Classical and Bayesian Inference for an Extension of the Exponential Distribution under Progressive Type-II Censored Data with Binomial Removals

S. K. Singh[1], U. Singh[1], M. Kumar[2,*] and P. K. Vishwakarma[1]

[1] Department of Statistics, Banaras Hindu University, Varanasi-221005, India
[2] Department of Statistics and DST-CIMS, Banaras Hindu University, Varanasi-221005, India

Abstract: Maximum likelihood and Bayes estimators of the unknown parameters of an extension of the exponential (EE) distribution have been obtained for Progressive Type-II Censored data with Binomial removals. Markov Chain Monte Carlo (MCMC) method is used to compute the Bayes estimates of the parameters of interest. The General Entropy Loss Function (GELF) and Squared Error Loss Function (SELF) have been considered for obtaining the Bayes estimators. Comparisons are made between Bayesian and Maximum likelihood estimators (MLEs) via Monte Carlo simulation. An example is discussed to illustrate its applicability.

Keywords: Maximum likelihood, Bayes estimator, Progressive Type-II Censored data with Binomial removals

1 Introduction

In life testing and reliability problems, the role of hazard rate is very important because many phenomenon in real situation are modeled by the probability distribution. In early age, exponential distribution was the most popular distribution and has been frequently used to analyze the life time data due to their constant hazard rate and computational ease. In real situation, constant hazard rate does not occurs commonly but it occurs in monotonic or non monotonic form as for example mortality of child with their age distribution and failure of electric products with respect to time etc, see Nelson [15], Lawless [14] and Barlow and Proschan [13]. Initially gamma and Weibull distribution have been proposed as a generalization of exponential distribution and extensively used for the situation when hazard rate is not constant. But both distributions have their own advantages and disadvantages see Murthy et al. [1]. Considering disadvantages of gamma distribution, Gupta and Kundu [10] proposed a new exponentiated exponential distribution as an alternative to gamma distribution and has many property like gamma distribution with addition to closed form of distribution function and hazard function. For more details see Gupta and Kundu [10]. In the same context, Haghighi and Sadeghi [4] proposed EE distribution which is an alternative to gamma, Weibull and exponentiated exponential distribution and having additional important feature of an increasing hazard function when their respective probability density functions are monotonically decreasing. However, gamma, Weibull and exponentiated exponential distribution only allow for decreasing or constant hazard rate when their respective probability density functions are monotonically decreasing. The more applicability of EE distribution has discussed by Haghighi and Sadeghi [4] and Nadarajah and Haghighi [18].

The survival function of EE distribution is given as

$$S(x) = exp[1 - (1 + \lambda x)^{\alpha}], \tag{1}$$

for $\alpha > 0$, $\lambda > 0$ and $x > 0$. The corresponding cumulative distribution function (cdf), probability density function (pdf) and the quantile function are given as.

$$F(x) = 1 - exp[1 - (1 + \lambda x)^{\alpha}], \tag{2}$$

* Corresponding author e-mail: manustats@gmail.com

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}exp[1-(1+\lambda x)^{\alpha}], \tag{3}$$

and

$$Q(p)=\frac{1}{\lambda}\left\{(1-log(1-p))^{1/\alpha}\right\}, 0 < p < 1,$$

respectively. The hazard function (hrf) is given by

$$h(x) = \alpha\lambda(1+\lambda x)^{\alpha}. \tag{4}$$

Now, For $\alpha = 1$, equation (3) reduced to exponential distribution (see, Nadarajah [18]). Equation (3) has showed the attractive feature of always having the zero mode and yet allowing for increasing, decreasing and constant hrfs. Haghighi and Sadeghi [4] and Nadarajah [18] have been obtained the MLEs for complete as well as censored case but none of them has paid attention for Bayes analysis under Progressive type-II censoring with Binomial removals. But now a days Progressive type-II censoring with Binomial removals becomes very popular and practicable in medical and engineering field.

In life testing experiments, situations do arise when units are lost or removed from the experiments while they are still alive; i.e, we get censored data from the experiment. The loss of units may occur due to time constraints giving type-I censored data. In such censoring scheme, experiment is terminated at specified time. Sometimes, the experiment is terminated after a prefixed number of observations due to cost constraints and we get type-II censored data. Besides these two controlled causes, units may drop out of the experiment randomly due to some uncontrolled causes. For example, consider that a doctor perform an experiment with n cancer patients but after the death of first patient, some patient leave the experiment and go for treatment to other doctor/ hospital. Similarly, after the second death a few more leave and so on. Finally the doctor stops taking observation as soon as the predetermined number of deaths (say m) are recorded.

It may be assumed here that each stage the participating patients may independently decide to leave the experiment and the probability (p) of leaving the experiment is same for all the patients. Thus the number of patients who leave the experiment at a specified stage will follow Binomial distribution with probability of success (p). The experiment is similar to a life test experiment which starts with n units. At the first failure X_1, r_1 (random) units are removed randomly from the remaining $(n-1)$ surviving units. At second failure X_2, r_2 units from remaining $n-2-r_1$ units are removed, and so on; till m^{th} failure is observed i.e. at m^{th} failure all the remaining $r_m = n-m-r_1-r_2\cdots r_{m-1}$ units are removed. Note that, here, m is pre-fixed and r_is are random. Such a censoring mechanism is termed as Progressive type-II censoring with Binomial removals. If we assume that probability of removals of a unit at every stage is p for each unit then r_i can be considered to follow a Binomial distribution i.e., $r_i \approx B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1,2,3,\cdots m-1$ and with $r_0 = 0$. For further details, readers are referred to Balakrishnan [2] and Singh et al.[8]. In last few years, the estimation of parameters of different life time distribution based on Progressive censored samples have been studied by several authors such as Childs and Balakrishnan [6], Balakrishnan and Kannan [3], Mousa and Jheen [17], Ng et al. [20]. The Progressive type-II censoring with random removals has been considered by Yang et al. [22] for Weibull distribution, Wu and Chang [26] for exponential distribution. Under the Progressive type-II censoring with random removals, Wu and Chang [27] and Yuen and Tse [28] developed the estimation problem for the Pareto distribution and Weibull distribution respectively, when the number of units removed at each failure time has a discrete uniform distribution, the expected time of this censoring plan is discussed and compared numerically.

In this paper, we have proposed Bayes estimators for the two parameter EE based on Progressive type-II censoring with Binomial removals. Bayes estimators are obtained under SELF and GELF. Rest of the paper is organized as follows: Section 2, provides the likelihood function. In section 3, MLE and Bayes estimators have been obtained. MCMC method is used to compute Bayes estimates of α and λ. The comparison of MLEs and corresponding Bayes estimators are given in section 4. Comparisons are based on simulation studies of risk (average loss over sample space) of the estimators. Section 5, illustrate an example by using the real data set. Finally, conclusions are presented in the last section.

2 Likelihood Function

Let $(X_1,R_1),(X_2,R_2),(X_3,R_3),\cdots,(X_m,R_m)$, denote a Progressive type-II censored sample with Binomial removals, where $X_1 < X_2 < X_3,\cdots,X_m$. With pre-determined number of removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional likelihood function can be written as, Cohen [7]

$$L(\alpha;\lambda;x|R=r) = c^* \prod_{i=1}^{m} f(x_i)[S(x_i)]^{r_i}, \tag{5}$$

where $c^* = n(n - r_1 - 1)(n - r_1 - r_2 - 2)(n - r_1 - r_2 - r_3 - 3) \cdots (n - r_1 - r_2 - r_3, \cdots, r_m - m + 1)$, and $0 \le r_i \le (n - m - r_1 - r_2 - r_3 \cdots r_{i-1})$, for $i = 1, 2, 3 \ldots, m - 1$. Substituting (1) and (3) into (5), we get

$$L(\alpha, \lambda; x | R = r) = c^* \prod_{i=1}^{m} \alpha \lambda (1 + \lambda x_i)^{\alpha-1} [exp[1 - (1 + \lambda x_i)^\alpha]]^{r_i+1} \tag{6}$$

Suppose that an individual unit being removed from the test at the ith failure, $i = 1, 2, \cdots (m - 1)$ is independent of the others but with the same probability p. That is no. R_i of the unit removed at i^{th} failure $i = 1, 2, \cdots (m - 1)$ follows a Binomial distribution with parameters $\left(n - m - \sum_{l=1}^{i-1} r_l, p\right)$ therefore,

$$P(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1} (1 - p)^{n-m-r_1}, \tag{7}$$

and for $i = 2, 3, \cdots, m - 1$,

$$\begin{aligned} P(R; p) &= P(R_i = r_i | R_{i-1} = r_{i-1}, \cdots R_1 = r_1) \\ &= \binom{n - m - \sum_{l=0}^{i-1} r_l}{r_i} p^{r_i} (1 - p)^{n-m-\sum_{l=0}^{i-1} r_l}. \end{aligned} \tag{8}$$

Now, we further assume that R_i is independent of X_i for all i. Then using above equations, we can write the full likelihood function as in the following form

$$L(\alpha, \lambda, p; x, r) = A L_1(\alpha, \lambda) L_2(p), \tag{9}$$

where

$$L_1(\alpha; \lambda) = \prod_{i=1}^{m} \alpha \lambda (1 + \lambda x_i)^{\alpha-1} [exp[1 - (1 + \lambda x_i)^\alpha]]^{r_i+1}, \tag{10}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1 - p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{11}$$

and $A = \frac{c^* (n-m)!}{\left(n - m - \sum_{l=1}^{i-1} r_i\right)! \prod_{i=1}^{m-1} r_i!}$, does not depend on the parameters α, λ and p.

3 Classical and Bayesian Estimation of Parameters

3.1 Maximum Likelihood Estimation

The MLE of α and λ are the simultaneous solution of following normal equations

$$\frac{m}{\alpha} + \sum_{i=1}^{m} \ln(1 + \lambda x_i) - \sum_{i=1}^{m} (1 + r_i)(1 + \lambda x_i)^\alpha \ln(1 + \lambda x_i) = 0 \tag{12}$$

and

$$\frac{m}{\lambda} + (\alpha - 1) \sum_{i=1}^{m} \frac{x_i}{1 + \lambda x_i} - \alpha \sum_{i=1}^{m} x_i (1 + r_i)(1 + \lambda x_i)^{\alpha-1} = 0. \tag{13}$$

It may be noted that (12) and (13) can not be solved simultaneously to provide a nice closed form for the estimators. Therefore, we use fixed point iteration method for solving these equations. For details about the proposed method readers may refer Jain et al. [16] and Rao [21].

3.2 Bayes procedure

Since the parameters α and λ both are unknown, a natural choice for the prior distributions of α and λ are independent gamma distributions as the following forms (14) and (15).

$$g_1(\alpha) = \frac{b_1{}^{a_1} e^{-b_1\alpha} \alpha^{a_1-1}}{\Gamma a_1} \quad ; \quad 0 < \alpha < \infty, \quad b_1 > 0, \quad a_1 > 0 \tag{14}$$

$$g_1(\lambda) = \frac{b_2{}^{a_2} e^{-b_2\lambda} \lambda^{a_2-1}}{\Gamma a_2} \quad ; \quad 0 < \lambda < \infty, \quad b_2 > 0, \quad a_2 > 0 \tag{15}$$

where a_1, b_1, and a_2, b_2, are chosen to reflect prior knowledge about α and λ. It may be noted that, the gamma prior $g_1(\alpha)$ and $g_2(\lambda)$ are chosen instead of the exponential prior of α and λ were used by Nassar and Eissa [19], Jung et al. [11] and Singh et al. [9] because the gamma prior is wealthy enough to cover the prior belief of the experimenter. Thus the joint prior pdf of α and λ is

$$g(\alpha, \lambda) = g_1(\alpha) g_2(\lambda) \quad ; \quad \alpha > 0, \quad \lambda > 0 \tag{16}$$

Combining the priors given by (14) and (15) with likelihood given by (9), we can easily obtain joint posterior pdf of (α, λ) as $\pi(\alpha, \lambda | x, r) = \frac{J_1}{J_0}$ where

$$J_1 = \alpha^{m+a_1-1} \lambda^{m+a_2-1} e^{-b_1\alpha} e^{-b_2\lambda} \left\{ \prod_{i=1}^{m} (1+\lambda x_i)^{\alpha-1} [exp[1-(1+\lambda x_i)^{\alpha}]]^{r_i+1} \right\}, \tag{17}$$

and $J_0 = \int_0^\infty \int_0^\infty J_1 d\alpha d\lambda$. Hence, the respective marginal posterior pdfs of α and λ are given by

$$\pi_1(\alpha | x, r) = \int_0^\infty \frac{J_1}{J_0} d\lambda, \tag{18}$$

and

$$\pi_2(\lambda | x, r) = \int_0^\infty \frac{J_1}{J_0} d\alpha. \tag{19}$$

Usually the Bayes estimators are obtained under SELF

$$l_1(\phi, \hat{\phi}) = \in_1 \left(\phi - \hat{\phi}\right)^2 ; \quad \in_1 > 0 \tag{20}$$

Where $\hat{\phi}$ is the estimate of the parameter ϕ and the Bayes estimator $\hat{\phi}_S$ of ϕ comes out to be $E_\phi[\phi]$, where E_ϕ denotes the posterior expectation. However, this loss function is symmetric loss function and can only be justified, if over estimation and under estimation of equal magnitude are of equal seriousness. A number of asymmetric loss functions are also available in statistical literature. Let us consider the GELF, proposed by Calabria and Pulcini [5], defined as follows :

$$l_2(\phi, \hat{\phi}) = \in_2 \left(\left(\frac{\hat{\phi}}{\phi}\right)^{\delta} - \delta \ln\left(\frac{\hat{\phi}}{\phi}\right) - 1 \right); \quad \in_2 > 0 \tag{21}$$

The constant δ, involved in (21), is its shape parameter. It reflects departure from symmetry. When $\delta > 0$, it considers over estimation (i.e., positive error) to be more serious than under estimation (i.e., negative error) and converse for $\delta < 0$. The Bayes estimator $\hat{\phi}_E$ of ϕ under GELF is given by,

$$\hat{\phi}_E = \left[E_\phi \left(\phi^{-\delta} \right) \right]^{\left(-\frac{1}{\delta}\right)} \tag{22}$$

provided the posterior expectation exits. It may be noted here that for $\delta = -1$, the Bayes estimator under loss (21) coincides with the Bayes estimator under SELF l_1. Expressions for the Bayes estimators $\hat{\alpha}_E$ and $\hat{\lambda}_E$ for α and λ respectively under GELF can be given as

$$\hat{\alpha}_E = \left[\int_0^\infty \alpha^{-\delta} \pi_1(\alpha | x, r) d\alpha \right]^{\left(-\frac{1}{\delta}\right)}, \tag{23}$$

and

$$\hat{\lambda}_E = \left[\int_0^\infty \lambda^{-\delta} \pi_1(\lambda | x, r) d\lambda \right]^{\left(-\frac{1}{\delta}\right)}, \tag{24}$$

It is to mention here that from equation (23) and (24), the Bayes estimators $\hat{\alpha}_E$ and $\hat{\lambda}_E$ are not reducible in nice closed form. Therefore, we use the numerical techniques for obtaining the estimates. We propose to use the MCMC method for obtaining the Bayes estimates of the parameters. In MCMC technique, Gibbs sampler and Metropolis-Hastings algorithm to generate samples from posterior distributions and compute the Bayes estimates. The Gibbs sampler is best applied on problems where the marginal distributions of the parameters of interest are difficult to calculate, but the conditional distributions of each parameter given all the other parameters and data have nice forms. If conditional distributions of the parameters have standard forms, then they can be simulated easily. But generating samples from full conditionals corresponding to joint posterior is not easily manageable, therefore we consider mixing of Metropolis-Hastings for those full conditional in the hybrid sampling i.e., Metropolis step is used to extract samples from some of the full conditional to complete a cycle in Gibbs chain. For more details about this method, see Chib and Greenberg [24], Gelfand and Smith [23] and Gamerman and Lopes[25]. Thus utilizing the concept of Gibbs sampling procedure as mentioned above, generate sample from the posterior density function under the assumption that parameter α and λ has independent gamma density function with hyper parameters a_1, b_1, and a_2, b_2, respectively. To corporate this technique we consider full conditional posterior densities of α and λ,

$$\pi_1^*(\alpha|\lambda,x,r) \propto \alpha^{m+a_1-1} e^{-b_1\alpha} \left\{ \prod_{i=1}^m (1+\lambda x_i)^\alpha \left[exp[1-(1+\lambda x_i)^\alpha] \right]^{r_i+1} \right\} \tag{25}$$

and

$$\pi_2^*(\lambda|\alpha,x,r) \propto \lambda^{m+a_2-1} e^{-b_2\lambda} \left\{ \prod_{i=1}^m (1+\lambda x_i)^{-1} \left[exp[1-(1+\lambda x_i)^\alpha] \right]^{r_i+1} \right\} \tag{26}$$

respectively. The Gibbs algorithm consist the following steps

  I. Set the initial guess of α and λ say α_0 and λ_0

  II. Set $i = 1$

 III. Generate α_i from $\pi_1^*(\alpha|\lambda_{i-1},x,r)$ and λ_i from $\pi_2^*(\lambda|\alpha_i,x,r)$

 IV. Repeat steps II-III, N times

  V. Obtain the Bayes estimates of α and λ under GELF as

    $\hat{\alpha}_E = \left[E(\alpha^{-\delta}|data) \right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0} \sum_{i=N_0+1}^N \alpha_i^{-\delta} \right]^{-\frac{1}{\delta}}$ and

    $\hat{\lambda}_E = \left[E(\lambda^{-\delta}|data) \right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0} \sum_{i=N_0+1}^N \lambda_i^{-\delta} \right]^{-\frac{1}{\delta}}$.

    Where, $(N_0 \approx 5000)$ is the burn-in-period of Markov Chain. Substituting δ equal to -1 in step V, we get Bayes estimates of α and λ under SELF.

 VI. To compute the HPD interval of α and λ, order the MCMC sample of α and λ (say $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N$ as $\alpha_{[1]}, \alpha_{[2]}, \alpha_{[3]}, \cdots, \alpha_{[N]}$) and $(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_N$ as $\lambda_{[1]}, \lambda_{[2]}, \lambda_{[3]}, \cdots, \lambda_{[N]})$. Then construct all the $100(1-\psi)\%$ credible intervals of α and λ say $((\alpha_{[1]}, \alpha_{[N(1-\psi)+1]}), \cdots, (\alpha_{[N\psi]}, \alpha_{[N]}))$ and $((\lambda_{[1]}, \lambda_{[N(1-\psi)+1]}), \cdots,$ $(\lambda_{[N\psi]}, \lambda_{[N]}))$ respectively. Here $[x]$ denotes the largest integer less than or equal to x. Then the HPD interval of α and λ are that interval which has the shortest length.

 VII. Using the asymptotic normality property of MLEs, we can construct approximate $100(1-\psi)\%$ confidence intervals for α and λ as

    $\hat{\alpha} \pm z_{\psi/2}(\sqrt{v\hat{a}r(\hat{\alpha})})$ and $\hat{\lambda} \pm z_{\psi/2}(\sqrt{v\hat{a}r(\hat{\lambda})})$ Where $z_{\psi/2}$ is the $100(1-\psi/2)\%$ upper percentile of standard normal variate.

4 Simulation Study

The estimators $\hat{\alpha}_M$ and $\hat{\lambda}_M$ denote the MLEs of the parameters α and λ respectively, while $\hat{\alpha}_S$ and $\hat{\lambda}_S$ are corresponding Bayes estimators under SELF and $\hat{\alpha}_E$ and $\hat{\lambda}_E$ are the corresponding Bayes estimators under GELF. We compare the estimators obtained under GELF with corresponding Bayes estimators under SELF and MLEs. The comparisons are based on the simulated risks (average loss over sample space) under GELF and SELF both. Here, $((\alpha_{L^c} \ \alpha_{U^c}), (\lambda_{L^c} \ \lambda_{U^c}))$ and $((\alpha_{L^h} \ \alpha_{U^h}), (\lambda_{L^h} \ \lambda_{U^h}))$ represent $100(1-\psi)\%$ CI and HPD intervals of α and λ respectively. It may be mentioned here that the exact expressions for the risks can not be obtained as estimators are not found in nice closed form. Therefore, the risks of the estimators are estimated on the basis of Monte-Carlo simulation study of 5000 samples. It may be noted that the risks of the estimators will depend on values of n, m, p, α, λ and δ. Also, the choice of hyper parameter α and λ can be taken in such a way that if we consider any two independent informations as prior mean and variance of α and λ are $(\mu_1 = \frac{a_1}{b_1}, \sigma_1 = \frac{a_1}{b_1^2})$ and $(\mu_2 = \frac{a_2}{b_2}, \sigma_2 = \frac{a_2}{b_2^2})$ respectively, whereas μ_1 and μ_2 are considered as true values of the parameters α and λ for different confidence in terms of smaller and larger variances. In order to consider variation in the values of these, we have obtained the simulated risks for effective samples $m = 15, 18, 21$ and 27, $\alpha = 2 = \mu_1$(say, prior mean of α), $\sigma_1 = 1, 10$ (say, prior variance of α) and $\delta = \pm 4$. Similarly, these variation is apply on the scale parameter $\lambda = 3 = \mu_2$(say, prior mean of λ), $\sigma_2 = 1, 10$ (say, prior variance of λ) and $\delta = \pm 4$. Figure 1 & 2 shows the risks of an estimators of α and λ for different values of δ under GELF and Figure $3-6$ shows the risk of estimators of α and λ for variation of the effective sample size m, where the other rest of the parameters are fixed, which is mention under the Figures. Table 1 & 2 represent the CI, HPD intervals and percentage of coverage probability in all considered situation. It is to be mention here that considered the value of hyper parameters such as prior mean is taken as guess value
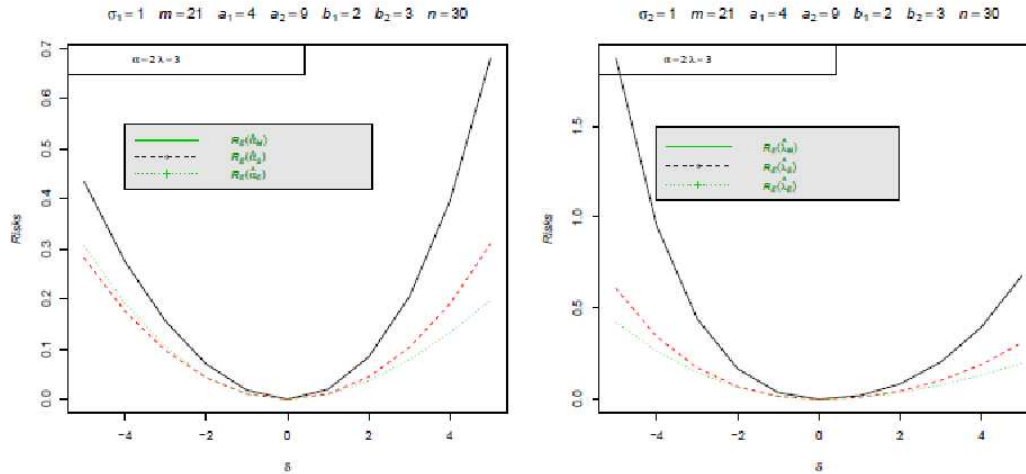
Fig. 1: Risks of Estimators of α and λ under GELF for different values of δ.

Table 1: Under smaller prior variance $\sigma_1 = 1$ and $\sigma_2 = 1$ the 95% CI, HPD intervals and % of coverage probability for different samples m for fixed $n = 30, \alpha = 2, \lambda = 3, a_1 = 4, a_2 = 9, b_1 = 2$ and $b_2 = 3$.

m	α					λ				
	α_{L^c}	α_{U^c}	α_{L^h}	α_{U^h}	% cov.prob	λ_{L^c}	λ_{U^c}	λ_{L^h}	λ_{U^h}	%cov.prob
15	0.5321	8.1728	1.4077	2.7687	93.9	0.7668	11.9473	1.8208	3.9305	93.6
18	0.6427	7.6259	1.4649	2.7355	94.2	1.5638	11.2735	1.8881	3.9000	93.6
21	0.6409	7.0113	1.4984	2.6761	95	1.5735	10.7623	1.9602	3.8880	94.1
27	1.0324	6.4232	1.5726	2.6427	96.7	2.0115	9.6552	2.0195	3.7595	97.4

of the parameters α and λ, when prior variance is small and large respectively. From Table 1 & 2, it is observed that HPD intervals are shorter length than CI and length of the intervals decreases as increment of the effective sample size m and also observed that, there is increment in coverage probability of CI and HPD.

5 Real data Analysis

For real data illustration, we have taken the following data from Linhart and Zucchini [12] which shows failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. We have obtained Kolmogrov-Smirnov (K-S) statistics, Akaike's information criterion (AIC) and Bayesian information criterion (BIC) for EE, Weibull, gamma and exponentiated exponential distributions for given data set and the values are summarized in Table 3. Considered criterion, we observed that EE distribution provide better fit than the other three distributions. Hence, EE model can be considered as an alternative to all three models. Therefore, we use this data to illustrate the our propose procedures. For this a Progressive type-II censoring with Binomial removals are generated from the given data set under various schemes, which are summarized in Table 4. We have obtained the MLEs, Bayes estimates (using non informative prior), 95% CI and HPD intervals for the parameters α and λ respectively under SELF and GELF for $\delta = \pm 4$ and value of the hyper parameters α and λ are taken as $a_1 = 0.00001, b_1 = 0.0001$ and $a_2 = 0.00001, b_2 = 0.0001$ respectively, which are summerized in Table 6 and Table 7. Table 5, shows the MLEs and Bayes estimators of α and λ under SELF, GELF and 95% CI/HPD intervals based on complete data set. On every censored sample schemes the length of HPD intervals are always less than CI.
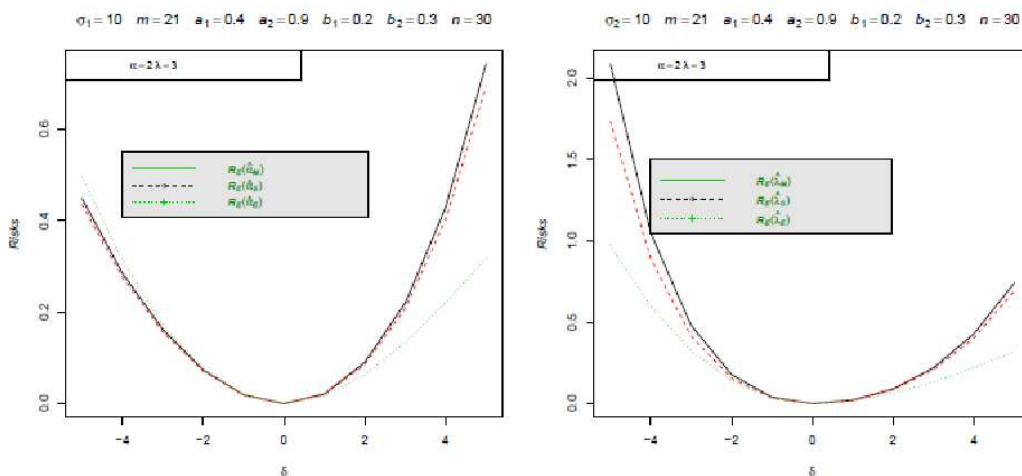
Fig. 2: Risks of Estimators of α and λ under GELF for different values of δ.
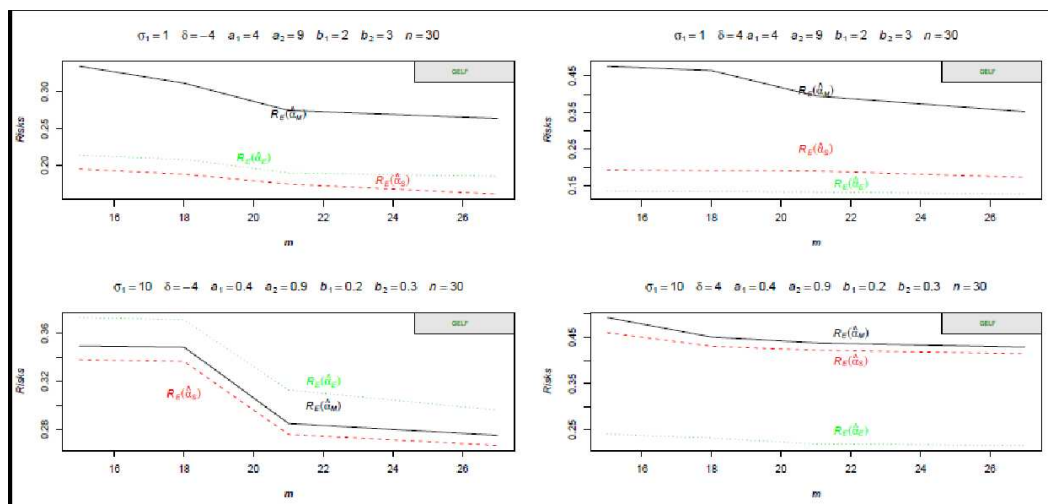

Fig. 3: Risks of Estimators of α under GELF for different values of m.

Table 2: Under larger prior variance $\sigma_1 = 10$ and $\sigma_2 = 10$ the 95% CI, HPD intervals and % of coverage probability for different samples m for fixed $n = 30, \alpha = 2, \lambda = 3, a_1 = 0.4, a_2 = 0.9, b_1 = 0.2$ and $b_2 = 0.3$.

m	α					λ				
	α_{L^c}	α_{U^c}	α_{L^h}	α_{U^h}	% cov.prob	λ_{L^c}	λ_{U^c}	λ_{L^h}	λ_{U^h}	%cov.prob
15	0	8.0839	1.3520	2.9879	92.3	0	12.1325	1.6044	4.1632	93.2
18	0	7.6110	1.4325	2.9406	94.3	0	11.3435	1.6920	4.0549	93.9
21	0	7.1227	1.4868	2.8800	94.9	0	10.5829	1.7599	3.9473	95
27	6.09E-05	6.5543	1.5714	2.8129	96.2	0	9.6984	1.8692	3.8262	97.1
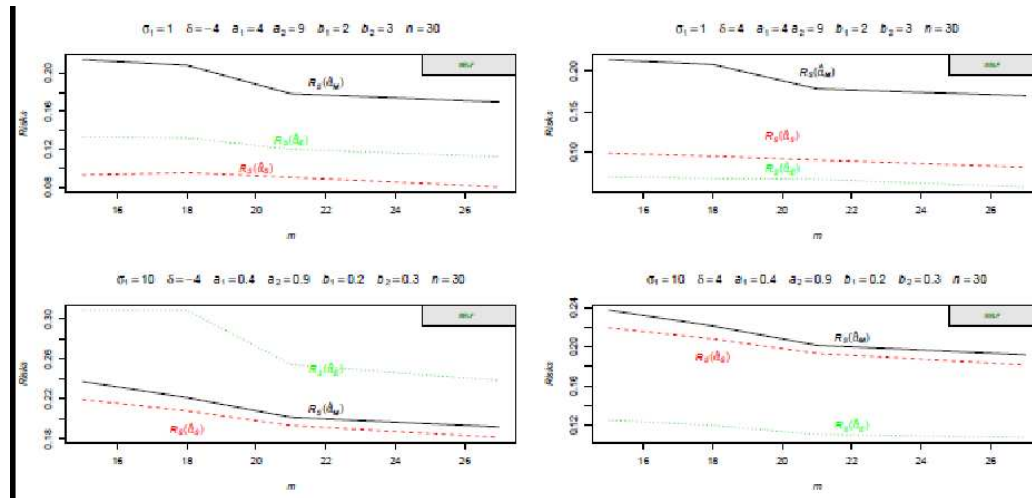
Fig. 4: Risks of Estimators of α under SELF for different values of m.
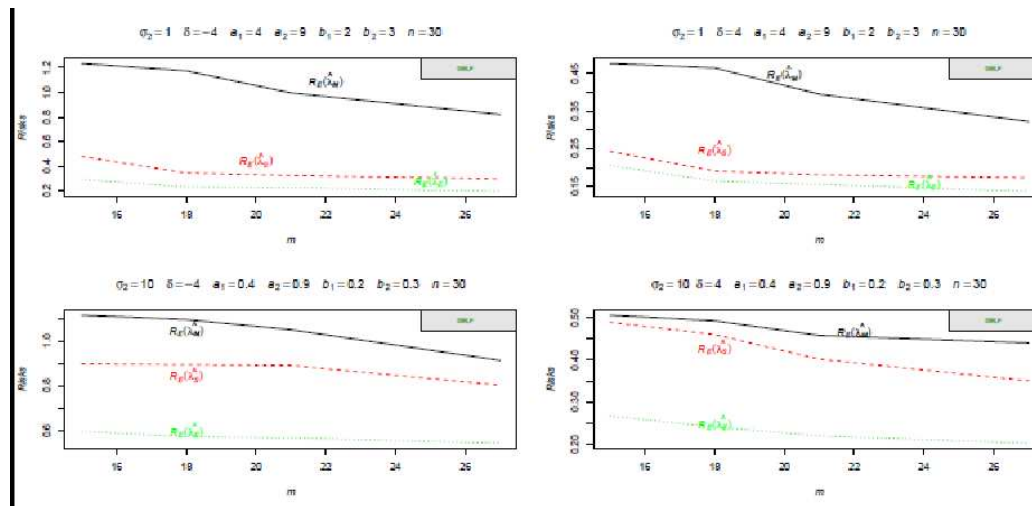

Fig. 5: Risks of Estimators of λ under GELF for different values of m.

Table 3: Goodness of fit for various data

	exponentiated exponential	Weibull	gamma	EE
Log-likelihood	-152.2013	-151.937	-152.943	-151.5815
K. S. statistics	0.29585	0.15390	0.17186	0.13187
AIC	308.4026	307.8740	309.8859	307.1630
BIC	311.2050	310.6764	312.6883	309.9654

Table 4: Failure time vector $Y = (y_2, ..., y_{30})$ under different PT-II CBR censoring schemes $S_j(n:m)$

Scheme	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$S_5(30:27)$	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	y_i	1	3	5	7	11	11	11	12	14	14	14	16	16	20	21	23	42	47
		19	20	21	22	23	24	25	26	27									
		0	1	0	0	1	0	1	0	0									
		52	62	71	87	90	120	120	246	261									
$S_4(30:24)$	R_i	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
	y_i	1	3	7	11	11	11	12	14	14	16	16	20	21	23	42	47	52	71
		2	0	0	1	0	0												
		71	95	120	120	246	261												
$S_3(30:21)$	R_i	0	1	0	0	0	0	1	0	0	1	0	1	0	0	0	0	2	0
	y_i	1	3	7	11	11	11	12	14	14	16	20	21	42	47	52	62	71	90
		2	1	0															
		95	225	261															
$S_2(30:18)$	R_i	0	1	0	1	0	0	1	0	0	0	0	0	2	2	2	2	1	0
	y_i	1	3	7	11	11	12	14	14	16	16	20	21	23	52	71	95	225	261
$S_1(30:15)$	R_i	0	0	0	0	0	0	0	0	0	0	2	0	6	7	0			
	y_i	1	3	5	7	11	11	11	12	14	14	14	20	21	71	261			

Table 5: Bayes and ML estimates based on real data set for n = 30; p = 0.5.

Parameter	MLE	Bayes Estimates(MCMC)			Inteval Estimates			
		SELF	GELF		95 CI		95 HPD	
			$\delta=4$	$\delta=-4$	L^c	U^c	L^h	U^h
α	0.59854	0.59732	0.59029	0.60147	0.23078	0.96631	0.51791	0.67731
λ	0.04339	0.04286	0.04060	0.04413	0.00000	0.09911	0.03133	0.05492

Table 6: Bayes and ML estimates, CI and HPD credible intervals for α with fixed n = 30 and p = 0.5 under PT-II CBR.

Scheme	MLE	Bayes Estimates(MCMC)			Interval Estimates			
		SELF	GELF		95% CI		95% HPD	
			$\delta=4$	$\delta=-4$	α_{L^c}	α_{U^c}	α_{L^h}	α_{L^h}
$S_1(30:15)$	0.309636	0.308854	0.301217	0.313206	0.047172	0.572101	0.247360	0.365271
$S_2(30:18)$	0.369187	0.368289	0.361281	0.372343	0.092626	0.445748	0.306766	0.411130
$S_3(30:21)$	0.451995	0.450888	0.442590	0.455687	0.114703	0.460988	0.384811	0.459988
$S_4(30:24)$	0.562338	0.561553	0.551595	0.567371	0.240013	0.571553	0.520692	0.567553
$S_5(30:27)$	0.528935	0.522843	0.515776	0.526962	0.289157	0.578935	0.550344	0.579776

Table 7: Bayes and ML estimates, CI and HPD credible intervals for λ with fixed n = 30 and p = 0.5 under PT-II CBR.

| Scheme	MLE	Bayes Estimates(MCMC)			Interval Estimates			
		SELF	GELF		95% CI		95% HPD	
			$\delta=4$	$\delta=-4$	λ_{L^c}	λ_{U^c}	λ_{L^h}	α_{L^h}
$S_1(30:15)$	0.095069	0.092577	0.082717	0.097497	0.000000	0.253197	0.057616	0.127248
$S_2(30:18)$	0.064836	0.063485	0.057888	0.066383	0.000000	0.164890	0.041442	0.085406
$S_3(30:21)$	0.047855	0.046958	0.043210	0.048955	0.000000	0.120469	0.030740	0.062327
$S_4(30:24)$	0.040020	0.039330	0.036510	0.040873	0.000000	0.099043	0.026832	0.051844
$S_5(30:27)$	0.051559	0.046062	0.042827	0.047844	0.000000	0.088054	0.036083	0.060970
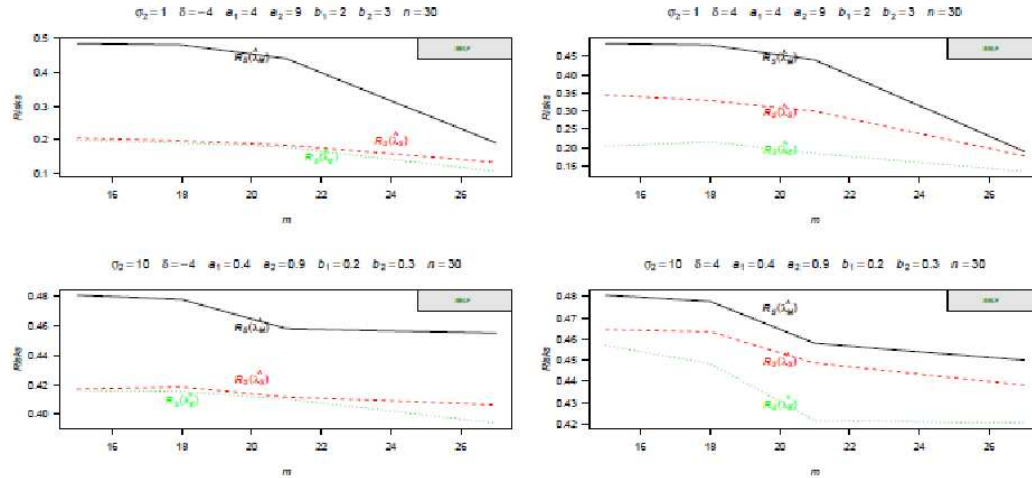
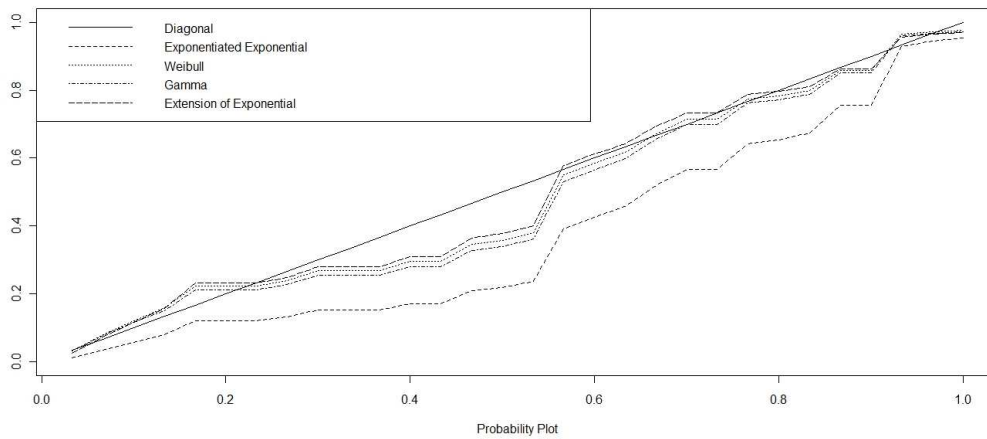**Fig. 6:** Risks of Estimators of λ under SELF for different values of m.

Fig. 7: Probability Plot for real data set example.

6 Conclusion

After an extensive study of the results of simulation, we may conclude that in most of the cases, under both losses, our proposed estimator $\hat{\alpha}_E$ and $\hat{\lambda}_E$ perform better than all the considered competitive estimators of α and λ respectively for $\delta > 0$ (when over estimation is more serious than under estimation). On the other hand for $\delta < 0$ (when under estimation is more serious than over estimation) $\hat{\alpha}_S$ and $\hat{\lambda}_E$ have minimum risk than all the competitive estimators of α and λ. Therefore, the proposed estimator $\hat{\lambda}_E$ is recommended for both losses, if under estimation is more serious than over estimation vice-versa.

References

[1] Murthy, D. N. P, Xie, M. and Jiang, R., 2004, Weibull Models *Wiley*, New Jersey.

[2] Balakrishnan, N., 2007, Progressive methodology : An appraisal (with discussion). *Test*, 16 (2):211 – 259.

[3] Balakrishnan, N. and Kannan, N., 2001, Point and Interval Estimation for Parameters of the Logistic Distribution Based on Progressively Type-II Censored Samples, in Handbook of Statisticsm N. Balakrishnan and C. R. Rao, volume 20. Eds. Amsterdam, North- Holand.
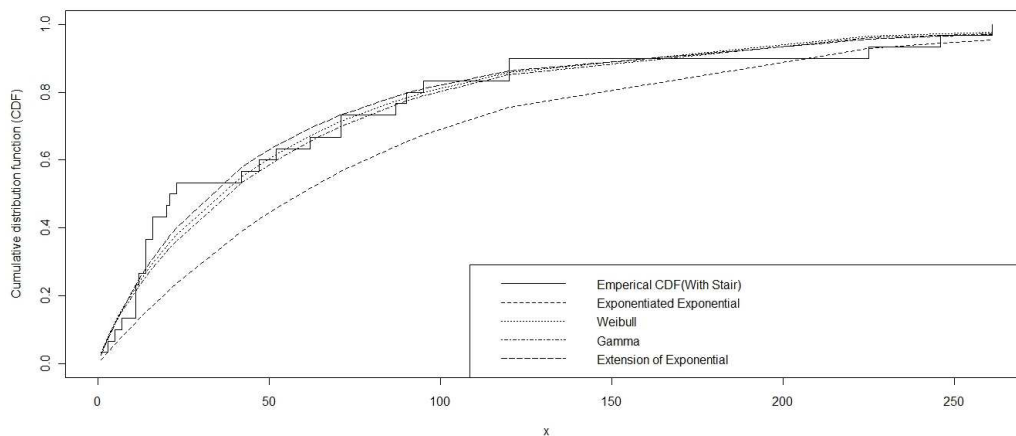
Fig. 8: CDF plot for real data set example.

[4] Haghighi, F. and Sadeghi, S., 2009, *An exponential extension* 41mes Journes de Statistique, SFdS, Bordeaux.

[5] Calabria, R. and Pulcini, G., 1996, Point estimation under - asymmetric loss functions for life - truncated exponential samples. *Commun. statist. Theory meth.*, 25(3):585–600.

[6] Childs, A. and Balakrishnan, N., 2000, Conditional inference procedures for the laplace distribution when the observed samples are progressively censored. *Metrika*, 52:253–265.

[7] Cohen, A. C., 1963, Progressively censored samples in life testing. *Technometrics*, pages 327–339.

[8] Singh, S. K., Singh, U. and Kumar M., 2012, Estimation of parameters of exponentiated Pareto distribution for progressive type-II censored data with Binomial random removals scheme. *Electronic Journal of Applied Statistical Analysis*, Vol. 06, Issue 02, 130-148.

[9] Singh, S. K., Singh, U. and Kumar M., 2013, Estimation of Parameters of Generalized Inverted Exponential Distribution for Progressive Type-II Censored Sample with Binomial Removals. *Journal of Probability and statistics*, Vol. 2013, Article ID 183652, 1-12.

[10] Gupta, R. D. and Kundu, D., 2001, Exponentiated exponential family; an alternative to gamma and Weibull. *Biometrical Journal*, vol. 43, 117 - 130.

[11] Jung, J., Kim, C. and Chung, Y., 2011, Bayesian estimation for the exponentiated Weibull model under type II progressive censoring. *Statistical Papers (accepted)*.

[12] Linhart, H. and Zucchini, W., 1986, *Model Selection*. Wiley, New York.

[13] Barlow, E. and Proschan, F., 1981, *Statistical Theory of Reliability and Life Testing: Probability Models. To Begin With*. Springer, USA.

[14] Lawless, J. F., 1982, *Statistical Method for Life Time Data*. Wiley, New York.

[15] Nelson, W, 1982, *Applied life data analysis*. Wiley, New York.

[16] Jain, M. K., Iyengar, S. R. K. and Jain, R. K., 1984. Numerical Methods for Scientific and Engineering Computation. New Age International (P) Limited, Publishers, New Delhi, fifth edition.

[17] Mousa, M. and Jaheen, Z., 2002, Statistical inference for the burr model based on progressively censored data. *An International Computers and Mathematics with Applications*, 43:1441–1449.

[18] Nadarajah, S. and Haghighi, F., 2011, An extension of the exponential distribution. *Statistics: A Journal of Theoretical and Applied Statistics*, Volume 45, Issue 6.

[19] Nassar, M. M. and Eissa, F. H., 2004, Bayesian estimation for the exponentiated Weibull model. *Communication in Statistics Theory and Methods*, 33:2343–2236.

[20] Ng, K., Chan, P. S. and Balakrishnan, N., 2002, Estimation of parameters from progressively censored data using an algorithm. *Computational Statistics and Data Analysis*, 39:371–386.

[21] Rao,S. G., 2006, *Numerical analysis*. New Age International (P) Ltd.

[22] Yang, C., Tse, S. K. and Yuen, H. K., 2000, Statistical analysis of Weibull distributed life time data under type II progressive censoring with binomial removals. *Jounal of Applied Statistics*, 27:1033–1043.

[23] Gelfand, A. E. and Smith A. F. M.,1990, Sampling-based approaches to calculating marginal densities. *J. Amer. Statist. Assoc.*, 85(410):398–409.

[24] Chib, S. and Greenberg, E., 1995, Understanding the Metropolis-Hastings algorithm. *J. Amer. Statist.*, 49:327–335.

[25] Gamerman. D and Lopes H. F., 2006, *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference*. 2nd ed., Chapman and Hall/CRC, Boca Raton, FL.

[26] Wu, S. J. and Chang, C. T., 2002, Parameter estimations based on exponential progressive type II censored with binomial removals. *International Journal of Information and Management Sciences*, 13:37–46.

[27] Wu, S. J. and Chang, C. T., 2003, Inference in the pareto distribution based on progressive type II censoring with random removals. *Journal of Applied Statistics*, 30:163–172.

[28] Yuen, H. K. and Tse, S. K., 1996, Parameters estimation for Weibull distribution under progressive censoring with random removals. *Journal Statis. Comput. Simul*, 55:57–71.