{(1-\alpha)^{t/\lambda} C_{r-1}}{(r-1)!(m+1)^r} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}$$

$$\times \Big(\frac{1-\alpha^\theta}{\alpha^\theta}\Big)^q B\Big(\frac{k}{m+1} + n - r + u + \frac{q}{m+1}, 1\Big), \tag{2.4}$$

where

$$(\alpha)_{(p)} = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+p-1), & p>0 \\ 1, & p=0 \end{cases}.$$

Since

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} B(a+k,c) = B(k,c+b) \tag{2.5}$$

where $B(a,b)$ is the complete beta function.
Therefore,

$$M_{X(r,n,m,k)}(t) = \frac{(1-\alpha)^{t/\lambda} C_{r-1}}{(m+1)^r} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!} \Big(\frac{1-\alpha^\theta}{\alpha^\theta}\Big)^q$$

$$\times \frac{\Gamma\Big(\frac{k+(n-r)(m+1)+q}{m+1}\Big)}{\Gamma\Big(\frac{k+n(m+1)+q}{m+1}\Big)}$$

$$= (1-\alpha)^{t/\lambda} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!} \Big(\frac{1-\alpha^\theta}{\alpha^\theta}\Big)^q \frac{1}{\prod_{a=1}^{r}\Big(1+\frac{q}{\gamma_a}\Big)}. \tag{2.6}$$

Special Cases
i) Putting $m = 0$, $k = 1$, in (2.6), the explicit formula for marginal moment generating functions of order statistics from the exponential-truncated negative binomial distribution can be obtained as

$$M_{X_{r:n}}(t) = \frac{(1-\alpha)^{t/\lambda} n!}{(n-r)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!} \Big(\frac{1-\alpha^\theta}{\alpha^\theta}\Big)^q \frac{\Gamma(n-r+1+q)}{\Gamma(n+1+q)},$$

for $r = 1$

$$M_{X_{1:n}}(t) = n(1-\alpha)^{t/\lambda} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!(n+q)} \Big(\frac{1-\alpha^\theta}{\alpha^\theta}\Big)^q.$$

ii) Setting $m = -1$ in (2.6), we get the explicit expression for marginal moment generating functions of $k-$th upper record values from exponential-truncated negative binomial distribution can be obtained as

$$M_{X(r,n,-1,k)}(t)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}\left(\frac{1-\alpha^{\theta}}{\alpha^{\theta}}\right)^{q}\frac{(1-\alpha)^{t/\lambda}}{\left(1+\frac{q}{k}\right)^{r}},\ \ t\neq 0.$$

A recurrence relation for marginal moment generating functions for *gos* from (2.1) can be obtained in the following theorem.

Theorem 2.1. For the distribution given in (1.1) and for $2\leq r\leq n,\ n\geq 2\ k=1,2,\ldots,$

$$\left(1-\frac{t}{\gamma_r}\right)M^{(j)}_{X(r,n,m,k)}(t)=M^{(j)}_{X(r-1,n,m,k)}(t)+\frac{j}{\gamma_r}M^{(j-1)}_{X(r,n,m,k)}(t)$$
$$-\frac{1}{\theta\gamma_r}\sum_{u=2}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}(1-\alpha)^{u-1}$$
$$\times\left\{tM^{(j)}_{X(r,n,m,k)}(t+\lambda(u-1))+jM^{(j-1)}_{X(r,n,m,k)}(t+\lambda(u-1))\right\}. \tag{2.7}$$

Proof. From (1.4), we have

$$M_{X(r,n,m,k)}(t)=\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}e^{tx}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{2.8}$$

Integrating by parts treating $[\bar{F}(x)]^{\gamma_r-1}f(x)$ for integration and rest of the integrand for differentiation, we get

$$M_{X(r,n,m,k)}(t)=M_{X(r-1,n,m,k)}(t)+\frac{tC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[\bar{F}(x)]^{\gamma_r}g_m^{r-1}(F(x))dx,$$

the constant of integration vanishes since the integral considered in (2.8) is a definite integral. On using (2.1), we obtain

$$M_{X(r,n,m,k)}(t)=M_{X(r-1,n,m,k)}(t)+\frac{t}{\gamma_r}M_{X(r,n,m,k)}(t)$$
$$-\frac{1}{\theta\gamma_r}\sum_{u=2}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}(1-\alpha)^{u-1}M_{X(r,n,m,k)}(t+\lambda(u-1)). \tag{2.9}$$

Differentiating both the sides of (2.9) j times with respect to t, we get

$$M^{(j)}_{X(r,n,m,k)}(t)=M^{(j)}_{X(r-1,n,m,k)}(t)+\frac{t}{\gamma_r}M^{(j)}_{X(r,n,m,k)}(t)+\frac{j}{\gamma_r}M^{(j-1)}_{X(r,n,m,k)}(t)$$
$$-\frac{t}{\theta\gamma_r}\sum_{u=2}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}(1-\alpha)^{u-1}M^{(j)}_{X(r,n,m,k)}(t+\lambda(u-1))$$
$$-\frac{j}{\theta\gamma_r}\sum_{u=2}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}(1-\alpha)^{u-1}M^{(j-1)}_{X(r,n,m,k)}(t+\lambda(u-1)).$$

The recurrence relation in equation (2.7) is derived simply by rewriting the above equation.
By differentiating both sides of equation (2.7) with respect to t and then setting $t=0$, we obtain the recurrence relations for single moments of *gos* from exponential-truncated negative binomial distribution in the form

$$E[X^{j}(r,n,m,k)]=E[X^{j}(r-1,n,m,k)]+\frac{j}{\gamma_r}E[X^{j}(r,n,m,k)]$$
$$-\frac{j}{\theta\gamma_r}\sum_{u=2}^{\theta+1}(-1)^{u}\binom{\theta+1}{u}(1-\alpha)^{u-1}E[\phi(X(r,n,m,k))], \tag{2.10}$$

where $\phi(x)=x^{j-1}e^{-\lambda(u-1)x}$.

Remark 2.1: Putting $m=0,\ k=1$ in (2.7) and (2.10), we can get the relations for marginal moment generating functions and moments of order statistics for exponential-truncated negative binomial distribution in the form

$$\left(1-\frac{t}{n-r+1}\right)M^{(j)}_{X_{r:n}}(t)=M^{(j)}_{X_{r-1:n}}(t)+\frac{j}{n-r+1}M^{(j-1)}_{X_{r:n}}(t)$$

$$-\frac{1}{\theta(n-r+1)}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$
$$\times\left\{tM_{X_{r:n}}^{(j)}(t+\lambda(u-1))+jM_{X_{r:n}}^{(j-1)}(t+\lambda(u-1))\right\}$$

and

$$E[X_{r:n}^j]=E[X_{r-1:n}^j]+\frac{j}{n-r+1}E[X_{r:n}^{j-1}]-\frac{1}{\theta(n-r+1)}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}$$
$$\times(1-\alpha)^{u-1}E[\phi(X_{r:n})].$$

Remark 2.2: Setting $m=-1$ and $k\geq 1$, in (2.7) and (2.10), relations for $k-$th record values can be obtained as

$$\left(1-\frac{t}{k}\right)M_{Z_r^{(k)}}^{(j)}(t)=M_{Z_{r-1}^{(k)}}^{(j)}(t)+\frac{j}{k}M_{Z_r^{(k)}}^{(j-1)}(t)$$
$$-\frac{1}{\theta k}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$
$$\times\left\{tM_{Z_r^{(k)}}^{(j)}(t+\lambda(u-1))+jM_{Z_r^{(k)}}^{(j-1)}(t+\lambda(u-1))\right\}$$

and

$$E[(Z_r^{(k)})^j]=E[(Z_{r-1}^{(k)})^j]+\frac{j}{k}E[(Z_r^{(k)})^{j-1}]-\frac{1}{\theta k}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}E[\phi(Z_r)].$$

For $k=1$

$$(1-t)M_{X_{U(r)}}^{(j)}(t)=M_{X_{U(r-1)}}^{(j)}(t)+jM_{X_{U(r)}}^{(j-1)}(t)$$
$$-\frac{1}{\theta k}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$
$$\times\left\{tM_{X_{U(r)}}^{(j)}(t+\lambda(u-1))+jM_{X_{U(r)}}^{(j-1)}(t+\lambda(u-1))\right\}$$

and

$$E[X_{U(r)}^j]=E[X_{U(r-1)}^j]+jE[X_{U(r)}^{j-1}]$$
$$-\frac{1}{\theta}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}E[\phi(X_{U(r)})].$$

Remark 2.3:
i) Putting $\theta=1$ in (2.6) and (2.7), the results for marginal moment generating functions of *gos* are deduced for Marshal-Olkin exponential distribution.
ii) Setting $\theta=1$, $\alpha=2$ in (2.6) and (2.7), we obtain the marginal moment generating functions of *gos* for the half-logistic distribution.
iii) When $\alpha\to 1$ in (2.6) and (2.7), we can get the results for marginal moment generating functions of *gos* for the exponential distribution.
iv) If $\theta\to 1$ and $0<\alpha<2$ in (2.6) and (2.7), then we obtain relations for marginal moment generating functions of *gos* for the exponential-Logarithmic distribution.

3 Relations for joint moment generating functions

For exponential-truncated negative binomial distribution, the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given as

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{x}^{\infty}e^{t_1x+t_2y}f_{X(r,n,m,k)X(s,n,m,k)}(x,y)dxdy.$$

On using (1.3) and binomial expansion, we have

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\int_{-\infty}^{\infty}e^{t_1x}[\bar{F}(x)]^{(s-r+u-v)(m+1)-1}f(x)G(x)dx, \quad (3.1)$$

where

$$G(x)=\int_x^{\infty}e^{t_2y}[\bar{F}(y)]^{\gamma_{s-v}-1}f(y)dy. \quad (3.2)$$

By setting $z=\bar{F}(y)$ in (3.2), we obtain

$$G(x)=(1-\alpha)^{t_2/\lambda}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}\left(\frac{1-\alpha^{\theta}}{\alpha^{\theta}}\right)^q\frac{[\bar{F}(x)]^{\gamma_{s-v}+q}}{(\gamma_{s-v}+q)}.$$

On substituting the above expression of $G(x)$ in (3.1), and simplifying the resulting equation, we get

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=\frac{(1-\alpha)^{(t_1+t_2)/\lambda}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s}}$$

$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}(t_1/\lambda)_{(l)}(l/\theta)_{(w)}}{p!q!l!w!}$$

$$\times\left(\frac{1-\alpha^{\theta}}{\alpha^{\theta}}\right)^{q+w}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}B\left(\frac{k}{m+1}+n-r+u+\frac{q+w}{m+1},1\right)$$

$$\times\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}B\left(\frac{k}{m+1}+n-s+v+\frac{q}{m+1},1\right). \quad (3.3)$$

On using relation (2.5) in (3.3), we get

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=\frac{(1-\alpha)^{(t_1+t_2)/\lambda}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s}}$$

$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}(t_1/\lambda)_{(l)}(l/\theta)_{(w)}}{p!q!l!w!}\left(\frac{1-\alpha^{\theta}}{\alpha^{\theta}}\right)^{q+w}$$

$$B\left(\frac{k}{m+1}+n-r+\frac{q+w}{m+1},r\right)B\left(\frac{k}{m+1}+n-s+\frac{q}{m+1},s-r\right)$$

Which after simplification yields

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=(1-\alpha)^{(t_1+t_2)/\lambda}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}$$

$$\times\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}(t_1/\lambda)_{(l)}(l/\theta)_{(w)}}{p!q!l!w!}\left(\frac{1-\alpha^{\theta}}{\alpha^{\theta}}\right)^{q+w}$$

$$\times\frac{1}{\prod_{a=1}^{r}\left(1+\frac{q+w}{\gamma_a}\right)\prod_{b=r+1}^{s}\left(1+\frac{q}{\gamma_b}\right)}. \quad (3.4)$$

Special Cases

i) Putting $m=0$, $k=1$ in (3.4), the explicit formula for joint moment generating functions of order statistics for the exponential-truncated negative binomial distribution can be obtained as

$$M_{X_{r:n}X_{s:n}}(t_1,t_2)=\frac{(1-\alpha)^{(t_1+t_2)/\lambda}n!}{(n-s)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}$$

$$\times \frac{(t_1/\lambda)_{(l)}(l/\theta)_{(w)}}{l!w!}\left(\frac{1-\alpha^\theta}{\alpha^\theta}\right)^{q+w}\frac{\Gamma(n-r+1+q+w)\Gamma(n-s+1+q)}{\Gamma(n+1+q+w)\Gamma(n-r+1+q)}.$$

ii) Setting $m=-1$ in (3.4), we deduce the explicit expression for joint moment generating functions of $k-$th upper record values for exponential-truncated negative binomial distribution in the form

$$M_{X_{U(r):k}X_{U(s):k}}(t_1,t_2)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{(t_2/\lambda)_{(p)}(p/\theta)_{(q)}(t_1/\lambda)_{(l)}(l/\theta)_{(w)}}{p!q!l!w!}$$

$$\times\left(\frac{1-\alpha^\theta}{\alpha^\theta}\right)^{q+w}\frac{(1-\alpha)^{(t_1+t_2)/\lambda}}{\left(1+\frac{q+w}{k}\right)^r\left(1+\frac{q}{k}\right)^{s-r}}.$$

Making use of (2.1), we can derive the recurrence relations for joint moment generating functions of *gos* from (1.5).

Theorem 3.1. For the distribution given in (1.1) and for $1\le r<s\le n,\ n\ge 2$ and $k=1,2...$

$$\left(1-\frac{t_2}{\gamma_s}\right)M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)=M^{(j)}_{X(r,n,m,k),X(s-1,n,m,k)}(t_1,t_2)$$

$$+\frac{j}{\gamma_s}M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)-\frac{1}{\theta\gamma_s}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$

$$\times\Big[t_2M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2+\lambda(u-1))$$

$$+jM^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2+\lambda(u-1))\Big]. \tag{3.5}$$

Proof: Using (1.5), the joint moment generating functions of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}$$

$$\times\int_{-\infty}^{\infty}[\bar F(x)]^m f(x)g_m^{r-1}(F(x))I(x)dx \tag{3.6}$$

where

$$I(x)=\int_x^\infty e^{t_1x+t_2y}[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar F(y)]^{\gamma_s-1}f(y)dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.6), we get

$$M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)=M_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,t_2)$$

$$+\frac{t_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_x^\infty e^{t_1x+t_2y}[\bar F(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar F(y)]^{\gamma_s}dydx,$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (2.1), we obtain

$$M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)=M_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,t_2)$$

$$+\frac{t_2}{\gamma_s}M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)-\frac{t_2}{\theta\gamma_s}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$

$$\times M_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2+\lambda(u-1)). \tag{3.7}$$

Differentiating both the sides of (3.7) i times with respect to t_1 and then j times with respect to t_2, we get

$$M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)=M^{(i,j)}_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,t_2)$$

$$+\frac{t_2}{\gamma_s}M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)+\frac{j}{\gamma_s}M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2}{\theta\gamma_s}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2+\lambda(u-1))$$

$$-\frac{j}{\theta\gamma_s}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,t_2+\lambda(u-1)),$$

which, when rewritten gives the recurrence relation in (3.5).
One can also note that Theorem 2.1 can be deduced from Theorem 3.1 by letting t_1 tends to zero.
By differentiating both sides of equation (3.5) with respect to $t_1,\ t_2$ and then setting $t_1=t_2=0$, we obtain the recurrence relations for product moments of *gos* from exponential-truncated negative binomial distribution in the form

$$E[X^i(r,n,m,k)X^j(s,n,m,k)]=E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

$$+\frac{j}{\gamma_s}E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)]-\frac{j}{\theta\gamma_s}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}$$

$$\times E[\phi(X(r,n,m,k)X(s,n,m,k))], \tag{3.8}$$

where

$$\phi(x,y)=x^iy^{j-1}e^{-\lambda(u-1)y}.$$

Remark 3.1 Putting $m=0$, $k=1$ in (3.5) and (3.8), we obtain the recurrence relations for joint moment generating functions and single moments of order statistics for exponential-truncated negative binomial distribution as

$$\left(1-\frac{t_2}{n-s+1}\right)M^{(i,j)}_{X_{r,s:n}}(t_1,t_2)=M^{(i,j)}_{X_{r,s-1:n}}(t_1,t_2)$$

$$+\frac{j}{n-s+1}M^{(i,j-1)}_{X_{r,s:n}}(t_1,t_2)-\frac{1}{\theta(n-s+1)}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}$$

$$\times(1-\alpha)^{u-1}\left[t_2M^{(i,j)}_{X_{r,s:n}}(t_1,t_2+\lambda(u-1))+jM^{(i,j-1)}_{X_{r,s:n}}(t_1,t_2+\lambda(u-1))\right]$$

and

$$E[X^{(i,j)}_{r,s:n}]=E[X^{(i,j)}_{r,s-1:n}]+\frac{j}{n-s+1}E[X^{(i,j-1)}_{r,s:n}]$$

$$-\frac{1}{\theta(n-s+1)}\sum_{u=2}^{\theta+1}(-1)^u\binom{\theta+1}{u}(1-\alpha)^{u-1}E[\phi(X_{r,s:n})].$$

Remark 3.2 Substituting $m=-1$ and $k\geq 1$, in (3.5) and (3.8), we get recurrence relation for joint moment generating functions and product moments of $k-$th upper record values for exponential-truncated negative binomial distribution.

Remark 3.3:
i) Putting $\theta=1$ in (3.4) and (3.5), the joint moment generating functions *gos* are deduced for Marshall-Olkin exponential distribution.
ii) Setting $\theta=1$, $\alpha=2$ in (3.4) and (3.5), we obtain the relations for joint moment generating functions of *gos* for the half-logistic distribution.
iii) When $\alpha\to 1$ in (3.4) and (3.5), we can get the relations for joint moment generating functions of for the exponential distribution.
iv) If $\theta\to 1$ and $0<\alpha<2$ in (3.4) and (3.5), we can get the relations for joint moment generating functions of *gos* for the exponential distribution.

4 Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos*, then from a continuous population with cdf $F(x)$ and pdf $f(x)$, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\leq r<s\leq n$, in view of (1.4) and (1.5), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}\times\frac{[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[F(x)]^{\gamma_{r+1}}}f(y).\ \ x<y \tag{4.1}$$

Theorem 4.1: Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]=(1-\alpha)^{t/\lambda}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}[1-\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta}]^q\prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+q}\Big), \tag{4.2}$$

if and only if

$$\bar{F}(x)=\frac{\alpha^\theta}{(1-\alpha^\theta)}[\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta}-1],\quad x>0,\quad \alpha,\theta>0\ \ and\ \ \lambda>0$$

Proof. From (4.1), we have

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\times\int_x^\infty e^{ty}\Big[1-\Big(\frac{\bar{F}(y)}{F(x)}\Big)^{m+1}\Big]^{s-r-1}\Big(\frac{\bar{F}(y)}{F(x)}\Big)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \tag{4.3}$$

By setting $u=\frac{\bar{F}(y)}{F(x)}$ from (1.2) in (4.3), we obtain

$$E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x]=\frac{(1-\alpha)^{t/\lambda}C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}[1-\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta}]^q\times\int_0^1 u^{\gamma_s+q-1}(1-u^{m+1})^{s-r-1}du. \tag{4.4}$$

Again by setting $t=u^{m+1}$ in (4.4) and simplifying the resulting expression, we derive the relation given in (4.2). To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^\infty e^{ty}[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-1}\times[\bar{F}(y)]^{\gamma_s-1}f(y)dy=[\bar{F}(x)]^{\gamma_{r+1}}H_r(x), \tag{4.5}$$

where

$$H_r(x)=(1-\alpha)^{t/\lambda}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t/\lambda)_{(p)}(p/\theta)_{(q)}}{p!q!}[1-\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta}]^q\prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}+q}\Big).$$

Differentiating (4.5) both sides with respect to x and rearranging the terms, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_x^\infty e^{ty}[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-2}$$

$$\times [\bar{F}(y)]^{\gamma_s - 1} f(y) dy = H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x)$$

or

$$-\gamma_{r+1} H_{r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m} f(x)$$
$$= H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = -\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]}$$
$$= \frac{(1-\alpha)\lambda\theta e^{-\lambda x}[1-(1-\alpha)e^{-\lambda x}]^{-(\theta+1)}}{[\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta} - 1]}$$

which proves that

$$\bar{F}(x) = \frac{\alpha^\theta}{(1-\alpha^\theta)}[\{1-(1-\alpha)e^{-\lambda x}\}^{-\theta} - 1], \quad x > 0, \quad \alpha, \theta > 0 \;\; and \;\; \lambda > 0$$

Remark 4.1. For $m = 0$, $k = 1$ and $m = -1$, $k = 1$, we obtain the characterization results of the exponential-truncated negative binomial distribution based on order statistics and record values, respectively.

5 Conclusions

In this study some explicit expressions and recurrence relations for marginal and joint moment generating functions of *gos* from the exponential-truncated negative binomial distribution have been established. Further, conditional expectation of *gos* has been utilized to obtain a characterization of the exponential-truncated negative binomial distribution.

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