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On Brauner's Angle and Frenet-Serret Formulae in Minkowski Space

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Abstract: In this paper, we define the *B-orthonormal frame* in Minkowski n-dimensional space M_B^n using the so-called Birkhoff orthogonality. Using the M-trigonometric functions, we insert the concept of Minkowski semi-inner product. Brauner's angle in M-space is given as a function of some ideal points in projective space. Finally, we redefine the Frenet-Serret formulae in three dimensional M-space using the concept of deformation vectors.

Keywords: Minkowski space, Birkhoff orthogonality, Semi-inner product, Brauner's theorem, Kruppa-Shonoda invariants, Frenet-Serret formulae.

1 Introduction

There are many open omitted problems of Minkowski space involving angle measure. Brauner [1] tries to define the concept of angle between two vectors $\mathbf{a},\mathbf{b}$ in projective space as a function of the ideal points of these vectors and its orthogonal vectors. In our work, we shall declare the Brauner's angle between two vectors in Minkowski space using the cross ratio of some ideal points in the plane at infinity. Because the concept of angle in Minkowski space is not symmetric, it is expected to obtain a relation of angle measure dependent on the angle from $\mathbf{a}$ to $\mathbf{b}$ and vice versa. This relation is based on the so-called M-cosine function [2] which is also not symmetric function. Besides, this function will allow us to calculate coefficients of derivative equations of type Frenet-Serret in a Minkowski 3-space M_B^3.

The main properties of a Minkowski space and its B-orthogonality as well as its relations to other orthogonality concepts are introduced by many authors. Thompson [2] and Alonso , for example, [3,4,5,6] introduce the concept of area orthogonality, which satisfies all orthogonality relations in an inner product space except the abelian and additive relations. The definition of the B-orthogonality is not in general a symmetric relation. In the case of Radon plane the relation is symmetric. The construction of Radon curve is presented by Radon [7]. Also, Birkhoff [8] and Day [9] give constructions of it in terms of polarity and a quarter rotations with respect to some Euclidean structure. Moreover, Martini and others [10,11] present it using only an "affine" bilinear form. Finally, Averkov and others [12] give some important characterizations of central symmetry of convex bodies in Minkowski spaces.

We will prefer a concept of Minkowski orthogonality due to Birkhoff [8], James [13,14,15] and Day [9] from a very geometric point of view which leads to a non-symmetric orthogonality as it is most naturally related to the geometry of the gauge Ball B of the Minkowski space. In general, it is no longer a symmetric relation between linear subspaces of the Minkowski space. Explicitly this construction of "left-orthogonality" reads as follows; the supporting plane of the unit ball B at a point x , contains all the lines y being "left-orthogonal" to vector x ; (and then x is right-orthogonal to y), see Figure 1, We use the symbols $y \dashv x$ for y left-orthogonal to x resp. $y \vdash x$ for y right-orthogonal to x.

Definition 1.1. If M_B^n is an n-dimensional normed linear space of unit ball B and if $x,y \in M_B^n$ then we say that x is B-orthogonal to y and write

$$x \dashv y \Leftrightarrow \|x\| \leq \|x+\lambda y\| \ \forall \lambda \in \mathbb{R} \tag{1}$$

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Geometrically, this means that $x \dashv y$ if and only if the line λy supports the unit ball B at x, see Figure 1.

The Hahn-Banach theorem [2] implies that $x+\lambda y$ lies in a hyperplane which supports B at x. Obviously, $\dashv$ is not a symmetric relation, i.e. if $x \dashv y$, it is not means that $y \dashv x$. In fact, for dimensions three or above, the only normed spaces for which normality is symmetric are the Euclidean spaces. In dimension two, normality is symmetric for the wide class of Radon planes [7,16].

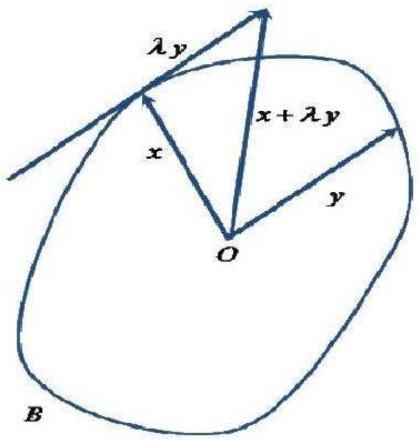

Fig. 1: Birkhoff (left) orthogonality, $\mathbf{x} \dashv \mathbf{y}$

Definition 1.2. Let x_1, x_2 are two vectors in a Minkowski space M_B^3 such that $\|x_1\| = \|x_2\| = 1$, then this pair is called *mutually normal* pair if $x_1 \dashv x_2$ and $x_2 \dashv x_1$.

Definition 1.3. (Thompson [2]), If is the unit ball in a Minkowski space then there exists a basis $(x_1, x_2, \ldots, x_d)$ such that $\|x_i\| = 1$ and $x_i \dashv x_j$ for all i and j with $i \neq j$; i.e. each pair of basis vectors is mutually normal.

Definition 1.4. (B-orthonormal frame in M_B^n): Let $e_1, e_2, \ldots, e_n \in M_B^n$, $\|e_i\| = 1 \forall i = 1,2,\ldots,n.$ If $e_j \dashv e_k \forall k = 1,2,\ldots,j-1$ then the ordered vector set $e_1, e_2, \ldots, e_n$ is called B-orthonormal frame in M_B^n .

2 Brauner's theorem (Angle measure in Projective space)

A famous formula of E. LAGUERRE (1853) describes the (Euclidean) angle between two lines a,b by means of Projective Geometry as follows: Let A,B be the ideal points of a,b and I,J be the pair of conjugate imaginary absolute points on the ideal line $u = A \vee B$, then the angle measure $\varphi = \sphericalangle \mathbf{a}, \mathbf{b}$ is calculated by

$$\sphericalangle \mathbf{a}, \mathbf{b} = \left| \frac{i}{2} lncr(I,J,A,B) \right| \tag{2}$$

This formula has, in spite of its importance for understanding Euclidean Geometry as a sub-geometry of Projective Geometry, two disadvantages: The first is the necessity of a complex extension of the places of action and the second is the necessity of a quadratic absolute figure, thus demanding the underlying vector space to be an inner product space.

H. BRAUNER'S angle formula [1] uses instead of Laguerres absolute points I,J the ideal points $A', B' \in u$ belongs to the directions $\mathbf{a}, \mathbf{b}$ which are orthogonal to $\mathbf{a}', \mathbf{b}'$ respectively. It reads as follows:

$$\tan\varphi = \sqrt{-cr(A,A',B,B',)} = \sqrt{-cr(\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}')} \tag{3}$$

($\mathbf{a}, \ldots$meaning the direction vectors to the ideal points $A, \ldots$). This formula avoids complex extension and it needs an orthogonality structure of the place of action. It is therefore possible to declare an angle concept with this formula in a large set of normed spaces.

3 M-trigonometric functions.

The Brauner's angle measure (3) is not clear to use in Minkowski space; however, it is almost projective space but it is not inner product space. Therefore, we try to use a well-defined trigonometric functions in M-space to define the so-called semi-inner product which allow us to find it in M-space.

Minkowski cosine (M-cosine) function.

We construct the more suitable definition of the Minkowski cosine function in M_B^n [2] between two vectors $x,y \in M_{B_2}^n \subset M_B^n$, where $B_2 = B \cap M^2$ is the unit ball of the subspace M^2 spanned by the two vectors x and y and pass through the origin of the main unit ball B of the space M_B^n . Mathematically, this function depends on the unique linear function if the unit ball is smooth at $\frac{x}{\|x\|}$ and hence this function is not symmetric function.

For all Minkowski spaces M_B^n with strictly convex smooth unit ball B , we have for all $x \in M_B^n, x \neq 0$, up to a positive scalar factor, a unique linear functional f_x attains its maximum at x . i.e.,

$$f_x(x) = \|x\| \,.\, \|f_x\| \tag{4}$$

using this function, we can define the M-cosine function as follows:

Definition 3.1. (Minkowski cosine function): For all two vectors $x,y \in M_B^n, y \neq 0$, the Minkowski cosine function from x to y is not symmetric function, denoted by $cm(x,y)$ with the definition

$$cm(x,y) := \frac{f_x(y)}{\|y\| \,.\, \|f_x\|} \tag{5}$$

substituting (4) into (5) we get

$$cm(x,y) = \frac{\|x\| \,.\, f_x(y)}{\|y\| \,.\, f_x(x)} \tag{6}$$

In general, if M^d is a subspace of M_B^n then we can define the cosine Minkowski function as

$$cm\left(x,M^d\right) := max\{cm(x,y)\,|y \in M^d - \{0\}\} \quad (7)$$

Minkowski sine (M-sine) function.

Let H and x be, respectively, a hyperplane through the origin and a non-zero vector in an oriented Minkowski space M_B^n with centrally symmetric unit ball B. Also, suppose also that H has a basis $(x_1, x_2, \ldots, x_{d-1})$ then the *Minkowski sine* $sm(x,y)$ is defined by

$$sm(H,x) := \frac{f(x)}{\|x\|.\tilde{\sigma}(f)} \quad (8)$$

where f is a linear function in dual space $(M_B^n)^*$ such that $f^\perp = H$ and whose sign is such that $f(x)$ has the same sign as the basis $(x_1, x_2, \ldots, x_{d-1}, x)$ for M_B^n and $\tilde{\sigma}$ is the norm in $(M_B^n)^*$ induced by the isoperimetrix I_B in M_B^n [16].

In general, if H is a hyperplane and if L is a subspace of M_B^n then we can define $sm(H,L)$ by

$$sm(H,L) := max\{sm(H,x)\,|x \in L - \{0\}\} \quad (9)$$

In the following proposition, we will give some interesting properties of the cosine function connecting with the B-orthogonality. We will use it later to define the Minkowski semi-inner product.

Proposition 3.2. For all $x_1, x_2 \in M_B^n, x_1, x_2 \neq 0$ we have

i. $cm(x_1,x_2) = 0$ iff $x_1 \dashv x_2$.

ii. $cm(\alpha x_1, \beta x_2) = cm(x_1,x_2)\, \forall \alpha, \beta > 0$.

iii. $cm(x_1,-x_2) = -cm(x_1,x_2)$.

iv. If f_{x_1} supports B at x_1 then $cm(-x_1,x_2) = -cm(x_1,x_2)$.

v. $cm(x_1,x_1) = 1$.

vi. For all $x_1 \neq x_2$, $|\, cm(x_1,x_2)\, | \leq 1$ with equality iff the line $\left[\frac{x_1}{\|x_1\|}\right], \left[\frac{x_2}{\|x_2\|}\right] \subset B$.

4 Minkowski semi-inner product.

We know that a normed linear space is not necessarily an inner product space. Therefore, a real normed linear space is an inner product space if and only if each two-dimensional linear subspace of it is also an inner product space. Equivalently, we can state that a normed linear space is an inner product space if and only if every plane section of the unit ball B pass through the origin o is an ellipse. For $n \geq 3$, t B is an ellipsoid and the Minkowski space is Euclidean space. As we cannot start with an inner product, we need to find a so called "semi-inner product", which is compatible with the B-orthogonality concept and the (non-Euclidean) Minkowski norm.

Theorem 4.1. (Dragomir [17]), In each real normed linear space M_B^n there exists at least one semi-inner product $[.,.]$ which generates the norm $\|.\|$. That is, $\|x\| = [.,.]^{1/2}$ for all $x \in M_B^n$, and it is unique if and only if M_B^n is smooth.

Definition 4.2. In Minkowski space M_B^n , we define the Minkowski semi-inner product of two vectors $x_1, x_2 \in M_B^n$ as follows:

$$\langle x_1, x_2 \rangle_M := \frac{f_{x_1}(x_2)}{f_{x_1}(x_1)} \|x_1\|^2 \quad (10)$$

By substitute (6) into (10), we have

$$\langle x_1, x_2 \rangle_M = \|x_1\|\,\|x_2\|\, cm(x_1,x_2) \quad (11)$$

Proposition 4.3. The Minkowski semi-inner product $\langle .,. \rangle_M : M_B^n \times M_B^n \to \mathbb{R}$ has the following properties for all $x_1, x_2, x_3 \in M_B^n$ and $\alpha, \beta \in \mathbb{R}$:

i. $\langle x_1, x_2 \rangle_M = 0$ iff $x_1 \dashv x_2$.

ii. In general $\langle x_1, x_2 \rangle_M \neq \langle x_1, x_2 \rangle_M$.

iii. $\langle x_1, \alpha x_2 + \beta x_3 \rangle_M = \alpha \langle x_1, x_2 \rangle_M + \beta \langle x_1, x_3 \rangle_M$ (Distributive law is not symmetric).

iv.$\langle \alpha x_1, x_2 \rangle_M = \alpha \langle x_1, x_2 \rangle_M$ *and* $\langle x_1, \beta x_2 \rangle_M = \beta \langle x_1, x_2 \rangle_M$

v. $\langle x_1, x_1 \rangle_M = 0$ iff $x = 0$

vi. $|\, \langle x_1, x_2 \rangle_M\, |^2 \leq \|x_1\|^2 . \|x_2\|^2$

5 Brauner's theorem in Minkowski space.

The definition of Brauner's angle in projective space (3) is not valid in Minkowski space, because the angle in M-space is not symmetric. Here we use the definition of left-orthogonality (1) and the definition of Minkowski semi-inner product to derive a suitable formula of the angle measure in M-space.

Theorem 5.1.

If $\mathbf{a}, \mathbf{b}$ are two vectors in a Minkowski space M_B^n the angle between $\mathbf{a}$ and $\mathbf{b}$ satisfies the equation $cm(u,v)\, cm(v,u) = cr\left(A_u, B_u, B_u^\dashv, A_u^\dashv\right)$, where $A_u = (0,u)\mathbb{R}$, $B_u = (0,v)\mathbb{R}$, $A_u^\dashv = \left(0,u^\dashv\right)\mathbb{R}$ and $B_u^\dashv = \left(0,v^\dashv\right)\mathbb{R}$ are the ideal points of the vectors $\mathbf{a}, \mathbf{a}^\dashv, \mathbf{b}, \mathbf{b}^\dashv$ respectively.

Proof. Assume that $\delta_M = cr\left(A_u, A_u^\dashv, B_u, B_u^\dashv\right)$ and $\mathbf{a}^\dashv, \mathbf{b}^\dashv$ are the left orthogonal vectors of $\mathbf{a}, \mathbf{b}$ respectively. From the collinearity, we get,

$$\left(0,u^\dashv\right)\mathbb{R} = \alpha(0,u)\mathbb{R} + \beta(0,v)\mathbb{R}, \alpha, \beta \in \mathbb{R}, \quad (12)$$

multiply both sides by u from left as a Minkowski semi-inner product, we get

$0 = \langle u, \alpha u + \beta v \rangle_M = \alpha \langle u,u \rangle_M + \beta \langle u,v \rangle_M \Rightarrow \alpha : \beta = -\langle u,v \rangle_M : \langle u,u \rangle_M$. Then the homogeneous coordinates of it are $(-\langle u,v \rangle_M : \langle u,u \rangle_M)$. Similarly the homogeneous coordinates of $B_u^\dashv$ are $(\langle v,v \rangle_M : -\langle v,u \rangle_M)$.

Now we can assume that the ideal points $A_u = (1:0)$ and $B_u = (0:1)$ then,

$$\delta_M = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} -\langle u\cdot v\rangle_M & 0 \\ \langle u\cdot u\rangle_M & 1 \end{vmatrix}} \frac{\begin{vmatrix} -\langle u\cdot v\rangle_M & \langle v\cdot v\rangle_M \\ \langle u\cdot u\rangle_M & -\langle v\cdot u\rangle_M \end{vmatrix}}{\begin{vmatrix} 1 & \langle v\cdot v\rangle_M \\ 0 & -\langle v\cdot u\rangle_M \end{vmatrix}},$$

without loss of generality we can assume that $\|u\| = \|v\| = 1$ then,

$$\delta_M = 1 - \frac{1}{cm(u,v)\,cm(v,u)} \text{ and hence,}$$

$$cm(u,v)\,cm(v,u) = cr\left(A_u, B_u, B_u^{\dashv}, A_u^{\dashv}\right) \tag{13}$$

This result is compatible with the fact that the angle measure in M-space is not symmetric; therefore, (13) gives a relation between the two orientation angles and the cross ratio of the two vectors and their orthogonal at infinity.

6 Frenet-Serret formulae in M-space.

The Frenet-Serret formulae in Minkowski space are modified by Shonoda [16], because the derivatives of the unit vectors of the frame $(\mathbf{t},\mathbf{h},\mathbf{b})$ in M_B^3 lie in the supporting planes of it, then $\mathbf{t}\dashv\mathbf{h}, \mathbf{h}\dashv\mathbf{b}$ and $\mathbf{b}\dashv\mathbf{t}$.

Moreover, we will define the so-called *deformation vector* in M^2 which helps us to find the more suitable Frenet-Serret formulae in M_B^3.

Definition 6.1. Deformation vector: Let $M_{B_2}^2$ be a Minkowski plane with smooth, strictly convex and centrally symmetric unit ball B_2. Then the deformation vector $\tilde{x}_y$ of the vector x in the plane (oxy), of any normed vectors $x, y \in M_{B_2}^2$ is defined as

$$\tilde{x}_y = \frac{\left(x^{\dashv}\right)^{\dashv}}{\left\|\left(x^{\dashv}\right)^{\dashv}\right\|} \tag{14}$$

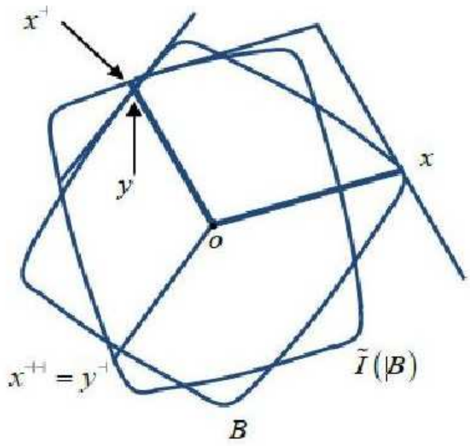

Fig. 2: Deformation vector $\mathbf{x}^{\dashv\dashv}$ of the vector $\mathbf{x}$ in Minkowski two dimensional unit ball B

Note that the pair $(x, x^{\dashv})$ is a B-orthonormal basis for $M_{B_2}^2$, see Figure 2.

Theorem 6.2.
Let $M_{B_2}^2 \subset M_B^n$ be a Minkowski plane with smooth, strictly convex and centrally symmetric unit ball B_2. Let $y = x^{\dashv}\left\|x^{\dashv}\right\|^{-1}$ be any normed vector of $M_{B_2}^2$, then its left-orthogonal vector $y^{\dashv}$ is described by the formula $y^{\dashv} = \left\|x^{\dashv}\right\|\left\{-x + y.cm\left(x^{\dashv}, x\right)\right\}$.

Proof. Assume that the vector $x \in M_{B_2}^2$. then, we have the vector $y^{\dashv}$ which can be considered as a linear combination of the normal basis $(x, x^{\dashv})$,

$$y^{\dashv} = A_1 x + A_2 x^{\dashv}, \tag{15}$$

multiply both sides of (15) by $x^{\dashv}$ from left as a Minkowski semi-inner product, then we have $\langle x^{\dashv}, y^{\dashv}\rangle_M = \langle x^{\dashv}, A_1 x + A_2 x^{\dashv}\rangle_M$, since $\langle x^{\dashv}, y^{\dashv}\rangle_M = 0$, then

$$A_2 = -\frac{A_1 cm\left(x^{\dashv}, x\right)}{\left\|x^{\dashv}\right\|}. \tag{16}$$

Remember that Minkowski plane $M_{B_2}^2$ is spanned by the pairs $(x, x^{\dashv})$ or $(y, y^{\dashv})$; therefore the Minkowski area $\mu(P)$ of the parallelogram spanned by y and $y^{\dashv}$ is given by $\mu(P) = \|y\|\left\|y^{\dashv}\right\| sm\left(y, y^{\dashv}\right) = \left\|y^{\dashv}\right\| sm\left(y, y^{\dashv}\right)$.

Since $sm\left(y, y^{\dashv}\right) = \frac{1}{\left\|y^{\dashv}\right\|}$, then by substituting $y^{\dashv}$ from (15) into (16), we have

$$\frac{1}{\left\|y^{\dashv}\right\|} = sm\left(y, y^{\dashv}\right) = sm\left(y, A_1 x + A_2 x^{\dashv}\right). \tag{17}$$

For all $x_1, x_2, x_3 \in M_{B_2}^2 - \{0\}$ we have

$$sm(x_1,x_3) = cm(x_2,x_3)\,sm(x_1,x_2) + cm(x_2,x_1)\,sm(x_2,x_3), \tag{18}$$

If we take $x_1 = y, x_2 = A_1 x$ and $x_3 = y^{\dashv}$, then (17) becomes

$$\frac{1}{\left\|y^{\dashv}\right\|} = sm\left(y, y^{\dashv}\right) = -cm\left(x, y^{\dashv}\right)\frac{1}{\left\|x^{\dashv}\right\|}, \tag{19}$$

multiply again both sides of (15) by x from left as a Minkowski semi-inner product, we have $A_1 = -\frac{1}{\left\|x^{\dashv}\right\|}$.

Substituting (19) into (20) and using (16), we get

$$y^{\dashv} = \left\|x^{\dashv}\right\|\left\{-x + y.cm\left(x^{\dashv}, x\right)\right\}, \tag{20}$$

which complete the proof.

Now, we assume that $\mathbf{e}(s_1)$ be a unit vector which depends only on a parameter s_1. By attaching this vector at the origin o of a fixed (affine) frame of a Minkowski space M_B^3 , we receive the spherical image c_1 of the ruled surface in consideration Φ at the unit sphere S and we call the cone $o \vee c_1$, the "direction cone" of Φ . Without loss of generality we can assume that the parameter s_1 is a "Minkowski arc length parameter" of the curve c_1 , i.e. the deviation vector $\mathbf{e}'(s_1)$ is normed all over the definition interval of s_1. The direction cone of Φ takes the form

$$\chi = v\mathbf{e}(s_1). \tag{21}$$

Based on the right handed B-orthonormal (affine) frame $\{\mathbf{e},\pi,\mathbf{z}\}$ we have $\mathbf{e} \dashv \mathbf{e}'$, $\mathbf{z} \dashv \mathbf{e}$ and $\mathbf{z} \dashv \mathbf{e}'$, and therefore $\langle \mathbf{e},\mathbf{e}'\rangle_M = 0$, $\langle \mathbf{z},\mathbf{e}\rangle_M = 0$ and $\langle \mathbf{z},\mathbf{e}'\rangle_M = 0$. The derivatives of the vectors $\{\mathbf{e},\mathbf{e}',\mathbf{z}\}$ should be linear combinations of these vectors. The formulae for these expressions are usually called the Frenet-Serret formulae of a moving frame.

We can assume that the vector $\mathbf{e}'(s_1) = \pi$ is defined by the first equation of the three Frenet-Serret equations. We state that the first equation must be the same for Minkowski cases as well as for the Euclidean case, because the unit vector $\mathbf{e}(s_1)$ is left-orthogonal to the derivative vector $\mathbf{e}'(s_1)$ as in the Euclidean case.

The derivative of the unit vector $\mathbf{z}$ can be obtained as a linear combination of the three B-orthonormal vectors $\{\mathbf{e},\pi,\mathbf{z}\}$ as follows:

$$\mathbf{z}' = B_1\mathbf{e} + B_2\pi + B_3\mathbf{z}. \tag{22}$$

Multiplying both sides of (22) by $\mathbf{z}$ from left as a Minkowski semi-inner product and using proposition 4.3, we get $B_3 = 0$, where $\mathbf{z} \dashv \mathbf{z}'$ ($\mathbf{z}'$ lies in the supporting plane of the unit ball B at $\mathbf{z}$). Therefore,

$$\mathbf{z}' = B_1\mathbf{e} + B_2\pi. \tag{23}$$

In the same manner, we multiply both sides of (23) by $\mathbf{e}$ from left side, we can find $B_1 = \langle \mathbf{e},\mathbf{z}'\rangle_M$. Then,

$$\mathbf{z}' = \langle \mathbf{e},\mathbf{z}'\rangle_M \mathbf{e} + B_2\pi. \tag{24}$$

By using the deformation vector $\tilde{\mathbf{e}}_\pi$ of the vector $\mathbf{e}$ in the plane $\{o\mathbf{e}\pi\}$, then the vector π becomes

$$\pi = \frac{\mathbf{e} + \tilde{\mathbf{e}}_\pi}{cm(\pi,\mathbf{e})}. \tag{25}$$

Then, we can rewrite (24) as follows:

$$\mathbf{z}' = \left[\langle \mathbf{e},\mathbf{z}'\rangle_M + \frac{B_2}{\mathrm{cm}(\pi,\mathbf{e})}\right]\mathbf{e} + B_2\frac{\tilde{\mathbf{e}}_\pi}{\mathrm{cm}(\pi,\mathbf{e})}. \tag{26}$$

Multiply again both sides of (26) by π from left as a Minkowski semi-inner product, we can easily compute the constant $B_2 = \|\mathbf{z}'\|\{cm(\pi,\mathbf{z}') - cm(\mathbf{e},\mathbf{z}')\,cm(\pi,\mathbf{e})\}$, we have from (24)

$$\mathbf{z}' = \|\mathbf{z}'\|\left\{cm(\mathbf{e},\mathbf{z}')\,\mathbf{e} + \left[cm(\pi,\mathbf{z}') - cm(\mathbf{e},\mathbf{z}')\,cm(\pi,\mathbf{e})\right]\pi\right\}. \tag{27}$$

By the same method, we can assume that the vector π' lies in the plane contains the vectors $\tilde{\mathbf{e}}_\pi$ and $\tilde{\mathbf{z}}_\pi$; hence, we can describe it as linear combination of that vectors,

$$\pi' = d_1\tilde{\mathbf{e}}_\pi + d_2\tilde{\mathbf{z}}_\pi, \tag{28}$$

where,

$$\left.\begin{aligned} \tilde{\mathbf{e}}_\pi &= -\mathbf{e} + \pi\,\mathrm{cm}(\pi,\mathbf{e}), \\ \tilde{\mathbf{z}}_\pi &= -\mathbf{z} + \pi\,\mathrm{cm}(\pi,\mathbf{z}) \end{aligned}\right\} \tag{29}$$

It is clear to calculate the constants d_1 and d_2 as follows:

$$d_1 = \frac{\|\pi'\|}{\|\tilde{\mathbf{e}}_\pi\|}\,\frac{cm(\tilde{\mathbf{e}}_\pi,\pi') - cm(\tilde{\mathbf{z}}_\pi,\pi')\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi)}{1 - cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi)}, \tag{30}$$

$$d_2 = \frac{\|\pi'\|}{\|\tilde{\mathbf{z}}_\pi\|}\,\frac{cm(\tilde{\mathbf{z}}_\pi,\pi') - cm(\tilde{\mathbf{e}}_\pi,\pi')\,cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)}{1 - cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi)}, \tag{31}$$

then (28) can be rewritten as follows:

$$\begin{aligned} \pi' = \|\pi'\| \Bigg(& \frac{cm(\tilde{\mathbf{e}}_\pi,\pi') - cm(\tilde{\mathbf{z}}_\pi,\pi')\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi)}{-H(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,\|\tilde{\mathbf{e}}_\pi\|}\mathbf{e} + \\ & \Bigg(\frac{(cm(\tilde{\mathbf{e}}_\pi,\pi') - cm(\tilde{\mathbf{z}}_\pi,\pi')\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi))\,cm(\pi,\mathbf{e})}{H(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,\|\tilde{\mathbf{e}}_\pi\|} + \\ & \frac{(cm(\tilde{\mathbf{z}}_\pi,\pi') - cm(\tilde{\mathbf{e}}_\pi,\pi')\,cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi))\,cm(\pi,\mathbf{z})}{H(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,\|\tilde{\mathbf{z}}_\pi\|}\Bigg)\pi + \\ & \frac{cm(\tilde{\mathbf{z}}_\pi,\pi') - cm(\tilde{\mathbf{e}}_\pi,\pi')\,cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)}{-H(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{z}}_\pi)\,\|\tilde{\mathbf{z}}_\pi\|}\mathbf{z}\Bigg) \end{aligned} \tag{32}$$

where, $H = (\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi) = 1 - cm(\tilde{\mathbf{z}}_\pi,\tilde{\mathbf{e}}_\pi)\,cm(\tilde{\mathbf{e}}_\pi,\tilde{\mathbf{z}}_\pi)$. Now, we consider $x(s)$ describing the space curve c with arc length s. The tangent vector $\mathbf{t} = x'$ can be moved into the unit sphere S to obtain the spherical image c_1 , the cone (oc_1) has generators which are parallel to the tangent of the curve c. Similarly as the previous construction, we have the right handed orthonormal (affine) frame $\{\mathbf{t},\mathbf{h},\mathbf{b}\}$, where $\mathbf{h}$ is called the principle normal vector and $\mathbf{b}$ is the binomial vector, without loss of generality. We can consider that the derivatives of these vectors are unit vectors, the plane $(\mathbf{th})$ is the osculating plane and the plane $(\mathbf{hb})$ is the normal plane. By using the derivatives of these vectors as before, we have

$$\frac{d\mathbf{t}}{ds_1} = \mathbf{h}, \tag{33}$$

$$\begin{aligned}\frac{d\mathbf{h}}{ds_1} &= \frac{cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right) - cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right) cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\tilde{\mathbf{b}}_{\mathbf{h}}\right)}{-H\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{t}}_{\mathbf{h}}\right\|}\mathbf{t}+ \\ &\left(\frac{\left(cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right) - cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right) cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\tilde{\mathbf{b}}_{\mathbf{h}}\right)\right) cm\left(\mathbf{h},\mathbf{t}\right)}{H\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{t}}_{\mathbf{h}}\right\|}+\right. \\ &\left.\frac{\left(cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right) - cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right) cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\right) cm\left(\mathbf{h},\mathbf{b}\right)}{H\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{b}}_{\mathbf{h}}\right\|}\right)\mathbf{h}+ \\ &\frac{cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right) - cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right) cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)}{-H\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{b}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{b}}_{\mathbf{h}}\right\|}\mathbf{b}\end{aligned} \tag{34}$$

$$\frac{d\mathbf{b}}{ds_1} = cm\left(\mathbf{t},\mathbf{b}'\right)\mathbf{t} + \left(cm\left(\mathbf{h},\mathbf{b}'\right) - cm\left(\mathbf{t},\mathbf{b}'\right) cm\left(\mathbf{h},\mathbf{t}\right)\right)\mathbf{h}, \tag{35}$$

We insert the following abbreviations and notations:

$\frac{ds_1}{ds} =: \chi \to$ M-curvature,

$-cm\left(\mathbf{h},\mathbf{b}'\right) =: \chi_1 \to$ conical curvature,

$cm\left(\mathbf{t},\mathbf{b}'\right) =: \chi_2 \to$ second conical curvature,

$cm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right) =: \chi_3 \to$ third conical curvature,

$cm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right) =: \chi_4 \to$ fourth conical curvature.

Multiplying both sides of the equations (33), (34) and (35) with the previous functions, we similarly get Minkowski analogues to the classical torsion functions as follows:

$\chi\chi_1 =: \tau_1 \to$ M-torsion,

$\chi\chi_2 =: \tau_2 \to$ second torsion,

$\chi\chi_3 =: \tau_3 \to$ third torsion,

$\chi\chi_4 =: \tau_4 \to$ fourth torsion.

The coefficient functions of the M-Frenet-Serret formulae are the 2^{nd} , 3^{rd} and 4^{th} curvatures and torsions. They have no geometric meaning in general but we can find such a meaning for some special unit balls B .

Then, The Frenet-Serret formulae can be written as follows

$$\begin{bmatrix}\mathbf{t}'\\ \mathbf{h}'\\ \mathbf{b}'\end{bmatrix} = \begin{bmatrix}0 & \chi & 0\\ -\overline{\tau} & \overline{\tau}\, cm\left(\mathbf{h},\mathbf{t}\right) + \overline{\tau}_1\, cm\left(\mathbf{h},\mathbf{b}\right) & -\overline{\tau}_1\\ \tau_2 & -\left(\tau_1 + \tau_2 cm\left(\mathbf{h},\mathbf{t}\right)\right) & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{h}\\ \mathbf{b}\end{bmatrix} \tag{36}$$

whereby $\overline{\tau}$ and $\overline{\tau}_1$ are the following functions of the 3^{rd} and 4^{th} Minkowski torsions [16]:

$$\overline{\tau} = \chi \frac{sm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\mathbf{h}'\right)}{sm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{t}}_{\mathbf{h}}\right\|}, \tag{37}$$

$$\overline{\tau}_1 = \chi \frac{sm\left(\tilde{\mathbf{t}}_{\mathbf{h}},\mathbf{h}'\right)}{sm\left(\tilde{\mathbf{b}}_{\mathbf{h}},\tilde{\mathbf{t}}_{\mathbf{h}}\right)\left\|\tilde{\mathbf{b}}_{\mathbf{h}}\right\|}. \tag{38}$$

7 Conclusion

We have shown that examples of Euclidean concepts can be translated into a Minkowski (normed) space. Of course there remain many open questions. In this paper we e.g. omitted those problems involving angle measures. By presenting just one topic of these problems, namely the Brauner's theorem, we want to point out the arbitrariness of finding and using a suitable angle concept. Moreover, the discussion of special ruled surfaces remains to another occasion. For example, it would be interesting to know how surfaces with constant M-curvatures and M-torsions look like.

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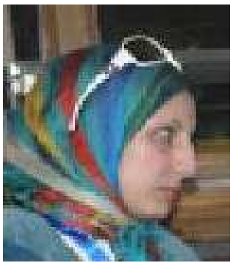

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