

A Contraction Approach for Noncoercive Hamilton-Jacobi-Bellman Equations

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We characterize both the continuous and finite element approximate solution of non-coercive Hamilton-Jacobi-Bellman equation as fixed points of contractions. We also derive L^∞ -error estimate of the approximation.

Keywords: HJB equations, contraction, finite element, error estimate.

1 Introduction

We are interested in the finite element approximation of the noncoercive problem associated with Hamilton-Jacobi-Bellman equation (HJB): find $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \max_{1 \leq i \leq M} (\mathcal{A}^i u - f^i) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where Ω is a bounded open set of \mathbb{R}^N , $N \geq 1$ with smooth boundary Γ , $\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^M$ denote uniformly second order elliptic operators assumed to be noncoercive, and f^1, \dots, f^M are M regular functions.

Problems of type (1.1) arise in many applications: stochastic control, management and economy, mechanics and optics, For example in stochastic control the solution of (1.1) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching (see [7]).

From the mathematical analysis point of view, this problem has been extensively studied in the eighties (see [3,6,8,9]). For numerical analysis and computational aspects of HJB equations and related variational inequalities (VI) and quasivariational inequalities (QVI) problems, we refer to [1,2,5,10–12].

In the present paper we propose to investigate furthermore the numerical analysis of the associated noncoercive problem.

More precisely, we show that both the continuous and the piecewise linear approximate solutions are fixed points of contractions in $L^\infty(\Omega)$. As a result of this, we derive L^∞ -error estimate of the approximation.

The paper is organized as follows. In section 2, we state the continuous problem and characterize its solution as the unique fixed point of a contraction. In Section 3, we define the discrete problem and characterize its solution as the unique fixed point of a contraction. In Section 4, we derive an L^∞ -error estimate of the approximation.

2 The Continuous Problem

We begin by laying down some notations and assumptions that will be needed in this paper.

2.1 Notations and assumptions

We define second order operators

$$\mathcal{A}^i = \sum_{1 \leq j, k \leq N} a_{jk}^i(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^N b_k^i(x) \frac{\partial}{\partial x_k} + a_0^i(x) \quad (2.1)$$

such that

$$a_{jk}^i(x), b_k^i(x), a_0^i(x) \in C^2(\bar{\Omega}), \quad x \in \bar{\Omega}$$

$$a_{jk}^i = a_{kj}^i; \quad a_0^i(x) \geq \beta > 0; \quad x \in \bar{\Omega}$$

$$\sum_{1 \leq j, k \leq N} a_{jk}^i(x) \xi_j \xi_k \geq \nu |\xi|^2, \nu > 0, \quad \forall x \in \bar{\Omega}, \quad \forall \xi \in R^N,$$

and the operators

$$\mathcal{B}^i = \sum_{1 \leq j, k \leq N} a_{jk}^i(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^N b_k^i(x) \frac{\partial}{\partial x_k} + (a_0^i(x) + \lambda), \quad (2.2)$$

where $\lambda > 0$ is large enough so that $\mathcal{B}^i = \mathcal{A}^i + \lambda I$ are strongly coercive on $H^1(\Omega)$.

We also define the associated bilinear forms

$$a^i(u, v) = \int_{\Omega} \left(\sum_{1 \leq j, k \leq N} a_{jk}^i \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^N b_k^i \frac{\partial u}{\partial x_k} v + a_0^i(x) uv \right) dx \quad (2.3)$$

and

$$b^i(u, v) = a^i(u, v) + \lambda(v, v), \quad (2.4)$$

where (\cdot, \cdot) denote the inner product in $L^2(\Omega)$.

Finally, let f^1, \dots, f^M be nonnegative right-hand sides in $W^{2,\infty}(\Omega)$.

2.2 Coercive HJB equation

Let g^1, \dots, g^M be given functions in $W^{2,\infty}(\Omega)$, and \mathcal{B}^i be the operators defined in (2.2). The following problem

$$\begin{cases} \max_{1 \leq i \leq M} (\mathcal{B}^i \zeta - g^i) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \quad (2.5)$$

is called coercive HJB equation.

It is shown in [9] that (2.5) can be approximated by the following weakly coupled system of QVIs

$$\begin{cases} b^i(\zeta^i, v - \zeta^i) \geq (g^i, v - \zeta^i) \quad \forall v \in H_0^1(\Omega) \\ \zeta^i \leq k + \zeta^{i+1}, \quad v \leq k + \zeta^{i+1}, \quad i = 1, \dots, M \\ \zeta^{M+1} = \xi^1, \end{cases} \quad (2.6)$$

where k is a positive constant. This is, precisely, stated in the following theorem.

Theorem 2.1 (cf. [9]). *The system (2.6) has a unique solution which belongs to $W^{2,p}(\Omega)^M$, $2 \leq p < \infty$. Moreover, as $k \rightarrow 0$, each component of this system converges uniformly in $C(\bar{\Omega})$ to the solution ζ of HJB equation (2.5), and $\zeta \in W^{2,\infty}(\Omega)$.*

2.3 Characterization of the solution of noncoercive HJB equation as the unique fixed of a contraction

One can observe that the noncoercive HJB equation can be solved by considering the following equivalent formulation

$$\begin{cases} \max_{1 \leq i \leq M} (\mathcal{B}^i u - \mathcal{F}^i(u)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (2.7)$$

where

$$\mathcal{F}^i(u) = f^i + \lambda u.$$

This can be achieved by characterizing the solution of HJB equation (1.1) as the unique fixed point of a contraction.

Indeed, let us introduce the mapping

$$\begin{aligned} \mathbb{T} : L^\infty(\Omega) &\rightarrow L^\infty(\Omega) \\ w &\rightarrow \mathbb{T}w = \zeta \end{aligned} \quad (2.8)$$

where ζ is the unique solution of the coercive HJB equations

$$\begin{cases} \max_{1 \leq i \leq M} (\mathcal{B}^i \zeta - \mathcal{F}^i(w)) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \Gamma, \end{cases} \quad (2.9)$$

with $\mathcal{F}^i(w) = f^i + \lambda w$.

Note that the $\mathcal{F}^i(w)$'s play the role of the g^i 's in (2.5). So, thanks to Theorem 2.1, (2.9) has a unique solution. It is also clear from the same theorem that (2.9) can be approximated by the following system of QVIs

$$\begin{cases} b^i(\zeta^i, v - \zeta^i) \geq (\mathcal{F}^i(w), v - \zeta^i) \quad \forall v \in H_0^1(\Omega) \\ \zeta^i \leq k + \zeta^{i+1}, \quad v \leq k + \zeta^{i+1}, \quad i = 1, \dots, M \\ \zeta^{M+1} = \xi^1 \end{cases} \quad (2.10)$$

and we have

$$\lim_{k \rightarrow 0} \|\zeta^i - \zeta\|_{C(\Omega)} = 0, \quad \forall i = 1, 2, \dots, M.$$

Lemma 2.1. *Let w, \tilde{w} be in $L^\infty(\Omega)$ and $(\zeta^1, \dots, \zeta^M)$ be the corresponding solutions to system (2.10) with right-hand sides $\mathcal{F}^i(w) = f^i + \lambda w$ and $\mathcal{F}^i(\tilde{w}) = f^i + \lambda \tilde{w}$, respectively. Then we have*

$$\max_{1 \leq i \leq M} \|\zeta^i - \tilde{\zeta}^i\|_\infty \leq \lambda / (\lambda + \beta) \|w - \tilde{w}\|_\infty.$$

Proof. Let us denote by $\|\cdot\|_\infty$ the norm of the space $L^\infty(\Omega)$ and set

$$\Phi^i = 1 / (\lambda + \beta) \|\mathcal{F}^i(w) - \mathcal{F}^i(\tilde{w})\|_\infty.$$

Then

$$\begin{aligned} \mathcal{F}^i(w) &\leq \mathcal{F}^i(\tilde{w}) + \|\mathcal{F}^i(w) - \mathcal{F}^i(\tilde{w})\|_\infty \\ &\leq \mathcal{F}^i(\tilde{w}) + (a_0^i(x) + \lambda) / (\lambda + \beta) \|\mathcal{F}^i(w) - \mathcal{F}^i(\tilde{w})\|_\infty \\ &\leq \mathcal{F}^i(\tilde{w}) + (a_0^i(x) + \lambda \Phi^i). \end{aligned}$$

So, making use of monotonicity result with respect to right-hand side for system of QVIs related to HJB equation (see [5]), we get

$$\zeta^i \leq \tilde{\zeta}^i + \Phi^i.$$

Similarly, interchanging the roles of w and \tilde{w} , we also get

$$\tilde{\zeta}^i \leq \zeta^i + \Phi^i.$$

Thus

$$\|\zeta^i - \tilde{\zeta}^i\|_{L^\infty(\Omega)} \leq \Phi^i,$$

which completes the proof. \square

Theorem 2.2. *Under conditions of Lemma 2.1, the mapping \mathbb{T} is a contraction.*

Proof. Indeed, Let $\zeta = \mathbb{T}w$ and $\tilde{\zeta} = \mathbb{T}\tilde{w}$ be solutions to HJB equation (2.9) with right-hand sides $\mathcal{F}^i(w) = f^i + \lambda w$ and $\mathcal{F}^i(\tilde{w}) = f^i + \lambda\tilde{w}$, respectively. Then, making use of both Theorem 2.1 and Lemma 2.1, we have

$$\begin{aligned} \|\mathbb{T}w - \mathbb{T}\tilde{w}\|_\infty &= \|\zeta - \tilde{\zeta}\|_\infty \\ &\leq \|\zeta - \zeta^i\|_{L^\infty(\Omega)} + \|\zeta^i - \tilde{\zeta}^i\|_\infty + \|\tilde{\zeta}^i - \tilde{\zeta}\|_\infty \\ &\leq \|\zeta - \zeta^i\|_\infty + \max_{1 \leq i \leq M} \|\zeta^i - \tilde{\zeta}^i\|_\infty + \|\tilde{\zeta}^i - \tilde{\zeta}\|_\infty \\ &\leq \lim_{k \rightarrow 0} \|\zeta - \zeta^i\|_\infty + \max_{1 \leq i \leq M} \|\zeta^i - \tilde{\zeta}^i\|_\infty + \lim_{k \rightarrow 0} \|\tilde{\zeta}^i - \tilde{\zeta}\|_\infty \\ &\leq \lambda/(\lambda + \beta) \|w - \tilde{w}\|_\infty, \end{aligned}$$

Thus, \mathbb{T} is a contraction, and therefore, the solution of HJB equation (1.1) is its unique fixed point. \square

3 The Discrete Problem

Let Ω be decomposed into triangles, τ_h denote the set of all those elements, and $h > 0$ be the mesh size. We assume that the family τ_h is regular and quasi-uniform. Let

$$V_h = \{v \in C(\bar{\Omega}) \cap H^1(\Omega) \text{ such that } v|_K \in P_1\}$$

be the finite element space, where K is a triangle of τ_h and P_1 is the space of polynomials with degree ≤ 1 . Let $\{\varphi_i\}$, $i = 1, \dots, m(h)$, be the basis functions of V_h , and \mathbb{A}^i the matrices with generic coefficients

$$(\mathbb{A}^i)_{ls} = a^i(\varphi_l, \varphi_s), \quad l = 1, \dots, m(h); \quad 1 \leq i \leq M. \tag{3.1}$$

Let us also define the discrete right-hand sides

$$F^i = (f^i, \varphi_l), \quad l = 1, \dots, m(h); \quad 1 \leq i \leq M \tag{3.2}$$

and the usual restriction operator r_h

$$\forall v \in C(\Omega) \cap H_0^1(\Omega), \quad r_h v = \sum_{l=1}^{m(h)} v_l \varphi_l. \tag{3.3}$$

3.1 The discrete HJB equation

Given the matrices \mathbb{A}^i , and the discrete right-hand sides F^i defined above, the discrete Hamilton-Jacobi-Bellman equation consists of solving the following problem: Find $u_h \in V_h$ solution to

$$\max_{1 \leq i \leq M} (\mathbb{A}^i u_h - F^i) = 0. \quad (3.4)$$

As in the continuous case, we shall handle the noncoercive problem by transforming (3.4) into

$$\max_{1 \leq i \leq M} (\mathbb{B}^i u_h - F^i(u_h)) = 0, \quad (3.5)$$

where

$$(F^i(u_h))_l = (f^i + \lambda u_h, \varphi_l), \quad l = 1, \dots, m(h), \quad 1 \leq i \leq M,$$

and \mathbb{B}^i are the matrices defined by

$$(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s), \quad l = 1, \dots, m(h), \quad 1 \leq i \leq M. \quad (3.6)$$

In the sequel of the paper a *discrete maximum principle (d.m.p)* assumption will be needed. More precisely, the matrices \mathbb{B}^i will be assumed to be M-matrices (see [4]).

As in the continuous case, we shall characterize the solution of the discrete HJB equation as the unique fixed point of a contraction. Let us first define the discrete counterpart of (2.5) by

$$\max_{1 \leq i \leq M} (\mathbb{B}^i \zeta_h - G^i) = 0 \quad (3.7)$$

with

$$G^i = (g^i, \varphi_l), \quad l = 1, \dots, m(h); \quad 1 \leq i \leq M. \quad (3.8)$$

It is shown in [5] that (2.5) can be approximated by the following discrete weakly coupled system of QVIs

$$\begin{cases} b^i(\zeta^i, v - \zeta^i) \geq (g^i, v - \zeta^i) & \forall v \in V_h \\ \zeta^i \leq k + \zeta_h^{i+1}, \quad v \leq k + \zeta_h^{i+1}, \quad i = 1, \dots, M \\ \zeta_h^{M+1} = \zeta_h^1 \end{cases} \quad (3.9)$$

Theorem 3.1 (cf. [5]). *Let the dmp hold. Then, the system (3.9) has a unique solution. Moreover, as $k \rightarrow 0$, each component of the solution of this system converges uniformly in $C(\bar{\Omega})$ to the solution ζ_h of (3.7).*

3.2 Characterization of the discrete solution of noncoercive HJB equation as a fixed point of a contraction

Let $(F^i(w))_l = (f^i + \lambda w, \varphi_l)$, $l = 1, \dots, m(h)$. We introduce the mapping

$$\mathbb{T}_h : L^\infty(\Omega) \rightarrow V_h \quad (3.10)$$

$$w \mapsto \mathbb{T}_h w = \zeta_h$$

where ζ_h is the unique solution of the following discrete coercive HJB equation

$$\max_{1 \leq i \leq M} (\mathbb{B}^i \zeta_h - F^i(w)) = 0 \quad (3.11)$$

with $(F^i(w))_l = (f^i + \lambda w, \varphi_l)$, $l = 1, \dots, m(h)$.

Note that the $F^i(w)$'s play the role of the G^i 's in (3.7). So, thanks to Theorem 3.1, (3.11) has a unique solution. It is also clear from the same theorem that (3.11) can be approximated by the following system of QVIs

$$\begin{cases} b^i(\zeta_h^i, v - \zeta_h^i) \geq (F^i(w), v - \zeta_h^i) & \forall v \in V_h \\ \zeta_h^i \leq k + \zeta_h^{i+1}, \quad v \leq k + \zeta_h^{i+1}, \quad i = 1, \dots, M \\ \zeta_h^{iM+1} = \zeta_h^1 \end{cases} \quad (3.12)$$

and we have $\lim_{k \rightarrow 0} \|\zeta_h - \zeta_h^i\| = 0$.

Lemma 3.1. *Let the dmp hold. Then we have*

$$\max_{1 \leq i \leq M} \|\xi_h^i - \tilde{\xi}_h^i\|_\infty \leq \lambda/(\lambda + \beta) \|w - \tilde{w}\|_\infty, \quad \forall w, \tilde{w} \in L^\infty(\Omega).$$

Proof. Exactly the same as that of Lemma 2.1. □

Theorem 3.2. *Under conditions of Lemma 3.1, the mapping \mathbb{T}_h is a contraction.*

Proof. Exactly the same as that of Theorem 2.2. □

3.3 L^∞ -error estimate

Now, we show that the fixed point approach developed in this paper leads to an L^∞ quasi-optimal convergence of the approximation. For that end, let us first introduce the following coercive discrete HJB equation

$$\max_{1 \leq i \leq M} (\mathbb{B}^i \bar{\zeta}_h - F^i(u)) = 0, \quad (3.13)$$

where $(F^i(u))_l = (f^i + \lambda u, \varphi_l)$, $l = 1, \dots, m(h)$, and u is the continuous solution of the HJB equation (1.1). So, in view of (3.10), we clearly have

$$\bar{\zeta}_h = \mathbb{T}_h u. \quad (3.14)$$

Therefore, as problem (3.13) is the discrete counterpart of problem (2.7), making use of [2], we have the following error estimate.

Theorem 3.3 (cf. [2]).

$$\| \bar{\zeta}_h - u \|_\infty \leq Ch^2 |\log h|^3, \quad (3.15)$$

where C is a constant independent of h .

Theorem 3.4. Let u and u_h be the solutions of HJB equations (1.1) and (3.4), respectively. Then

$$\| u - u_h \|_\infty \leq Ch^2 |\log h|^3,$$

where C is a constant independent of h .

Proof. Since $\bar{\zeta}_h = \mathbb{T}_h u$ and $u_h = \mathbb{T}_h u_h$, making use of Theorems 2.2, 3.2 and estimate (3.15), we obtain

$$\begin{aligned} \| u - u_h \|_{L^\infty(\Omega)} &\leq \| u - \bar{\zeta}_h \|_{L^\infty(\Omega)} + \| \bar{\zeta}_h - u_h \|_\infty \\ &\leq \| u - \bar{\zeta}_h \|_\infty + \| \mathbb{T}_h u - \mathbb{T}_h u_h \|_\infty \\ &\leq Ch^2 |\log h|^3 + \frac{\lambda}{\lambda + \beta} \| u - u_h \|_\infty. \end{aligned}$$

Thus,

$$\| u - u_h \|_\infty \leq \frac{Ch^2 |\log h|^3}{\lambda/(\lambda + \beta)}. \quad \square$$

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