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On Fractional Ultra-Hyperbolic Kernel Related to the Spectrum

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Abstract: In this paper, we study the equation $(I-\Box)^{\frac{\alpha}{2}}u(x)=f(x), x\varepsilon R^n, 0<\alpha<n$. The operator $\Box$ is named ultra-hyperbolic operator defined by

$$\Box=(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}),$$

$p+q=n$ is the dimension of Euclidean spaceR^n, $f(x)$is given generalized function.We define the fractional ultra-hyperbolic kernel $E_\alpha$and obtain the solution of such equation which is related to the spectrum of E_α.Moreover,such $E_\alpha$and $u(x)$ are estimated,and then we show that they are bounded.Then we study the non linear equation

$$(I-\Box)^{\frac{\alpha}{2}}u(x)=f(x,u(x)).$$

And on suitable conditions for f,u and for the spectrum of the kernel E_α we can obtain a unique bounded solution for the nonlinear equation in a compact subset of R^n.

Keywords: Fractional ultra-hyperbolic kernel, Solution, Estimations, Spectrum.

1 Introduction

It is well known that the solution of equation

$$-(\Delta-I)u=f in R^n \tag{1.1}$$

where $f\varepsilon L^2(R^n)$, was investigated before,see e.g. [1]. Its Fourier transform is

$$\hat{u}=\frac{\hat{f}}{1+|y|^2}$$

And its inverse Fourier transform is

$$u=(\frac{\hat{f}}{1+|y|^2})\check{}=\frac{f*E}{(2\pi)^{\frac{n}{2}}}, where \hat{E}=\frac{1}{1+|y|^2}$$

As is known in [1] , [2]

$$E(x)=\frac{1}{2^{\frac{n}{2}}}\int_0^\infty\frac{e^{\frac{-t-|x|^2}{(4t)}}}{t^{\frac{n}{2}}}dt, x\varepsilon R^n \tag{1.2}$$

where E is called a Bessel potential ,and

$$u(x)=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty\int_{R^n}\frac{e^{\frac{-t-|x-y|^2}{(4t)}}}{t^{\frac{n}{2}}}f(y)dydt, \tag{1.3}$$

Also the solution of the problems $-(\Delta-I)^k u=f, k\geq 1$ and $-(\Delta-I)^{\frac{\alpha}{2}}u=f, 0<\alpha<n$ were considered in [1], [3] and [4].

Recently a published work dealing with fractional differential equations can be found in [10].

Now, the purpose of this work is to study the solution of the equation

$$(I-\Box)^{\frac{\alpha}{2}}u(x)=f(x), \tag{1.4}$$

$f(x)$ is the given generalized function.We obtain $u(x)=\frac{f*E_\alpha}{(2\pi)^{\frac{n}{2}}}$ as a solution of (1.4),where

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$$E_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})} \cdot \int_\Omega \int_0^\infty e^{-t-\|\xi\|^2 t + i(\xi,x)} . t^{\frac{\alpha}{2}-1} dt d\xi \qquad (1.5)$$

and $\Omega \subset R^n$ is the spectrum of $E_\alpha(x)$. The function $E_\alpha(x)$ is called fractional ultra -hyperbolic kernel or the elementary solution of (1.4).

If we put $q = 0, \alpha = 2$, then (1.4) and (1.5) reduce to (1.1) and (1.2) respectively. Also under certain conditions on f and u,we study the solution of the following nonlinear equation of the form:

$$(I - \Box)^{\frac{\alpha}{2}} u(x) = f(x, u(x)) \qquad (1.6)$$

Also on suitable conditions for f, u and for the spectrum of the kernel we can find unique solution for the non linear equation in the compact subset of R^n, see [5], [6] and [7].

2 preliminaries

Definition 2.1*Let $f(x)\varepsilon L^1(R^n)-$ the space of integrable function inR^n.*

The fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-i(\xi,x)} f(x) dx$$

where

$$\xi = (\xi_1, \xi_2, \cdots, \xi_n), x = (x_1, x_2, \cdots, x_n)\varepsilon R^n,$$

$(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots\cdots + \xi_n x_n$ is usual inner product in R^n and $dx = dx_1 dx_2 \cdots dx_n$.

Also the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{i(\xi,x)} \hat{f}(\xi) d\xi,$$

see [2],[8] and [9]

Definition 2.2*let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ be a point in R^n and we write $\mu = \| \xi \|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2, p + q = n$.*

Denote by $\Gamma_+ = \{\xi\varepsilon R^n : \xi_1 > 0 \text{ and } \mu > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma_+}$ denotes the closure of Γ_+.

Definition 2.3*(Bipoolar coordinates)*

let $\xi_1 = r\omega_1$, $\xi_2 = r\omega_2$,$\cdots$,$\xi_p = r\omega_p$
and
$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \cdots$, $\xi_{p+q} = s\omega_{p+q}$
where $\sum_{i=1}^{p} \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$ where $d\xi = r^{p-1}s^{q-1} dr ds d\Omega_p d\Omega_q$,and $d\Omega_p, d\Omega_q$ are the elements of surface area of the unit sphere in R^p, R^q respectively. Since $\Omega \subset R^n, \Omega \subset \overline{\Gamma_+}$ is the spectrum of E_α and we suppose $0 \le r \le R$ and $0 \le s \le L$ where R and L are constants.

Definition 2.4*The Fourier transform of a function f which is sufficiently smooth, and small at infinity, and its Laplacean,$\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$, are related by $(-\Delta \hat{f})(y) = 4\pi^2 \mid y \mid^2 \hat{f}(y)$, and thus the fractional power of the Laplacean by $((-\Delta)^{\frac{\alpha}{2}} \hat{f})(y) = (2\pi \mid y \mid)^\alpha \hat{f}(y)$ and by replacing the "non-negative" operator $-\Delta$, by the "strictly positive" operator $I - \Delta$, (I = identity), then we get*

$((I - \Delta)^{\frac{\alpha}{2}} \hat{f}) = (1 + 4\pi^2 \mid y \mid^2)^{\frac{\alpha}{2}} \hat{f}(y)$, see[4].

3 Main results

Theorem 3.1*Given the equation*

$$(I - \Box)^{\frac{\alpha}{2}} u(x) = f(x) \qquad (3.1)$$

we obtain

$$u(x) = \frac{f * E_\alpha}{(2\pi)^{\frac{n}{2}}} \qquad (3.2)$$

as a solution of (3.1) where $E_\alpha(x)$ is given as

$$E_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega e^{i(\xi,x)} .(\frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-t-\|\xi\|^2 t} . t^{\frac{\alpha}{2}-1} dt) d\xi$$

where $\Omega \subset R^n$ is the spectrum of E_α

proof:*Taking the Fourier transform to both sides of (3.1), we obtain*

$$F[(I - \Box)^{\frac{\alpha}{2}} u(x)] = F[f(x)]$$

But from properties of Fourier transform $F(D^\alpha u) = (i\xi)^\alpha \hat{u}$ for each multiindex α such that $D^\alpha u\varepsilon L^2(R^n)$

Then, we get

$$\hat{u}(\xi) = \frac{\hat{f}}{(1 + \| \xi \|^2)^{\frac{\alpha}{2}}} = \hat{f}.\hat{E_\alpha} \qquad (3.3)$$

Where

$$\| \xi \|^2 = \sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 > 0$$

But from the definition of gamma function we have,

$$r^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-rt} t^{a-1}, a = \frac{\alpha}{2}, r = 1 + \| \xi \|^2 \qquad (3.4)$$

Then,

$$\hat{E_\alpha}(\xi) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-(1+\|\xi\|^2)t} t^{\frac{\alpha}{2}-1} dt$$

From the definition of inverse Fourier transform, we get

$$E_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{i(\xi,x)} (\frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-t-\|\xi\|^2 t} . t^{\frac{\alpha}{2}-1} dt) d\xi$$

Since $\Omega \subset R^n, \Omega$ is the spectrum of E_α

$$E_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})} \int_\Omega \int_0^\infty e^{-t-\|\xi\|^2 t+i(\xi,x)}.t^{\frac{\alpha}{2}-1} dt d\xi \tag{3.5}$$

Thus (3.3) *can be written in the convolution form* $u(x) = \frac{f*E_\alpha}{(2\pi)^{\frac{n}{2}}}$. *Then*

$$u(x) = \frac{1}{(2\pi)^n\Gamma(\frac{\alpha}{2})}$$

$$.\int_\Omega \int_\Omega \int_0^\infty e^{-t-\|\xi\|^2 t+i(\xi,x-y)} t^{\frac{\alpha}{2}-1}$$

$$.f(y)dydtd\xi$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_\alpha(x-y)f(y)dy \tag{3.6}$$

Lemma 3.1 (Estimation of E_α)

$$| E_\alpha(x) | \leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}.M_\alpha \tag{3.7}$$

where

$$M_\alpha = \int_0^\infty \int_0^R \int_0^L exp[-t+ t(s^2-r^2)] .t^{\frac{\alpha}{2}-1} r^{p-1} s^{q-1} drdsdt, \quad \Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} and \Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} \tag{3.8}$$

proof: *Using* (3.5) *we get*

$$| E_\alpha(x) | \leq \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}$$

$$.\int_\Omega \int_0^\infty e^{-t-\|\xi\|^2 t}.t^{\frac{\alpha}{2}-1} dt d\xi$$

by changing to bipolar, we get

$$| E_\alpha(x) | \leq \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}$$

$$.\int_\Omega \int_0^\infty exp[-t+t(s^2-r^2)]$$

$$.t^{\frac{\alpha}{2}-1} r^{p-1} s^{q-1} drdsdtd\Omega_p d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, *and* $d\Omega_p$ *and* $d\Omega_q$ *are the elements of surface area of the unit sphere in* R^p *and* R^q *respectively. Since* $\Omega \subset R^n, \Omega \subset \overline{\Gamma}$ *is the spectrum of* E_α *and we suppose* $0 \leq r \leq R$ *and* $0 \leq s \leq L$ *where* R *and* L *are constants.*

Thus we obtain,

$$| E_\alpha(x) | \leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}$$

$$.\int_0^\infty \int_0^R \int_0^L \exp[-t+t(s^2-r^2)]$$

$$.t^{\frac{\alpha}{2}-1} r^{p-1} s^{q-1} drdsdt$$

$$= \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}.M_\alpha$$

i.e. E_α *is bounded.*

Lemma3.2 (Estimation of u)

$$| u(x) | \leq \frac{\Omega_p\Omega_q}{(2\pi)^n\Gamma(\frac{\alpha}{2})}.M_\alpha.N \tag{3.9}$$

where $M_\alpha, \Omega_p, \Omega_q$ *defined in* (3.8) *and* $N = \int_{R^n} | f(y) | dy$

proof: *Using* (3.6) *we get*

$$u(x) = \frac{1}{(2\pi)^n\Gamma(\frac{\alpha}{2})} \int_\Omega \int_\Omega \int_0^\infty e^{-t-\|\xi\|^2 t+i(\xi,x-y)}$$

$$.t^{\frac{\alpha}{2}-1} f(y)dydtd\xi = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_\alpha(x-y)f(y)dy$$

then

$$| u(x) | \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega | E_\alpha(x-y)f(y) | dy$$

$$\leq \frac{\Omega_p\Omega_q}{(2\pi)^n\Gamma(\frac{\alpha}{2})}M_\alpha.N$$

i.e. u is bounded.

Theorem 3.2*Given the nonlinear equation*

$$(I-\Box)^{\frac{\alpha}{2}} u(x) = f(x,u(x)), \quad for x \varepsilon R^n, 0 < \alpha < n. \tag{3.10}$$

Then we obtain

$$u(x) = \frac{f(x,u(x)) * E_\alpha(x)}{(2\pi)^{\frac{n}{2}}}$$

as a solution of (3.10) *where* $E_\alpha(x)$ *is defined in* (3.5)

proof:*Taking the Fourier transform to both sides of* (3.10), *and similar theorem* 3.1, *we obtain*

$$\hat{u}(x) = \hat{f}(x,u(x)).\hat{E_\alpha}(x) \tag{3.11}$$

Where E_α *is defined in* (3.5)

Thus (3.11) *can be written in the convolution form*

$$u(x) = \frac{f(x,u(x)) * E_\alpha(x)}{(2\pi)^{\frac{n}{2}}} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega E_\alpha(x-y)f(y,u(y))dy \tag{3.12}$$

Lemma 3.3(Estimation of $u(x)$ for the nonlinear equation)

$$| u(x) | \leq \frac{\Omega_p\Omega_q}{(2\pi)^n\Gamma(\frac{\alpha}{2})}M_\alpha.N \tag{3.13}$$

where $M_\alpha, \Omega_p, \Omega_q$ *defined in* (3.8) *and* $N = \int_{R^n} | f(y,u(y)) | dy$

proof: *Using* (3.7) *and* (3.12) *and similar to Lemma* 3.2, *we obtain the result and then* $u(x)$ *is bounded.*

Theorem 3.3 *Given the nonlinear equation*

$$(I-\Box)^{\frac{\alpha}{2}}u(x)=f(x,u(x))$$

for $x\varepsilon R^n, 0<\alpha<n$, *and with the following conditions*

1) *f satisfies the Lipchitz condition , that is*

$$|\ f(x,u)-f(x,w)\ |\leq A\ |\ u-w\ |$$

Where A is constant,

$$A<\frac{(2\pi)^n\Gamma(\frac{\alpha}{2})}{\Omega_p^2\Omega_q^2M_\alpha S}, M_\alpha,\Omega_p,\Omega_q$$

defined in (3.8)*and* $S=\frac{R^p}{p}\frac{R^q}{q}$.

2)

$$\int_{R^n}|\ f(x,u(x))\ |\ dx=N$$

for $x=(x_1,x_2,\cdots,x_n)\varepsilon R^n$.

Then, for the spectrum of $E_\alpha(x)$ *we obtain* $u(x)=\frac{f(x,u(x))*E_\alpha(x)}{(2\pi)^{\frac{n}{2}}}$ *is bounded on* R^n *and also* $u(x)$ *is a unique solution of* (3.10) *for* $x\varepsilon\Omega_0$ *where* Ω_0 *is a compact subset of* R^n *and* $E_\alpha(x)$ *is defined by* (3.5).

proof: *The formula*

$$u(x)=\frac{f(x,u(x))*E_\alpha(x)}{(2\pi)^{\frac{n}{2}}}$$

was obtained in Theorem (3.2) *and also boundness of u was shown in Lemma* (3.3) *as*

$$|\ u(x)\ |\leq\frac{\Omega_p\Omega_q}{(2\pi)^n\Gamma(\frac{\alpha}{2})}M_\alpha.N$$

where $M_\alpha,\Omega_p,\Omega_q$ *defined in* (3.8) *and* $N=\int_{R^n}|\ f(y,u(y))\ |\ dy$

To show that $u(x)$ *is unique.Let*

$$L(u)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_\Omega E_\alpha(x-y)f(y,u(y))dy$$

and suppose there is another solution $w(x)$ *of equation* (3.10).

Then,

$|\ L(u)-L(w)\ |\leq\frac{1}{(2\pi)^{\frac{n}{2}}}$

$$.\int_\Omega E_\alpha(x-y)\ |\ f(y,u(y))-f(y,w(y))\ |\ dy$$

But f satisfies the Lipchitz condition,then

$$|\ f(x,u)-f(x,w)\ |\\ \leq A\ |\ u(x)-w(x)\ |, \tag{3.14}$$

where A is constant,then

$|\ L(u)-L(w)\ |\leq\frac{1}{(2\pi)^{\frac{n}{2}}}A\ |\ u-w\ |$

$.\int_\Omega|\ E_\alpha(x-y)\ |\ dy$, *and using the estimation of* E_α *in* (3.7) *we obtain*

$$|\ L(u)-L(w)\ |\leq\frac{1}{(2\pi)^{\frac{n}{2}}}A\ |\ u-w\ |$$

$$.\frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}.M_\alpha\Omega_p\Omega_qS=K\ |\ u-w\ |$$

where

$K=\frac{A}{(2\pi)^n\Gamma(\frac{\alpha}{2})}\Omega_p^2\Omega_q^2M_\alpha S$ *and*

$S=\frac{R^p}{p}\frac{L^q}{q}$

It is clear that by Banach contraction fixed point theorem that if $A<\frac{(2\pi)^n\Gamma(\frac{\alpha}{2})}{\Omega_p^2\Omega_q^2M_\alpha S}$,

Then $u=L(u)$ *has a unique solution* $u(x)$ *and is defined by* (3.12)

References

[1] A. S. Abdel-Rady, and H. Beghr, On higher order Bessel potentials, *Arch. Math,* **82**, 2004, 361-370.

[2] L. C. Evans, Partial Differential Equations, *Math. Soc. Providence,RI*,**19**, 1998.

[3] Lima,Dezhong Chen,Radial,Symetry and uniqueness for positive solutions of Schrodinger type system, *Mathematical and computer Modelling*, 2008.

[4] E. M. Stien , princeton,lectures in analysis(I),Singular Integrals and differentiability properties of functions, *princeton university press,princeton NJ,* ***30****,1970.*

[5] K. Nolaopon and A. Kananthai, On the Ultra-hyperbolic heat kernel related to the spectrum. *International Journal of Pure and Applied Mathematics*, **17**, 2004, 19-28.

[6] K. Nolaopon and A. Kananthai, On the generalized Ultra - hyperbolic heat kernel related to the spectrum. *Science Asia*, **32**, 2006, 21-24.

[7] K. Nolaopon and A. Kananthai, On the generalized nonlinear Ultra -hyperbolic heat kernel related to the spectrum. *Computational and Applied Mathematics*, **28**, 2009, 157-166.

[8] B. Osgood, Fourier Transform and its Applications. *Stanford University*.

[9] E. M. Stien and R. Shakarchi, princeton,lectures in analysis(I),fourier analysis,an introduction, *princeton university press,princeton 2002*.

[10] A.A.Kilbas,H.M.Srivastava and J.J.Trujillo, theorey and applications of fractional differential equations. *Elsevier (North-Holland) Science publishers,Amsterdam,London and New York*, **204**, 2006.

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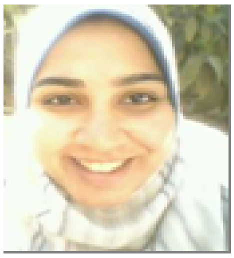

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