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Some Special Curves in Normed Three Dimensional Space

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Abstract: In this paper, we consider the place of action is an affine 3-space endowed with a metric which is well known by a centrally symmetric, convex and smooth body B as a unit ball, such a space is called Minkowski normed space. We defined a semi-inner product in M-space using the Cosine-Minkowski function. Finally, we redefine Bertrand curves and insert the concepts of *Right and Left involutes* in Minkowski three dimensional space with new related theorems

Keywords: Minkowski space, Birkhoff orthogonality, Semi-inner product, Left and right involute, Bertrand curves.

1 Introduction

Minkowski space is a real normed linear finite space endowed with a metric based on a centrally symmetric unit ball B. Such "Minkowsi-spaces" are, in general, not inner product spaces and therefore one has to modify the classical concepts of orthogonality and angle measure [1]. There are many authors attempts to define Minkowski orthogonality, most of them justified by and applied to just a single geometric problem [2], [3]. Most commonly used is "Birkhoff-orthogonality" (B-orthogonality) with the disadvantage of being a non-symmetric relation, see Birkhoff [4], James [5,6,7] and Day [8].

Let B be a centrally symmetric (gauge) convex body in an affine 3-space E^3, then we can define a norm whose unit ball is B, see [9]. In the following, we will consider only Minkowski spaces with a strictly convex unit ball B, that means the boundry ∂B contains no line segments.

If M_B^n is n-dimensional Minkowski space with unit ball B, let $x, y \in M_B^n$, then we say that x is left orthogonal (Birkhoff orthogonal) to y, $(x \dashv y)$, if $\|x + \lambda y\| \geq \|x\|$ for all λ in $\mathbb{R}$. In Figure 1, we see the vector **a** passing through the origin and its conjugate line $\mathbf{a}^{\dashv}$ which is left orthogonal to **a** in a Minkowski plane (M-plane) with the unit ball B.

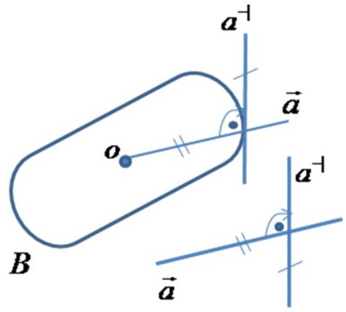

Fig. 1: $\mathbf{a}^{\dashv}$ is a left-orthogonal line to **a** in a Minkowski plane M_B^2 .

A. C. Thompson [10] introduce an angle measure between two lines using Birkhoff orthogonal projection, see Figure 2, and delivering an analogue to the cosine function in the Euclidean case. It is called *Cosine Minkowski* function which can be defined as follows:

If $x_1, x_2 \in M_B^n, x_2 \neq 0$, then,

$$cm(x_1, x_2) := \frac{f_1(x_2)}{\|x_2\| \|f_1\|} \quad (1)$$

where, f_1 is a unique linear functional attains its norm at x_1.

Thereby, $cm(x_1, x_2) = 0 \Leftrightarrow x_1 \dashv x_2$ and $cm(x_1, x_1) = 1$. Also, for $x_1 \neq x_2$, $|cm(x_1, x_2)| \leq 1$ with $cm(x_1, x_2) = 1$ iff

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the line segment $[x_1/\|x_1\|, x_2/\|x_2\|] \subset \partial B$, for more details see [10].

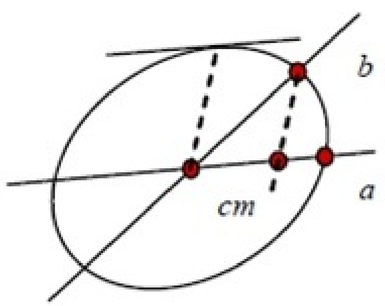

Fig. 2: Cosine-Minkowski $cm(\mathbf{b},\mathbf{a})$ of an angle spanned by an ordered pair of oriented lines **a** and **b** with another unit ball B .

Definition 1.1 In Minkowski space M_B^n, we define the Minkowski semi-inner product of two vectors $x_1, x_2 \in M_B^n$ as follows:

$$\langle x_1, x_2\rangle_M = \frac{f_1(x_2)}{f_1(x_1)}\|x_1\|^2. \tag{2}$$

By substitute (1) into (2), we have

$$\langle x_1, x_2\rangle_M = \|x_1\|\|x_2\| cm(x_1, x_2), \tag{3}$$

which is a suitable definition of the Minkowski semi-inner product of two vectors $x_1, x_2 \in M_B^n$, see [11,12].

The Minkowski semi-inner product $\langle .,.\rangle_M$ has the following properties:

i. $\langle x_1, x_2\rangle_M = 0$ if and only if $x_1 \dashv x_2$.

ii. In general, $\langle x_1, x_2\rangle_M \neq \langle x_2, x_1\rangle_M \forall x_1, x_2 \in M_B^n$, $x_1 \neq x_2$.

iii. $\langle x_1, ax_2 + bx_3\rangle_M = a\langle x_1, x_2\rangle_M + b\langle x_1, x_3\rangle_M$, for all $a, b \in \mathbb{R}$ and all $x_1, x_2, x_3 \in M_B^n$.

iv. $\langle cx_1, x_2\rangle_M = c\langle x_1, x_2\rangle_M$ and $\langle x_1, dx_2\rangle_M = d\langle x_1, x_2\rangle_M$ for all $x_1, x_2 \in M_B^n$ and $c, d \in \mathbb{R}$.

v. $\langle x, x\rangle_M \geq 0$ and $\langle x, x\rangle_M = 0$ if and only if $x = 0$.

vi. $|\langle x_1, x_2\rangle_M|^2 \leq \|x_1\|^2\|x_2\|^2 \forall x_1, x_2 \in M_B^n$.

Dragomir [11] proof that, in each real normed linear space M_B^n there exists at least one semi-inner product $[.,.]$ which generates the norm $\|.\|$. That is, $\|x\| = [.,.]^{1/2}$ for all $x \in M_B^n$, and it is unique if and only if M_B^n is smooth. Therefore, definition (3) is unique which generate the norm which is generate by the unit ball B.

Definition 1.2. Let x_1, x_2 are two vectors in a Minkowski space M_B^3 such that $\|x_1\| = \|x_2\| = 1$, then this pair is called *mutually normal* pair if $x_1 \dashv x_2$ and $x_2 \dashv x_1$.

Definition 1.3. (Thompson [10]), If B is the unit ball in a Minkowski space then there exists a basis $(x_1, x_2, ..., x_d)$ such that $\|x_i\| = 1$ and $x_i \dashv x_j$ for all i and j with $i \neq j$; i.e. each pair of basis vectors is mutually normal.

Definition 1.4. (B-orthonormal frame in M_B^n): Let $e_1, e_2, ..., e_n \in M_B^n$, $\|e_i\| = 1 \forall i = 1,2,...,n$. If $e_j \dashv e_k \forall k = 1,2,..., j-1$ then the ordered vector set $e_1, e_2, ..., e_n$ is called B-orthonormal frame in M_B^n.

In our work, we shall use the previous definitions to introduce and modify some concepts of curve which depend on left and right B-orthogonal, e.g. theorems about *Involute* and *Bertrand curves* in three dimensional M-space.

2 Involute.

2.1. In Euclidean space.

The tangent lines of a space curve c generate a surface called the tangent ruled surface of c. A curve c^* which lies on the tangent ruled surface of c and intersects the tangent lines orthogonally is called *an involute* of c.

If c is given by $\mathbf{x} = \mathbf{x}(s)$, (where s is the arc length of the curve c), and if $\mathbf{x}^*$ is a point on an involute c^*, where c^* crosses the tangent line at $\mathbf{x}(s)$, then $\mathbf{x}^* - \mathbf{x}(s)$ is proportional to $\mathbf{t}(s)$. Thus c^* have a representation of the form $\mathbf{x}^* = \mathbf{x}(s) + k(s)\,\mathbf{t}(s)$. Moreover, on an involute, the tangent vector is given by

$$\frac{d\mathbf{x}^*}{ds} = (1+\dot{k})\,\mathbf{t} + k\,\chi\,\mathbf{h}, \tag{4}$$

where, $\mathbf{t}$ and $\mathbf{h}$ and are the tangent and principal normal vectors on c respectively. χ is the curvature of the curve c, $\frac{d\mathbf{x}^*}{ds}$ is orthogonal to the tangent vector $\mathbf{t}$ on C; that is

$$\frac{d\mathbf{x}^*}{ds} \cdot \mathbf{t} = 1 + \dot{k} = 0 \tag{5}$$

Integrating both sides of (5) gives $k = -s + \alpha$, $\alpha = constant$. Thus there exists an infinity family of involute, one for each α, $\mathbf{x}^* = \mathbf{x} + (\alpha - s)\mathbf{t}$.

In the following subsection, we try to insert new involute concepts in Minkowski space depend on left and right-orthogonality.

2.2. In Minkowski space.

Definition 2.1 ***(left involute)***

If c is a space curve with a parametric representation $\mathbf{x} = \mathbf{x}(s)$. Then, the corresponding space curve c_1^* is called *left involute* of c if and only if for all points $a \in \mathbf{x}(s)$ there exists $\mathbf{x}_{\dashv}^*$ upon c_1^* such that the tangent line $\mathbf{t}_a$ at a is B-orthogonal to $\frac{d\mathbf{x}_{\dashv}^*}{ds}$ at the point of intersection, i.e., $\mathbf{t}_a \dashv \frac{d\mathbf{x}_{\dashv}^*}{ds}$ for all $a \in \mathbf{x}(s)$, see Figure 3.

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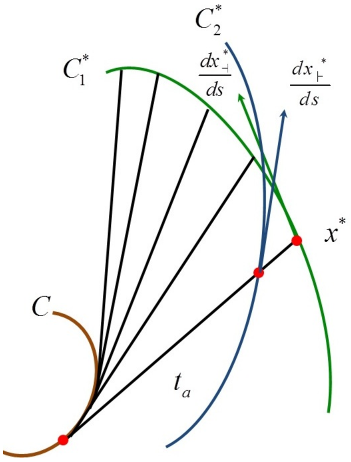

Fig. 3: Left and right involute in M-Space.

By the above definition we have

$$\mathbf{x}^* = \mathbf{x}(s) + \lambda(s)\, \mathbf{t}(s),$$

$$\frac{d\mathbf{x}^*}{ds} = \frac{d\mathbf{x}}{ds} + \lambda^{\cdot}\, \mathbf{t}(s) + \lambda\, \mathbf{t}^{\cdot}(s)\ .$$

From Frenet-Serret equations [1,12], we get

$$\mathbf{t}^{\cdot}(s) = \chi_M\, \mathbf{h}(s), \tag{6}$$

and,

$$\frac{d\mathbf{x}^*}{ds} = (\|\mathbf{x}^{\cdot}\| + \lambda^{\cdot})\, \mathbf{t} + \lambda\, \chi_M\, \mathbf{h}$$

where χ_M is the M-curvature and $\mathbf{t}, \mathbf{h}$ are the tangent and M-principle normal vectors with respect to B-orthogonality and the unit ball B.

Since, $\mathbf{t} = \mathbf{t}_a \dashv \frac{d\mathbf{x}^*}{ds}$, then $\langle \mathbf{t}_a, \frac{d\mathbf{x}^*}{ds}\rangle_M = 0$, $\|\mathbf{x}^{\cdot}\| + \lambda^{\cdot} + \lambda\, \chi_M\, cm(\mathbf{t}_a, \mathbf{h}) = 0$, where $cm(\mathbf{t}_a, \mathbf{h}) = 0$ because $\mathbf{t}_a \dashv \mathbf{h}$, then,

$$\|\mathbf{x}^{\cdot}\| + \lambda^{\cdot} = 0. \tag{7}$$

Without loss of generality, we can take $\|\mathbf{x}^{\cdot}\| = 1$. Then we get

$$1 + \lambda^{\cdot} = 0. \tag{8}$$

Integrating both sides of (8), we get $\lambda(s) = -s + \beta$ where β is a constant, we have

$$\frac{d\mathbf{x}^*}{ds} = \chi_M(-s+\beta)\mathbf{h}. \tag{9}$$

Definition 2.2. ***(Right involute)***

If c is a space curve given by the representation $\mathbf{x} = \mathbf{x}(s)$ then the corresponding space curve c_2^* is called *right involute* of c if and only if for all points $a \in \mathbf{x}(s)$ there exists $\mathbf{x}_{\vdash}^*$ upon c_2^* such that the tangent line $\mathbf{t}_a$ is right-orthogonal to $\frac{d\mathbf{x}_{\vdash}^*}{ds}$ at the point of intersection, i.e., $\mathbf{t}_a \vdash \frac{d\mathbf{x}_{\vdash}^*}{ds}$ for all $a \in \mathbf{x}(s)$, see Figure 3. In the following theorem, we try to get the condition that the two vectors $\frac{d\mathbf{x}_{\dashv}^*}{ds}$ and $\frac{d\mathbf{x}_{\vdash}^*}{ds}$ are mutually normal at any point of the parameter $s = s_0$. Also, if it is valid for all points on all curves in the space, then the space is Euclidean one.

Theorem 2.3

If c is a curve with a vector representation $\mathbf{x} = \mathbf{x}(s)$ and $\mathbf{x}_{\dashv}^*, \mathbf{x}_{\vdash}^*$ are the left and right involutes of $\mathbf{x}(s)$ respectively, such that

$$\frac{d\mathbf{x}_{\dashv}^*}{ds} = \rho_1 \frac{d\mathbf{x}_{\vdash}^*}{ds}\ at\ s = s_0, \tag{10}$$

where, ρ_1 is a constant. Then $\frac{d\mathbf{x}}{ds}$ and $\frac{d\mathbf{x}_{\dashv}^*}{ds}$ are mutually normal vectors at $s = s_0$. If (10) is valid for all points a belongs to a curve $\mathbf{x}(s)\ \forall\ \mathbf{x}(s) \in M_B^3$ then, the space is Euclidean space with ellipsoid unit ball B.

Proof. Since

$$\frac{d\mathbf{x}_{\dashv}^*}{ds} = \rho_1 \frac{d\mathbf{x}_{\vdash}^*}{ds},$$

from the definition 2.1, $\frac{d\mathbf{x}_{\dashv}^*}{ds} \vdash \frac{d\mathbf{x}}{ds}$. Therefore, $\rho_1 \frac{d\mathbf{x}_{\vdash}^*}{ds} \vdash \frac{d\mathbf{x}}{ds}$. Thus we get,

$$\frac{d\mathbf{x}_{\vdash}^*}{ds} \perp \frac{d\mathbf{x}}{ds}, \tag{11}$$

and hence $\frac{d\mathbf{x}}{ds}$ and $\frac{d\mathbf{x}_{\dashv}^*}{ds}$ are mutually normal vectors. ■

3 Bertrand curves.

3.1. In Euclidean space.

An interesting problem in the theory of curves is whether it is possible for several curves to share the same family of tangents, principal normals or binormals. For the tangents, the answer is easily seen to be negative: the family of tangents uniquely determines the curve. For the principle normals, the problem, was answered by Joseph Bertrand, who discovered that, for an arbitrary curve, the answer is negative. However, there are special curves for which there might be, also, other curves with the same family of principle normals. These curves are called Bertrand curves.

Usually, for a Bertrand curve, there is only one curve having the same principle normals. We will say that, the two curves are *Bertrand mates*, or that they are *associated*, or *conjugated Bertrand curves*. It turns out that if a Bertrand curve has more than one Bertrand mate, then it has an infinity and the curve (and all of its mates) is a circular cylindrical helix.

We have the following results related to Bertrand curves:

- The angle of the tangents of two associated Bertrand curves at corresponding points is constant.

- A curve r is a Bertrand curve if and only if its torsion and curvature verify a relation of the form

$$\mathbf{a} \cdot \tau + \mathbf{b} \cdot \chi = 1 \tag{12}$$

with constants vectors $\mathbf{a}$ and $\mathbf{b}$.

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3.2. In Minkowski space.

Let $\mathbf{r}, \mathbf{r}^*$ be two Bertrand mates. We assume that the first curve is parametrized with the arc length parameter s. The second curve also depend on s of the first curve. We assume that there exist a point, belongs to $\mathbf{r}^*$, which is on the Bertrand mate having the same Minkowski-principle normal as the first one at s. The two points are called corresponding Bertrand points,

$$\mathbf{r}^*(s) = \mathbf{r}(s) + \lambda_1(s)\mathbf{h}(s) ,$$
$$\mathbf{h}(s) = \pm\mathbf{h}^*(s) ,$$

then,

$$\frac{d\mathbf{r}^*}{ds} = \frac{d\mathbf{r}}{ds} + \lambda_1(s)\frac{d\mathbf{h}}{ds} + \frac{d\lambda_1}{ds}\mathbf{h}. \quad (13)$$

Using Frenet matrix [1], we get

$$\frac{d\mathbf{r}^*}{ds} = (1-\lambda_1\bar{\tau})\mathbf{t} + \left(\frac{d\lambda_1}{ds} + \lambda_1(\bar{\tau}\, cm(\mathbf{h},\mathbf{t}) + \bar{\tau}_1 cm(\mathbf{h},\mathbf{b}))\right)\mathbf{h} - \lambda_1\bar{\tau}_1\mathbf{b}, \quad (14)$$

where $\bar{\tau}, \bar{\tau}_1$ are given as functions of M-torsions, see [1] and $\frac{d\mathbf{r}^*}{ds}$ is tangent to the second curve, therefore it is B-orthogonal both on $\mathbf{h}$ and $\mathbf{h}^*$.

Multiply both sides of (14) by $\mathbf{h}$ from left as a semi-inner product, we get

$$\frac{d\lambda_1}{ds} = cm(\mathbf{h},\mathbf{t}^*)\frac{ds^*}{ds} - cm(\mathbf{h},\mathbf{t}). \quad (15)$$

Since,

$$\frac{d\mathbf{r}^*}{ds} = \mathbf{t}^*\frac{ds^*}{ds} ,$$

then we get,

$$\mathbf{t}^* = (1-\lambda_1\bar{\tau})\mathbf{t}\frac{ds}{ds^*} + \left(cm(\mathbf{h},\mathbf{t}^*) - \frac{ds}{ds^*}(1-\lambda_1\bar{\tau})cm(\mathbf{h},\mathbf{t}) + \frac{ds}{ds^*}\lambda_1\bar{\tau}_1 cm(\mathbf{h},\mathbf{b})\right)\mathbf{h} - \frac{ds}{ds^*}\lambda_1\bar{\tau}_1\mathbf{b}. \quad (16)$$

Multiply both sides of (16) by t from left we get,

$$cm(\mathbf{t},\mathbf{t}^*) = (1-\lambda_1\bar{\tau})\frac{ds}{ds^*} - \frac{ds}{ds^*}\lambda_1\bar{\tau}_1 cm(\mathbf{t},\mathbf{b}).$$

We say that the angle between $\mathbf{t}$ and $\mathbf{t}^*$ is constant with respect to Birkhoff orthogonality for first argument $\mathbf{t}$ if $cm(\mathbf{t},\mathbf{t}^*) = const$.

Then the condition of $\mathbf{r}$ and $\mathbf{r}^*$ to be two Bertrand mates is:

$$\frac{ds}{ds^*}((1-\lambda_1\bar{\tau}) - \lambda_1\bar{\tau}_1 cm(\mathbf{t},\mathbf{b})) = const. \quad (17)$$

Conclusion

For Bertrand curves in Minkowski space, we assume that the angle between the tangents of two associated Bertrand curves is constant when the Minkowski cosine is also constant. We can improve result (17) by using Brauner's theorem (Angle measure in M-space) using the ideal plane at the projective space.

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