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New Types of Common Fixed Point Theorems in Fuzzy Metric Spaces

H. M. Abu-Donia[1] *and Khaled Abd-Rabou*[1,2,*]

[1] Department of mathematics, Faculty of science, Zagazig University, Egypt
[2] Department of mathematics,Faculty of science, Shaqra University, Al-qawwiya, K.S.A

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Abstract: The purpose of this paper is to study common fixed point theorems for set-valued and single-valued mappings in different fuzzy metric spaces, namely; fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces. We extend some definitions to the three aforementioned fuzzy metric spaces. The results of Fisher [1], Sharma [2] and Tiwari [3] have been extended throughout the paper. We prove common fixed point theorems for hybrid mappings in fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces.

Keywords: Fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces, weakly compatible mappings, common fixed point.

1 Introduction

The emergence of fuzzy set theory to human knowledge has opened the way for advanced methods of reasoning. The fuzzy thinking and reasoning is scientifically appropriate for solving real life problems in various fields such as medicine, economic and engineering. The metric functions are used to derive general method for measuring similarity between elements. The concept of two metrics considers two view points for similarities. These concepts assist in many directions especially in the process of information technology where fuzzy concepts play an important role in decision making.

Heilpern [4], Chang [5,6] and others studied fixed point theorems for fuzzy contraction mappings. Kaleva and Seikkala [7] studied fuzzy sets and systems on fuzzy metric space. This work was extended to a pair of fuzzy contraction mappings by Bose and Sahani [8]. Park and Jeong [9] proved the existence of common fixed point for fuzzy mappings in complete metric space, which are the fuzzy extensions of theorems as proved by Beg and Azam [10]. Tantawy and Abu-Donia [11] presented common fixed point for a pair of fuzzy mapping in 2-metric space. Singh and Chauhan [12] and Vasuki [13] introduced the concept of *R*-weakly commuting and compatible maps in fuzzy metric space. The notions of weak compatible and semi compatible maps in fuzzy metric spaces was introduced by Singh and Jain [14]. . Further, Sharma [15], Sharma and Tiwari [3] studied unique common fixed point for three mappings in fuzzy 2-metric and fuzzy 3-metric spaces. Likewise, Abu-Donia and Abdrabou [16,17] studied unique common fixed point for four self and hybrid mappings in fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces. Abdrabou [18] studied unique common fixed point for sequences of mappings in fuzzy metric spaces. In this paper we extend new definitions in fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces. We also extend the results of Fisher [1], Sharma [15], Sharma and Tiwari [3]. We prove common fixed point theorems for hybrid mappings in fuzzy metric, fuzzy 2-metric and fuzzy 3-metric spaces.

2 Preliminaries

Now we begin with some definitions

Definition 2.1 **[19]** A binary operation $* : [0,1]^2 \to [0,1]$is called a continuous *t*-norm if $([0,1],*)$ is an Abelian topological monoid with the unit 1 such that $a_1 * b_1 \le a_2 * b_2$ whenever $a_1 \le a_2$ and $b_1 \le b_2$ for all $a_1,\ a_2,\ b_1,\ b_2 \in [0,1]$.
For example, $a_1 * b_1 = a_1 b_1$ or $a_1 * b_1 = \min\{a_1, b_1\}$.
Definition 2.2 [20] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-

* Corresponding author e-mail: k_abdrabo@yahoo.com

norm and $M$is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X$and $t_1,t_2 > 0$
(FM-1) $M(x,y,0) = 0$,
(FM-2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$,
(FM-4) $M(x,z,t_1+t_2) \geq M(x,y,t_1) * M(y,z,t_2)$,
(FM-5) $M(x,y,\ .) : [0,\infty) \to [0,1]$ is left continuous.
(FM-6) $\lim_{t\to\infty} M(x,y,t) = 1$, for all $x,y \in X$.

Note that $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.3 [13] Let $(X,M,*)$ be a fuzzy metric space:
A sequence $\{x_n\}$ in fuzzy metric space $X$is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$, for all $t > 0$.
A sequence $\{x_n\}$ in fuzzy metric space $X$is called a Cauchy sequence, if
$\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$, for all $t > 0$and $p > 0$.
A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4 [22,3]A binary operation $* : [0,1]^3 \to [0,1]$is called a continuous t-norm if $([0,1],*)$ is an Abelian topological monoid with the unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2$ and $c_1 \leq c_2$ for all $a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2 \in [0,1]$.

Definition 2.5 [22,3] The 3-tuple $(X,M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and $M$is a fuzzy set in $X^3 \times (0,\infty)$ satisfying the following conditions for all $x,y,z,u \in X$and $t_1,t_2,t_3 > 0$
(F2M-1) $M(x,y,z,0) = 0$,
(F2M-2) $M(x,y,z,t) = 1$ for all $t > 0$ when at least two of the three points are equal,
(F2M-3) $M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)$,
(Symmetry about three variables)
(F2M-4) $M(x,y,z,t_1+t_2+t_3)$
$\geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$,
(This corresponds to tetrahedron inequality in 2-metric space)
The function $M(x,y,z,t)$ may be interpreted as the probability that the area of triangle is less than t.
(F2M-5) $M(x,y,z,\ .) : [0,\infty) \to [0,1]$ is left continuous.
(F2M-6) $\lim_{t\to\infty} M(x,y,z,t) = 1$, for all $x,y,z \in X$.

Definition 2.6 [22,3] Let $(X,M,*)$ be a fuzzy 2-metric space:
A sequence $\{x_n\}$ in fuzzy 2-metric space $X$is said to be convergent to a point
$x \in X$ if $\lim_{n\to\infty} M(x_n,x,z,t) = 1$, for all $z \in X$ and $t > 0$.
A sequence $\{x_n\}$ in fuzzy 2-metric space $X$is called a Cauchy sequence, if
$\lim_{n\to\infty} M(x_{n+p},x_n,z,t) = 1$, for all $z \in X$ and $t > 0$, $p > 0$.
A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.7 [3]A binary operation $* : [0,1]^4 \to [0,1]$is called a continuous t-norm if $([0,1],*)$ is an Abelian topological monoid with the unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2,\ b_1 \leq b_2,\ c_1 \leq c_2$ and $d_1 \leq d_2$ for all $a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2,\ d_1,\ d_2 \in [0,1]$.
For example, $a_1 * b_1 * c_1 * d_1 = a_1 b_1 c_1 d_1$ or $a_1 * b_1 * c_1 * d_1 = \min\{a_1,b_1,c_1,d_1\}$.

Definition 2.8 [3] The 3-tuple $(X,M,*)$ is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm and $M$is a fuzzy set in $X^4 \times (0,\infty)$ satisfying the following conditions for all $x,y,z,w,u \in X$and $t_1,t_2,t_3,t_4 > 0$
(F3M-1) $M(x,y,z,w,0) = 0$,
(F3M-2) $M(x,y,z,w,t) = 1$ for all $t > 0$,
Only when the three simplex $< x,y,z,w >$ degenerate,
(F3M-3) $M(x,y,z,w,t) = M(x,y,w,z,t) = M(x,w,y,z,t) = \ldots$,
(F3M-4) $M(x,y,z,w,t_1+t_2+t_3+t_4) \geq M(x,y,z,u,t_1)$
$* M(x,y,u,w,t_2) * M(x,u,z,w,t_3) * M(u,y,z,w,t_4)$,
(This corresponds to tetrahedron inequality in 3-metric space)
(F3M-5) $M(x,y,z,w,\ .) : [0,\infty) \to [0,1]$ is left continuous.
(F3M-6) $\lim_{t\to\infty} M(x,y,z,w,t) = 1$, for all $x,y,z,w \in X$.

In the following example, we know that every metric induces a fuzzy metric.

Example 2.1. Let (X,d) be a 3-metric space. Define $a_1 * b_1 * c_1 * d_1 = a_1 b_1 c_1 d_1$ (or $a_1 * b_1 * c_1 * d_1 = \min\{a_1,b_1,c_1,d_1\}$) and for all $x,y,z,w \in X$ and $t > 0$,

$$M(x,y,z,w,t) = \frac{t}{t+d(x,y,z,w)}.$$

Then $(X,M,*)$ is a fuzzy 3-metric space. We call this fuzzy 3-metric M induced by the 3-metric d.

Definition 2.9 [3] Let $(X,M,*)$ be a fuzzy 3-metric space:
A sequence $\{x_n\}$ in fuzzy 3-metric space $X$is said to be convergent to a point
$x \in X$ if $\lim_{n\to\infty} M(x_n,x,z,w,t) = 1$, for all $z,w \in X$ and $t > 0$.
A sequence $\{x_n\}$ in fuzzy 3-metric space $X$is called a Cauchy sequence, if
$\lim_{n\to\infty} M(x_{n+p},x_n,z,w,t) = 1$, for all $z,w \in X$ and $t > 0$, $p > 0$.
A fuzzy 3-mtric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.10 [2]Any two maps S and T from a fuzzy metric space $(X,M,*)$ into itself are said to be compatible if $\lim_{n\to\infty} M(STx_n,TSx_n,t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r$ and $r \in X$.

Definition 2.11 [16]Any two maps S and T from a fuzzy 2-metric space $(X,M,*)$ into itself are said to be compatible if $\lim_{n\to\infty} M(STx_n,TSx_n,z,t) = 1$ for all

$z \in X, t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r$ and $r \in X$.

Definition 2.12 [16] Any two maps S and T from a fuzzy 3-metric space $(X,M,*)$ into itself are said to be compatible if $\lim_{n\to\infty} M(STx_n, TSx_n, z, w, t) = 1$ for all $z, w \in X, t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r$ and $r \in X$.

Definition 2.13 [16] Any two maps S and T from a fuzzy metric space or fuzzy 2-metric space or fuzzy 3-metric space $(X,M,*)$ into itself are said to be weakly compatible [12] if they commute at their coincidence points.

Remark 2.1 [16]. Since * is continuous, it follows from (FM-4), (F2M-4) and (F3M-4) that the limit of a sequence in a fuzzy metric space or a fuzzy 2-metric space or a fuzzy 3-metric space is unique.

Lemma 2.1 [23] Let $\{y_n\}$be a sequence in a fuzzy metric space $(X,M,*)$. If there exists a number $k \in (0,1)$ such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, for all $t > 0$and $n = 1,2,....$ Then $\{y_n\}$is a Cauchy sequence in X.

Lemma 2.2 [2] If for all $x, y \in X$, $t > 0$ and for a number $k \in (0,1)$,

$M(x,y,kt) \geq M(x,y,t)$. Then $x = y$.

Remark 2.2 [16]. Lemma 2.1 and Lemma 2.2 hold for fuzzy 2-metric spaces and fuzzy 3-metric spaces.

Fisher [1] proved the following theorem for three mappings in complete metric space:

Theorem 1. Let S and T be continuous mappings of a complete metric space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping $A$of Xinto $S(X)\bigcap T(X)$ which commute with S and T and satisfy $d(Ax,Ay) \leq \alpha\, d(Sx,Ty)$for all $x,y \in X$, $0 < \alpha < 1$. Indeed S,T and A have a unique common fixed point.

Sharma [3] extended Theorem 1 to fuzzy metric as the following.

Theorem 2. Let $(X,M,*)$ be a complete fuzzy metric space and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists a continuous mapping $A$of Xinto $S(X)\bigcap T(X)$ which commute with S and T and

$M(Ax,Ay,qt)$
$\geq \min\{M(Ty,Ay,t), M(Sx,Ax,t), M(Sx,Ty,t)\}$

for all $x,y \in X$, $t > 0$ and $0 < q < 1$. Then S,T and A have a unique common fixed point.

Sharma [3] also extended Theorem 2 in fuzzy 2-metric and fuzzy 3-metric spaces.

Sharma and Tiwari [12] improved results of Sharma [3] and proved the following.

Theorem 3. Let $(X,M,*)$ be a complete fuzzy metric space with $t*t > t$for all $t \in [0,1]$. Let A, S and T be mappings from X into itself such that: $S(X)\bigcup T(X) \subset A(X)$,

The pairs $\{A,T\}$ and $\{A,S\}$ are weakly compatible,

There exists a number $k \in (0,1)$such that

$$M(Sx,Ty,kt)$$

$$\geq M(Ax,Ay,t)*M(Sx,Ax,t)*M(Ay,Ty,t)$$
$$*M(Ay,Sx,t)*M(Ax,Ty,(2-\alpha)t)$$

for all $x,y \in X, t > 0, \alpha \in (0,2)$. Then S,T and A have a unique common fixed point.

Sharma and Tiwari [12] also extended Theorem 3 in fuzzy 2-metric and fuzzy 3-metric spaces.

3 Main Results

Let $CB(X)$be the class of all nonempty bounded closed subsets of X.

Definition 3.1 The mappings $I: X \to X$ and $F: X \to CB(X)$ in fuzzy metric space $(X,M,*)$ are said to be compatible if $IFx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} M(IFx_n, FIx_n, t) = 1$ for all $t > 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = r \in A = \lim_{n\to\infty} Fx_n$, $A \in CB(X)$.

Definition 3.2 The mappings $I: X \to X$ and $F: X \to CB(X)$ in a fuzzy metric space $(X,M,*)$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $IFx = FIx$ whenever $Ix \in Fx$.

Theorem 3.1 Let $(X,M,*)$ be a fuzzy metric space with $t*t \geq t$ for all $t \in [0,1]$. Let I and J be mappings from X into itself and $G,F: X \to CB(X)$ set-valued mappings such that

$\bigcup FX \subseteq JX$and $\bigcup GX \subseteq IX$. (1)

Also, the mappings I,J,F and G satisfy the following inequality

$$M(Fx,Gy,kt)$$

$\geq M(Ix,Jy,t)*M(Ix,Fx,t)*M(Jy,Gy,t)$
$*M(Ix,Gy,t)*M(Jy,Fx,t)$ (2)

for all $x,y \in X$, $t > 0$ and $k \in (0,1)$. Suppose that the pairs $\{F,I\}$ and $\{G,J\}$are weakly compatible. If the range of one of the mappings I,J,F and G is complete subspace of X. Then I,J,F and G have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and by using (1), there exists a point $x_1 \in X$ such that $Jx_0 \in Fx_1$, for this point x_1, we can choose a point $x_2 \in X$ such that $Ix_1 \in Gx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$Jx_{2n+1} \in Fx_{2n} = y_{2n}$ and $Ix_{2n+2} \in Gx_{2n+1} = y_{2n+1}$ (3)

We shall prove that $\{y_n\}$is Cauchy sequence in X.

On using (2) and (3), we obtain

$$M(y_{2n},y_{2n+1},kt) = M(Fx_{2n},Gx_{2n+1},kt)$$

$$\geq M(Ix_{2n},Jx_{2n+1},t)*M(Ix_{2n},Fx_{2n},t)*M(Jx_{2n+1},Gx_{2n+1},t)$$
$$*M(Ix_{2n},Gx_{2n+1},t)*M(Jx_{2n+1},Fx_{2n},t)$$
$$\geq M(y_{2n-1},y_{2n},t)*M(y_{2n-1},y_{2n},t)*M(y_{2n},y_{2n+1},t)$$

$$*M(y_{2n-1},y_{2n+1},t)*M(y_{2n},y_{2n},t)$$
$$\geq M(y_{2n-1},y_{2n},t)*M(y_{2n},y_{2n+1},t)$$
$$*M(y_{2n-1},y_{2n},t/2)*M(y_{2n},y_{2n+1},t/2)$$
$$\geq M(y_{2n-1},y_{2n},t)*M(y_{2n},y_{2n+1},t).$$

Similarly, we have also

$$M(y_{2n+1},y_{2n+2},kt)\geq M(y_{2n},y_{2n+1},t)*M(y_{2n+1},y_{2n+2},t)$$

Thus, $M(y_{n+1},y_{n+2},kt)\geq M(y_n,y_{n+1},t)*M(y_{n+1},y_{n+2},t)$ for all $n=1,2,\ldots$ and so for positive integers n,p

$$M(y_{n+1},y_{n+2},kt)\geq M(y_n,y_{n+1},t)*M(y_{n+1},y_{n+2},t/k^p)$$

Since $M(y_{n+1},y_{n+2},t/k^p)\to 1$ as $p\to\infty$.
Then, $M(y_{n+1},y_{n+2},kt)\geq M(y_n,y_{n+1},t)$.
On using Lemma 2.1, we obtain the sequence $\{y_n\}$is Cauchy sequence. Suppose that JX is complete, therefore by the above, the sequence $\{Jx_{2n+1}\}$ is Cauchy and hence $Jx_{2n+1}\to z=Jv$ for some $v\in X$. Hence, the sequence $\{y_n\}$ converges also to z and the subsequence $\{Ix_{2n+2}\}$, $\{Fx_{2n+1}\}$ and $\{Gx_{2n+1}\}$ converge to z. We shall prove that $z=Jv\in Gv$. On using (2), we obtain that

$$M(Fx_{2n},Gv,kt)$$
$$\geq M(Ix_{2n},Jv,t)*M(Ix_{2n},Fx_{2n},t)*M(Jv,Gv,t)$$
$$*M(Ix_{2n},Gv,t)*M(Jv,Fx_{2n},t).$$

Taking the limit $n\to\infty$, we obtain

$$M(z,Gv,kt)$$
$$\geq M(z,z,t)*M(z,z,t)*M(z,Gv,t)*M(z,Gv,t)*M(z,z,t)$$
$$\geq M(z,Gv,t).$$

On using Lemma 2.2, we obtain that $Gv=\{z\}=\{Jv\}$.
Since $\bigcup GX\subseteq IX$, so $u\in X$ exists such that $\{Iu\}=Gv=\{z\}=\{Jv\}$. Now if $Fu\neq Gv$, so that we have

$$M(Fu,Gv,kt)$$
$$\geq M(Iu,Jv,t)*M(Iu,Fu,t)*M(Jv,Gv,t)$$
$$*M(Iu,Gv,t)*M(Jv,Fu,t)$$

Hence,

$$M(Fu,z,kt)$$
$$\geq M(z,z,t)*M(z,Fu,t)*M(z,z,t)$$
$$*M(z,z,t)*M(z,Fu,t)$$
$$\geq M(z,Fu,t).$$

Then, $Fu=\{z\}=\{Iu\}=Gv=\{Jv\}$.
Since $Fu=\{Iu\}$ and the pair $\{F,I\}$ is weakly compatible, we obtain

$$Fz=FIu=IFu=\{Iz\}$$

On using inequality (2), we have

$$M(Fz,Gv,kt)\geq M(Fz,z,kt)$$
$$\geq M(Iz,Jv,t)*M(Iz,Fz,t)*M(Jv,Gv,t)$$
$$*M(Iz,Gv,t)*M(Jv,Fz,t)$$
$$\geq M(z,Fz,t).$$

Hence $\{z\}=Fz=\{Iz\}$. Similarly, $\{z\}=Gz=\{Jz\}$where the pair $\{G,J\}$ is weakly compatible. Therefore, we obtain that $\{z\}=\{Iz\}=\{Jz\}=Fz=Gz$.

To see z is unique, suppose that $\{p\}=\{Ip\}=\{Jp\}=Fp=Gp$.
If $p\neq z$, then

$$M(Fz,Gp,kt)\geq M(z,p,kt)$$
$$\geq M(Iz,Jp,t)*M(Iz,Fz,t)*M(Jp,Gp,t)$$
$$*M(Iz,Gp,t)*M(Jp,Fz,t)$$
$$\geq M(z,p,t).$$

Thus $z=p$. Then, I,J,F and G have a unique common fixed point.

Theorem 3.2 Let $(X,M,*)$ be a fuzzy metric space with $t*t\geq t$ for all $t\in[0,1]$. Let I be mapping from X into itself and $G,F:X\to CB(X)$ set-valued mappings such that

$$\bigcup FX\bigcup\left(\bigcup GX\right)\subseteq IX$$

Also, the mappings I,F and G satisfy the following inequality

$$M(Fx,Gy,kt)$$
$$\geq M(Ix,Iy,t)*M(Ix,Fx,t)*M(Iy,Gy,t)$$
$$*M(Ix,Gy,t)*M(Iy,Fx,t)$$

for all $x,y\in X$, $t>0$ and $k\in(0,1)$. Suppose that the pairs $\{F,I\}$ and $\{G,I\}$are weakly compatible. If the range of one of the mappings I,F and G is complete subspace of X. Then I,F and G have a unique common fixed point.

Proof. It is obvious if we take $I\equiv J$ in Theorem 3.1

Remark 3.1 Theorem 3.2 is extension for Theorem 3 (Sharma and Tiwari [21]) in fuzzy metric space.

Theorem 3.3 Let S be mapping from fuzzy metric space $(X,M,*)$ into itself and $T:X\to CB(X)$ set-valued mapping such that $\bigcup TX\subseteq SX$ and

$$M(Tx,Ty,kt)$$
$$\geq M(Sx,Sy,t)*M(Sx,Tx,t)*M(Sy,Ty,t)$$
$$*M(Sx,Ty,t)*M(Sy,Tx,t)$$

for all $x,y\in X$, $t>0$ and $k\in(0,1)$. Suppose that the pair $\{T,S\}$ is weakly compatible, if the range of one of the mappings T and S is complete subspace of X. Then T and Shave a unique common fixed point.

Proof. It is obvious if we take $F=G=T$ and $I=J=S$ in Theorem 3.1.

Now we prove that I,J,F and G have a unique common fixed point in fuzzy 2-metric space.

Definition 3.3 The mappings $I : X \to X$ and $F : X \to CB(X)$ in fuzzy 2-metric space $(X,M,*)$ are said to be compatible if $IFx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} M(IFx_n, FIx_n, z, t) = 1$ for all $z \in X, t > 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = r \in A = \lim_{n\to\infty} Fx_n$, $A \in CB(X)$.

Definition 3.4 The mappings $I : X \to X$ and $F : X \to CB(X)$ in a fuzzy 2-metric space $(X,M,*)$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $IFx = FIx$ whenever $Ix \in Fx$.

Theorem 3.4 Let $(X,M,*)$ be a fuzzy 2-metric space with $t*t \geq t$ for all $t \in [0,1]$. Let I and J be mappings from X into itself and $G,F : X \to CB(X)$ set-valued mappings satisfying condition (1). Also, the mappings I,J,F and G satisfy the following inequality

$$M(Fx,Gy,w,kt)$$

$$\geq M(Ix,Jy,w,t)*M(Ix,Fx,w,t)*M(Jy,Gy,w,t)$$
$$*M(Ix,Gy,w,t)*M(Jy,Fx,w,t) \quad (4)$$

for all $x,y,w \in X$, $t > 0$ and $k \in (0,1)$. Suppose that the pairs $\{F,I\}$ and $\{G,J\}$are weakly compatible. If the range of one of the mappings I,J,F and G is complete subspace of X. Then I,J,F and G have a unique common fixed point.

Proof. We can define a sequence $\{y_n\}$ in X such that

$Jx_{2n+1} \in Fx_{2n} = y_{2n}$ and $Ix_{2n+2} \in Gx_{2n+1} = y_{2n+1}$

Now, we shall prove that $\{y_n\}$is Cauchy sequence in X. On using (4), we obtain

$$M(y_{2n},y_{2n+1},w,kt) = M(Fx_{2n},Gx_{2n+1},w,kt)$$
$$\geq M(Ix_{2n},Jx_{2n+1},w,t)*M(Ix_{2n},Fx_{2n},w,t)$$
$$*M(Jx_{2n+1},Gx_{2n+1},w,t)*M(Ix_{2n},Gx_{2n+1},w,t)$$
$$*M(Jx_{2n+1},Fx_{2n},w,t)$$
$$\geq M(y_{2n-1},y_{2n},w,t)*M(y_{2n-1},y_{2n},w,t)*M(y_{2n},y_{2n+1},w,t)$$
$$*M(y_{2n-1},y_{2n+1},w,t)*M(y_{2n},y_{2n},w,t)$$
$$\geq M(y_{2n-1},y_{2n},w,t)*M(y_{2n},y_{2n+1},w,t)*M(y_{2n-1},y_{2n},w,t/3)$$
$$*M(y_{2n},y_{2n+1},w,t/3)*M(y_{2n-1},y_{2n+1},y_{2n},t/3)$$
$$\geq M(y_{2n-1},y_{2n},w,t)*M(y_{2n},y_{2n+1},w,t)*M(y_{2n-1},y_{2n},w,t/3)$$
$$*M(y_{2n},y_{2n+1},w,t/3)*M(y_1,y_3,y_2,t\Big/9k^{2n})$$
$$*M(y_1,y_2,y_2,t\Big/9k^{2n})*M(y_2,y_3,y_2,t\Big/9k^{2n}).$$

Since $M(y_1,y_3,y_2,t/9k^{2n}) \to 1$ as $n \to \infty$.
Thus, we have

$$M(y_{2n},y_{2n+1},w,kt) \geq M(y_{2n-1},y_{2n},w,t)*M(y_{2n},y_{2n+1},w,t)$$

Similarly, we have also

$$M(y_{2n+1},y_{2n+2},w,kt) \geq M(y_{2n},y_{2n+1},w,t)*M(y_{2n+1},y_{2n+2},w,t)$$

Thus,

$$M(y_{n+1},y_{n+2},w,kt) \geq M(y_n,y_{n+1},w,t)*M(y_{n+1},y_{n+2},w,t)$$

for all $n = 1,2,...$ and so for positive integers n,p

$$M(y_{n+1},y_{n+2},w,kt) \geq M(y_n,y_{n+1},w,t)*M(y_{n+1},y_{n+2},w,t/k^p)$$

Since $M(y_{n+1},y_{n+2},w,t/k^p) \to 1$ as $p \to \infty$.
Then, $M(y_{n+1},y_{n+2},w,kt) \geq M(y_n,y_{n+1},w,t)$.
On using Lemma 2.1, we obtain the sequence $\{y_n\}$is Cauchy sequence. Suppose that JX is complete, therefore by the above, the sequence $\{Jx_{2n+1}\}$ is Cauchy and hence $Jx_{2n+1} \to z = Jv$ for some $v \in X$. Hence, the sequence $\{y_n\}$ converges also to z and the subsequence $\{Ix_{2n+2}\}$, $\{Fx_{2n+1}\}$ and $\{Gx_{2n+1}\}$ converge to z. We shall prove that $z = Jv \in Gv$. On using (4), we obtain that

$$M(Fx_{2n},Gv,w,kt)$$
$$\geq M(Ix_{2n},Jv,w,t)*M(Ix_{2n},Fx_{2n},w,t)*M(Jv,Gv,w,t)$$
$$*M(Ix_{2n},Gv,w,t)*M(Jv,Fx_{2n},w,t).$$

Taking the limit $n \to \infty$, we obtain
$M(z,Gv,w,kt)$
$\geq M(z,z,w,t)*M(z,z,w,t)*M(z,Gv,w,t)$

$$*M(z,Gv,w,t)*M(z,z,w,t)$$

$\geq M(z,Gv,w,t)$.
On using Lemma 2.2, we obtain that $Gv = \{z\} = \{Jv\}$.
Since $\bigcup GX \subseteq IX$, so $u \in X$ exists such that $\{Iu\} = Gv = \{z\} = \{Jv\}$. Now if $Fu \neq Gv$, so that we have

$$M(Fu,Gv,w,kt)$$
$$\geq M(Iu,Jv,w,t)*M(Iu,Fu,w,t)*M(Jv,Gv,w,t)$$
$$*M(Iu,Gv,w,t)*M(Jv,Fu,w,t)$$

Hence,

$$M(Fu,z,w,kt)$$
$$\geq M(z,z,w,t)*M(z,Fu,w,t)*M(z,z,w,t)$$
$$*M(z,z,w,t)*M(z,Fu,w,t)$$
$$\geq M(z,Fu,w,t).$$

Then,

$$Fu = \{z\} = \{Iu\} = Gv = \{Jv\}.$$

Since $Fu = \{Iu\}$ and the pair $\{F,I\}$ is weakly compatible, we obtain

$$Fz = FIu = IFu = \{Iz\}$$

On using inequality (4), we have

$$M(Fz,Gv,w,kt) \geq M(Fz,z,w,kt)$$
$$\geq M(Iz,Jv,w,t)*M(Iz,Fz,w,t)*M(Jv,Gv,w,t)$$
$$*M(Iz,Gv,w,t)*M(Jv,Fz,w,t)$$
$$\geq M(z,Fz,w,t).$$

Hence $\{z\} = Fz = \{Iz\}$. Similarly, $\{z\} = Gz = \{Jz\}$where the pair$\{G,J\}$ is weakly compatible. Therefore, we obtain that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

To see z is unique, suppose that $\{p\} = \{Ip\} = \{Jp\} = Fp = Gp$.

If $p \neq z$, then

$$M(Fz,Gp,w,kt) \geq M(z,p,w,kt)$$

$$\geq M(Iz,Jp,w,t)*M(Iz,Fz,w,t)*M(Jp,Gp,w,t)$$
$$*M(Iz,Gp,w,t)*M(Jp,Fz,w,t)$$
$$\geq M(z,p,w,t).$$

Thus $z = p$. Then, I,J,F and G have a unique common fixed point.

Theorem 3.5 Let $(X,M,*)$ be a fuzzy 2-metric space with $t*t \geq t$ for all $t \in [0,1]$. Let I be mapping from X into itself and $G,F : X \to CB(X)$ set-valued mappings such that

$$\bigcup FX \bigcup \left(\bigcup GX\right) \subseteq IX$$

Also, the mappings I,F and G satisfy the following inequality

$$M(Fx,Gy,w,kt)$$

$$\geq M(Ix,Iy,w,t)*M(Ix,Fx,w,t)*M(Iy,Gy,w,t)$$
$$*M(Ix,Gy,w,t)*M(Iy,Fx,w,t),$$

for all $x,y,w \in X$, $t > 0$ and $k \in (0,1)$. Suppose that the pairs $\{F,I\}$ and $\{G,I\}$are weakly compatible. If the range of one of the mappings I,F and G is complete subspace of X. Then I,F and G have a unique common fixed point.

Proof. It is obvious if we take $I \equiv J$ in Theorem 3.4

Remark 3.2 Theorem 3.5 is extension for Sharma and Tiwari [12] in fuzzy 2-metric space.

Theorem 3.6 Let S be mapping from fuzzy 2-metric space $(X,M,*)$ into itself and $T : X \to CB(X)$ set-valued mapping such that $\bigcup TX \subseteq SX$ and

$$M(Tx,Ty,w,kt)$$

$$\geq M(Sx,Sy,w,t)*M(Sx,Tx,w,t)*M(Sy,Ty,w,t)$$
$$*M(Sx,Ty,w,t)*M(Sy,Tx,w,t),$$

for all $x,y,w \in X$, $t > 0$ and $k \in (0,1)$. Suppose that the pair $\{T,S\}$ is weakly compatible, if the range of one of the mappings T and S is complete subspace of X. Then T and Shave a unique common fixed point.

Proof. It is obvious if we take $F = G = T$ and $I = J = S$ in Theorem 3.4.

Now we prove that I,J,F and G have a unique common fixed point in fuzzy 3-metric space.

Theorem 3.7 Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t \geq t$ for all $t \in [0,1]$. Let I and J be mappings from X into itself and $G,F : X \to CB(X)$ set-valued mappings satisfying condition (1). Also, the mappings I,J,F and G satisfy the following inequality

$$M(Fx,Gy,w,q,kt)$$

$$\geq M(Ix,Jy,w,q,t)*M(Ix,Fx,w,q,t)*M(Jy,Gy,w,q,t)$$
$$*M(Ix,Gy,w,q,t)*M(Jy,Fx,w,q,t) \qquad (5)$$

for all $x,y,w,q \in X$, $t > 0$ and $k \in (0,1)$. Suppose that the pairs $\{F,I\}$ and $\{G,J\}$are weakly compatible, if the range of one of the mappings I,J,F and G is complete subspace of X. Then I,J,F and G have a unique common fixed point.

Proof. Theorem 3.7 can be proved in the similar manner as Theorem 3.4.

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H. M. Abu-Donia Assistant Professor of Mathematics , Faculty of Science Zagazig University, Egypt He got degree of MSC and PHD in Mathematics from Faculty of Science, Zagazig, University, Egypt He obtained a degree of assistant professor in mathematics ”Topology” in 2008 He participated in several local and international conferences in mathematics His research interests are His multiple research directions: in general topology fixed point theorem and Rough set theory He has published research articles in reputed international journals of mathematical sciences He is referee of many mathematical journals

Khaled Abd-Rabou is assistant Professor of Mathematics , Faculty of Science Shaqra University, Al-qawwiya, Kingdom of Saudi Arabia. He has got MSC degree in Mathematics from Faculty of Science, Zagazig University, Egypt. He has got his PHD in Mathematics from Faculty of Science, Tanta University, Egypt. His research interests are the Function analysis , Specially, Fixed point theorem and its applications . He has published research articles in reputed international journals of mathematical sciences. He is referee and editor of mathematical journals.