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Some Relations between Certain Classes of Analytic Multivalent Functions Involving Generalized Sălăgean Operator

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Abstract: The aim of this paper is to introduce and study two subclasses of multivalent functions involving generalized Sălăgean operator. Our classes $\mathscr{M}_{p,n}^{m,\sigma}(\gamma;\eta)$ and $\mathscr{N}_{p,n}^{m,\sigma}(\alpha,\beta;\eta)$ unify the standard classes of multivalent starlike functions of order η, multivalent convex functions of order η, and Bazilević functions. Some connections between our classes are obtained and several consequences of main results are discussed.

Keywords: Analytic functions, starlike function, close-to-convex functions, multivalent functions.

1 Introduction

Let $\mathscr{U} = \{z \in C : |z| < 1\}$ be the unit disk and let $\mathscr{A}(p,n)$ be the class of all analytic functions in $\mathscr{U}$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \qquad (p,n \in \mathbb{N}) \tag{1}$$

and let denote $\mathscr{A} := \mathscr{A}(1,1)$.

A function $f \in \mathscr{A}(p,n)$ is said to be multivalent starlike functions of order α in $\mathscr{U}$, if it satisfies the following inequality

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathscr{U}, \qquad (0 \le \alpha < p, p \in \mathbb{N})$$

and we denote this class by $S^*_{p,n}(\alpha)$. A function $f \in \mathscr{A}(p,n)$ is said to be multivalent convex functions of order α in $\mathscr{U}$, if it satisfies the following inequality

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathscr{U}, \qquad (0 \le \alpha < p, p \in \mathbb{N})$$

and we denote this class by $C_{p,n}(\alpha)$.

A function $f \in \mathscr{A}(p,n)$ is said to be multivalent close-to-convex functions of order α in $\mathscr{U}$, if it satisfies the following inequality

$$Re\frac{f'(z)}{z^{p-1}} > \alpha,\ z \in \mathscr{U}, \qquad (0 \le \alpha < p, p \in \mathbb{N})$$

and we denote this class by $K_{p,n}(\alpha)$.

In the recent paper of Aouf et al. [1], the authors introduced the subclass $\mathscr{K}_p^{\lambda}(\alpha)$ of $\mathscr{A}(p) := \mathscr{A}(p,1)$, consisting on the functions $f \in \mathscr{A}(p)$ that satisfy the inequality

$$\mathrm{Re}\frac{[\lambda + p(1-\lambda]zf'(z) + \lambda z^2 f'(z)}{(1-\lambda)pf(z) + \lambda zf'(z)} > \alpha,\ z \in \mathscr{U},$$

with $0 \le \lambda \le 1;\ 0 \le \alpha < p,\ p \in \mathbb{N}$.

For a function f in $\mathscr{A}(p,n)$, we define the following generalized Sălăgean differential operator:

$$D^0_\sigma f(z) = f(z) \tag{2}$$

$$D^1_\sigma f(z) = (1-\sigma)f(z) + \frac{\sigma}{p}zf'(z) = D_\sigma f(z), \sigma \ge 0 \tag{3}$$

$$D^m_\sigma f(z) = D_\sigma\left(D^{m-1}f(z)\right), \qquad (m \in \mathbb{N}) \tag{4}$$

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If f is given by (1), then from (3) and (4), we see that

$$D_\sigma^m f(z) = z^p + \sum_{k=p+n}^{\infty} \left[1+\left(\frac{k}{p}-1\right)\sigma\right]^m a_k z^k \qquad (5)$$

For $\sigma = p = 1$, we get the well-known Sălăgean operator [16].
Motivated by the subclass $\mathcal{K}_p^\lambda(\alpha)$ due to Aouf et al. [1] and two subclasses $\mathcal{M}_{p,n}^\lambda(\gamma;\beta)$ and $\mathcal{N}_{p,n}^\lambda(\mu,\eta;\delta)$ due to Goswami et al. [4], we introduce the next two new subclasses of $\mathcal{A}(p,n)$.
Definition 1. Let $\mathcal{M}_{p,n}^{m,\sigma}(\gamma;\eta)$ be the class of functions $f \in \mathcal{A}(p,n)$ that satisfy the condition

$$\mathrm{Re}\left[(1-\gamma)\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \gamma\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right] > \eta,\ z \in \mathcal{U},$$

$$(0 \le \sigma \le 1; 0 \le \eta < p; \gamma \in \mathbb{R}; m,p \in \mathbb{N})$$

and let $\mathcal{N}_{p,n}^{m,\sigma}(\alpha,\beta;\eta)$ be the class of functions $f \in \mathcal{A}(p,n)$ that satisfy the conditions

$$\frac{(D_\sigma^m f(z))(D_\sigma^m f(z))'}{z^p} \ne 0, z \in \mathcal{U}\setminus\{0\}$$

and

$$\mathrm{Re}\left[\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta\right] > \delta,\ z \in \mathcal{U}$$

$$(\alpha,\beta \in \mathbb{R}; 0 \le \delta < 1; m,p \in \mathbb{N})$$

From above definition, the following subclasses of the classes $\mathcal{A}(p,n)$ and $\mathcal{A}(n) = \mathcal{A}(1,n)$ emerge from the families of the functions $\mathcal{M}_{p,n}^{m,\sigma}(\gamma;\eta)$ and $\mathcal{N}_{p,n}^{m,\sigma}(\alpha,\beta;\eta)$:

$$\mathcal{M}_{p,n}^{0,\sigma}(0;\eta) = \mathcal{N}_{p,n}^{0,\sigma}(-1,1;\eta) = \mathcal{S}_{p,n}^*(\eta)\,(0 \le \eta < p);$$

$$\mathcal{M}_{1,n}^{0,\sigma}(0;\eta) = \mathcal{N}_{1,n}^{0,\sigma}(-1,1;\eta) = \mathcal{S}_{1,n}^*(\eta)$$
$$= \mathcal{S}_n^*(\eta)\,(0 \le \eta < 1);$$

$$\mathcal{M}_{p,n}^{1,1}(0;\eta) = \mathcal{C}_{p,n}(\eta)\,(0 \le \eta < p);$$

$$\mathcal{M}_{1,n}^{1,1}(0;\eta) = \mathcal{C}_{1,n}(\eta) =: \mathcal{C}_n(\eta)\,(0 \le \eta < 1);$$

$$\mathcal{M}_{p,n}^{1,\sigma}(0;\eta) = \mathcal{K}_p^\sigma(\eta)\,(0 \le \eta < p);$$

$$\mathcal{N}_{1,n}^{1,1}(1,\beta;\eta) =: \mathcal{B}_n(\beta;\eta)\,(\beta \ge -1, 0 \le \eta < 1)$$

Note that $\mathcal{S}_{p,n}^*(\eta)$, $\mathcal{C}_{p,n}(\eta)$, $\mathcal{S}_n^*(\eta)$, $\mathcal{C}_n(\eta)$ and $\mathcal{B}_n(\beta;\eta)$ are said to be the classes of multivalent starlike functions of order η, multivalent convex functions of order η, univalent starlike functions of order η, univalent convex functions of order η, and a subclass of Bazilević functions, respectively. Further, for $m = 1$, we get the subclasses $\mathcal{M}_{p,n}^\lambda(\gamma;\eta)$ and $\mathcal{N}_{p,n}^\lambda(\alpha,\beta;\eta)$ which is similar to the classes studied recently by Goswami et al. [4]

Also let denote by $\mathcal{H}[a,n]$ the class

$$\mathcal{H}[a,n] = \{p \in \mathcal{H}(\mathcal{U}) : p(z) = a + a_n z^n + ..., z \in \mathcal{U}\}.$$

For studies related to multivalent functions, (see, e.g. [5]-[8],[12],[14]). Singh and Singh [17] obtained several interesting conditions for functions $f \in \mathcal{A}$ satisfying inequalities involving $f'(z)$ and $zf''(z)$ to be univalent or starlike in $\mathcal{U}$. Owa et al. [15] generalized the results of Singh and Singh [17] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of function $f \in \mathcal{A}$. Further, Lee et al. [10] extended the results obtained by Owa et al. [15] for $f \in \mathcal{A}(p,n)$. Also, Goswami et al. [4] have obtained similar type of results.

In this paper we will extend the results of Irmak et al. [9] and Goswami et al. [4] for multivalent functions, by defining the differential operator $\mathcal{J}_{p,n}^{m,\sigma}(\alpha,\beta) : A_{p,n} \to \mathcal{H}[(\alpha+\beta), p+n]$,

$$\mathcal{J}_{p,n}^{m,\sigma}(\alpha,\beta)f(z) = \alpha\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \beta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)$$

and further find its relationship with $\mathcal{N}_{p,n}^{m,\sigma}(\alpha,\beta;\eta)$.
In our proposed investigation of the class $\mathcal{A}(p,n)$, we need the following lemmas:
Lemma 1.1..(See [13]). Let the (nonconstant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then

$$z_0 w'(z_0) = m w(z_0)$$

where m is a real number and $m \ge n$ where $n \ge 1$.
Lemma 1.2.. (See [11]) Let $h(z)$ be analytic in $\mathcal{U}$ with $h(0) \ne 0$ $(z \in \mathcal{U})$. Further suppose that $\mu, \nu \in \mathbb{R}^+ = (0,\infty)$ and

$$|\arg(h(z) + \nu z h'(z))| < \frac{\pi}{2}\left(\mu + \frac{2}{\pi}\arctan(\nu\mu)\right)$$

then

$$|\arg h(z)| < \frac{\pi}{2}\mu, \qquad z \in \mathcal{U}$$

2 Main Results

Theorem 2.1.. *Let the function $f \in \mathcal{A}(p,n)$, satisfies the inequality*

$$\mathrm{Re}\left[\mathcal{J}_{p,n}^{m,\sigma}(\alpha,\beta)f(z)\right] > \frac{[2(\alpha+\beta)p-n]+\lambda[2(\alpha+\beta)p+n]}{2(1+\lambda)} \qquad (6)$$

then

$$\mathrm{Re}\left[\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta\right] > \frac{1+\lambda}{2} \qquad (7)$$

where $(\alpha,\beta \in \mathbb{R}; 0 \le \lambda < 1; p,n \in \mathbb{N})$.

Proof. Let the function w be defined by

$$\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta = \frac{1+\lambda w(z)}{1+w(z)} \tag{8}$$

Then, clearly, w is analytic in $\mathscr{U}$ with $w(0)=0$. We also find from (8) that

$$\alpha\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \beta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)$$

$$= p(\alpha+\beta) + \frac{\lambda z w'(z)}{1+\lambda w(z)} - \frac{z w'(z)}{1+w(z)},\ z\in\mathscr{U}. \tag{9}$$

Suppose there exists a point $z_0 \in \mathscr{U}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$, when $|z| < |z_0|$.

Then, by applying Lemma 1.1, there exists $m \geq n$ such that

$$z_0 w'(z_0) = m w(z_0), \qquad \left(m \geq n \geq 1; w(z_0) = e^{i\theta}; \theta \in \mathbb{R}\right). \tag{10}$$

Using (9) and (10), it follows that

$$\mathrm{Re}\left[\alpha\frac{z(D_\sigma^m f(z_0))'}{D_\sigma^m f(z_0)} + \beta\left(1+\frac{z(D_\sigma^m f(z_0))''}{(D_\sigma^m f(z_0))'}\right)\right]$$

$$= p(\alpha+\beta) + \mathrm{Re}\left(\frac{\lambda m e^{i\theta}}{1+\lambda e^{i\theta}}\right) - \mathrm{Re}\left(\frac{m e^{i\theta}}{1+e^{i\theta}}\right)$$

$$= p(\alpha+\beta) + \frac{\lambda m(\lambda+\cos\theta)}{1+\lambda^2+2\lambda\cos\theta} - \frac{m}{2}$$

$$= p(\alpha+\beta) - \frac{m(1-\lambda^2)}{2(1+\lambda^2+2\lambda\cos\theta)}$$

$$\leq p(\alpha+\beta) - \frac{n}{2}\left(\frac{1-\lambda}{1+\lambda}\right)$$

$$\leq \frac{[2(\alpha+\beta)p-n] + \lambda[2(\alpha+\beta)p+n]}{2(1+\lambda)}$$

which contradicts the given hypothesis. Hence $|w(z)| < 1$, for all $z \in \mathscr{U}$, which implies

$$\left|\frac{1-\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha\left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta}{\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha\left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta - \lambda}\right| < 1,\ z\in\mathscr{U}, \tag{11}$$

or equivalently

$$\mathrm{Re}\left[\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha\left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta\right] > \frac{1+\lambda}{2},\ z\in\mathscr{U},$$

and this completes the proof of the theorem.
Setting $\alpha = 0, \beta = 1, m = 0$ in above theorem, we get:
Corollary 2.2.. *If the function $f \in \mathscr{A}(p,n)$ satisfies the inequality*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{(2p-n)+\lambda(2p+n)}{2(1+\lambda)},\ z\in\mathscr{U},$$

then

$$\mathrm{Re}\frac{f'(z)}{pz^{p-1}} > \frac{1+\lambda}{2},\ z\in\mathscr{U},$$

which is the result obtained earlier by Lee et al. [10].
Setting $p = n = 1$ in above corollary, the result reduces to:
Corollary 2.3.. *If the function $f \in \mathscr{A}$ satisfies the inequality*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1+3\lambda}{2(1+\lambda)},\ z\in\mathscr{U},$$

then

$$\mathrm{Re}\, f'(z) > \frac{1+\lambda}{2},\ z\in\mathscr{U},$$

which is the same result obtained earlier by Owa et al. [15].
Setting $\alpha = 1, \beta = 0, m = 0$, Theorem 2.1 gives
Corollary 2.4.. *Let the function $f \in \mathscr{A}(p,n)$, satisfies the inequality*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{(2p-n)+\lambda(2p+n)}{2(1+\lambda)},\ z\in\mathscr{U},$$

then

$$\mathrm{Re}\frac{f(z)}{z^p} > \frac{1+\lambda}{2},\ z\in\mathscr{U}.$$

Setting $p = n = 1$ in corollary 2.4, the result reduces to
Corollary 2.5.. *Let the function $f \in \mathscr{A}$, satisfies the inequality*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{1+3\lambda}{2(1+\lambda)},\ z\in\mathscr{U},$$

then

$$\mathrm{Re}\frac{f(z)}{z} > \frac{1+\lambda}{2},\ z\in\mathscr{U}.$$

Setting $m = 0, \alpha = 1-\gamma$ and $\beta = \gamma$ in above theorem, we obtain the following special case:
Corollary 2.6.. *Let the function $f \in \mathscr{A}(p,n)$, satisfies the inequality*

$$\mathrm{Re}\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > p + \frac{n}{2}\left(\frac{\lambda-1}{\lambda+1}\right),\ z\in\mathscr{U},$$

then

$$\mathrm{Re}\left[\left(\frac{f(z)}{z^p}\right)^{1-\gamma}\left(\frac{f'(z)}{pz^{p-1}}\right)^\gamma\right] > \frac{1+\lambda}{2},\ z\in\mathscr{U}.$$

Theorem 3.1. *Let the function $f \in \mathscr{A}(p,n)$, satisfies the inequality*

$$\mathrm{Re}\left[\mathscr{I}_{p,n}^{m,\sigma}(\alpha,\beta)f(z)\right] < \frac{\{(\alpha+\beta)p+n\}\lambda + \{2p(\alpha+\beta)+n\}}{\lambda+2},\ z\in\mathscr{U}, \tag{12}$$

then

$$\left|\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha\left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta - 1\right| < 1+\lambda,\ z\in\mathscr{U}, \tag{13}$$

where $(\alpha,\beta \in \mathbb{R};\ 0 \le \lambda < 1;\ p,n \in \mathbb{N})$.

Proof. Let the function w be defined by

$$\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta = (1+\lambda)\,w(z)+1. \quad (14)$$

Then, clearly, w is analytic in $\mathscr{U}$ with $w(0)=0$, and using the logarithmic differentiation (14) yields

$$\alpha\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \beta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)$$

$$= p(\alpha+\beta) + \frac{(1+\lambda)\,zw'(z)}{1+(1+\lambda)\,w(z)},\ z\in\mathscr{U}. \quad (15)$$

Suppose there exists a point $z_0 \in \mathscr{U}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$, with $|z| < |z_0|$

Then by applying Lemma 1.1, there exists $m \ge n$ such that

$$z_0 w'(z_0) = m w(z_0),\ \left(m \ge n \ge 1; w(z_0) = e^{i\theta}; \theta \in \mathbb{R}\right) \quad (16)$$

Then by using (15) and (16), it follows that

$$\operatorname{Re}\left[\alpha\frac{z(D_\sigma^m f(z_0))'}{D_\sigma^m f(z_0)} + \beta\left(1+\frac{z(D_\sigma^m f(z_0))''}{(D_\sigma^m f(z_0))'}\right)\right]$$

$$= (\alpha+\beta)\,p + \operatorname{Re}\left(\frac{(1+\lambda)z_0 w'(z_0)}{(1+\lambda)w(z_0)+1}\right)$$

$$= (\alpha+\beta)\,p + \operatorname{Re}\left(\frac{(1+\lambda)me^{i\theta}}{(1+\lambda)e^{i\theta}+1}\right)$$

$$= (\alpha+\beta)\,p + \left(\frac{m(1+\lambda)(1+\lambda+\cos\theta)}{1+(1+\lambda)^2+2(1+\lambda)\cos\theta}\right)$$

$$\ge \frac{\{(\alpha+\beta)p+n\}\lambda + \{2p(\alpha+\beta)+n\}}{\lambda+2},\ z\in\mathscr{U}$$

which contradicts the hypothesis (12). It follows that $|w(z)| < 1,\ z\in\mathscr{U}$, that is

$$\left|\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta - 1\right| < 1+\lambda,\ z\in\mathscr{U}.$$

This evidently completes the proof of the theorem.

Setting $\alpha = 0, \beta = 1, m = 0$ in above theorem, we get

Corollary 3.2. *If the function $f \in \mathscr{A}(p,n)$ satisfies the inequality*

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] < \frac{(p+n)\lambda+(2p+n)}{\lambda+2},\ z\in\mathscr{U},$$

then

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1+\lambda,\ z\in\mathscr{U},$$

which is the result obtained earlier by Lee et al. [10].

Setting $p = n = 1$ in above corollary, the result reduces to

Corollary 3.3. *If the function $f \in \mathscr{A}$ satisfies the inequality*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{2\lambda+3}{\lambda+2},\ z\in\mathscr{U},$$

then

$$\left|f'(z)-1\right| < 1+\lambda,\ z\in\mathscr{U},$$

which is the same result obtained earlier by Owa et al. [15].

Setting $\alpha = 1, \beta = 0, m = 0$, the above theorem gives

Corollary 3.4. *Let the function $f \in \mathscr{A}(p,n)$, satisfies the inequality*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{(p+n)\lambda+(2p+n)}{\lambda+2},\ z\in\mathscr{U},$$

then

$$\left|\frac{f(z)}{z^p} - 1\right| < 1+\lambda,\ z\in\mathscr{U}.$$

Setting $p = n = 1$ in corollary 3.4, the result reduces to:

Corollary 3.5. *Let the function $f \in \mathscr{A}$, satisfies the inequality*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{3+2\lambda}{2+\lambda},\ z\in\mathscr{U},$$

then

$$\left|\frac{f(z)}{z} - 1\right| < 1+\lambda,\ z\in\mathscr{U}.$$

For the next result, we assume that $\alpha,\beta \in \mathbb{R}$ s.t. $\alpha+\beta > 0$

Theorem 4.1. *Let the function $f \in \mathscr{A}(p,n)$, satisfies the inequality*

$$\left|\arg\left[\frac{\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta \times}{\left\{\frac{\alpha}{p(\alpha+\beta)}\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \frac{\beta}{p(\alpha+\beta)}\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right\}}\right]\right|$$

$$< \frac{\pi}{2}\left[\gamma + \frac{2}{\pi}\arctan\left(\frac{\gamma}{p(\alpha+\beta)}\right)\right],\ z\in\mathscr{U},$$

where $\gamma > 0$, then

$$\left|\arg\left\{\left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta\right\}\right| < \frac{\pi}{2}\gamma,\ z\in\mathscr{U}.$$

Proof. If we define the function

$$h(z) = \left(\frac{D_\sigma^m f(z)}{z^p}\right)^\alpha \left(\frac{(D_\sigma^m f(z))'}{pz^{p-1}}\right)^\beta \quad (17)$$

then $h(z) = 1 + c_1 z + \ldots$ is analytic in $\mathscr{U}$ and $h(0) = 1$, $h'(0) \ne 0$.

Differentiating (17) logarithmically with respect to z and by simple calculation, we get

$$zh'(z) = h(z)$$
$$\left[\alpha\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \beta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right) - p(\alpha+\beta)\right]$$

Thus,

$$h(z) + \frac{1}{p(\alpha+\beta)}zh'(z)$$
$$= \frac{h(z)}{p(\alpha+\beta)}\left[\alpha\frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \beta\left(1+\frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'}\right)\right]$$

and by using lemma (1.2), we obtain the desired result. Setting $\alpha = 1, \beta = 0, m = 0$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.2. *If $f \in \mathscr{A}(p,n)$ satisfies the inequality*

$$\left|\arg\left(\frac{f'(z)}{pz^{p-1}}\right)\right| < \frac{\pi}{2}\left[\gamma + \frac{2}{\pi}\arctan\left(\frac{\gamma}{p}\right)\right],\ z \in \mathscr{U},$$

then

$$\left|\arg\left(\frac{f(z)}{z^p}\right)\right| < \frac{\pi}{2}\gamma,\ z \in \mathscr{U}.$$

Setting $p = 1$ in above corollary 4.2, we obtain the following corollary:

Corollary 4.3. *If $f \in \mathscr{A}(1,n)$ satisfies the inequality*

$$\left|\arg\left(f'(z)\right)\right| < \frac{\pi}{2}\left[\gamma + \frac{2}{\pi}\arctan(\gamma)\right],\ z \in \mathscr{U},$$

then

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\gamma,\ z \in \mathscr{U}.$$

Setting $\alpha = 0$, $\beta = 1, m = 0$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.4. *If $f \in \mathscr{A}(p,n)$ satisfies the inequality* $\left|\arg\left(\frac{1}{pz^{p-1}}\{f'(z)+zf''(z)\}\right)\right| < \frac{\pi}{2}\left(\gamma+\frac{2}{\pi}\arctan\left(\frac{\gamma}{p}\right)\right),\quad z \in \mathscr{U},$
then

$$\left|\arg\left\{\frac{f'(z)}{pz^{p-1}}\right\}\right| < \frac{\pi}{2}\gamma,\ z \in \mathscr{U}.$$

Setting $p = 1$ in above corollary, we obtain

Corollary 4.5. *If $f \in \mathscr{A}(1,n)$ satisfies the inequality*

$$\left|\arg\left\{f'(z)+zf''(z)\right\}\right| < \frac{\pi}{2}\left(\gamma + \frac{2}{\pi}\arctan\gamma\right),\ z \in \mathscr{U},$$

then

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\gamma,\ z \in \mathscr{U}.$$

Setting $\beta = m = 0, p = 1$ in in Theorem 4.1, we obtain the following corollary:

Corollary 4.6. *$f \in \mathscr{A}(1,n)$ satisfies the inequality* $\left|\arg\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha}\right\}\right| < \frac{\pi}{2}\left[\gamma+\frac{2}{\pi}\arctan\left(\frac{\gamma}{\alpha}\right)\right],\ z \in \mathscr{U},$
then

$$\left|\arg\left(\frac{f(z)}{z}\right)^{\alpha}\right| < \frac{\pi}{2}\gamma,\ z \in \mathscr{U},$$

which is the same result obtained earlier by Lashin [11].

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