Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.12785/sjm/020103

Generalized Fractional Calculus of the M-Series Involving F_3 Hypergeometric function

Dinesh Kumar[1,*] *and R. K. Saxena*[2]

[1] Department of Applied Sciences, Chandawaji, Pratap University, Jaipur-302103, India
[2] Department of Mathematics & Statistics Jai Narain Vyas University, Jodhpur - 342005, India

Received: 2 Apr. 2014, Revised: 5 Jun. 2014, Accepted: 7 Jun. 2014
Published online: 1 Sep. 2015

Abstract: In this paper we investigate the generalized fractional integration and differentiation of the generalized M-Series. We give representations of the generalized M-Series in terms of the Wright generalized hypergeometric function ${}_p\psi_q$, and formulas for generalized fractional calculus operators. Several other new and known results can also be obtained from our main theorems. Some results derived by Sharma [17], Kilbas [3], and Sharma and Jain [18] are special cases of the main findings.

Keywords: Generalized fractional calculus operators, generalized hypergeometric function, generalized M-Series, Wright generalized hypergeometric function.
2010 Mathematics Subject Classification: 26A33, 33C20, 33C65, 33E12.

1 Introduction

The Wright generalized hypergeometric function [20] is given by

$$ {}_p\psi_q(z) = {}_p\psi_q\left[z\left|\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix}\right.\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+nA_i)}{\prod_{j=1}^{q}\Gamma(\beta_j+nB_j)}\frac{z^n}{n!} \tag{1}$$

where $A_i > 0\ (i=\overline{1,p}),\ B_j > 0\ (j=\overline{1,q});\ \alpha_i,\beta_j \in C$, and $\sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j \le 1$.
When $A_1 = \ldots = A_p = B_1 = \ldots = B_q = 1$, then (1) reduces to a generalized hypergeometric function ${}_pF_q(.)$ as shown below

$$ {}_p\psi_q\left[z\left|\begin{matrix}(\alpha_1,1),\ldots,(\alpha_p,1)\\(\beta_1,1),\ldots,(\beta_q,1)\end{matrix}\right.\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z) \tag{2}$$

where $p \le q, |z| < \infty; p = q+1,\ |z| < 1;\ p = q+1,$ $|z| = 1;\ Re\left(\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j\right) > 0.$
It is observed that the Riemann-Liouville fractional integral and derivative of the Wright function is also the Wright function but of greater order. Conditions for the existence of the series (1) together with its presentation in terms of the Mellin-Barnes integral and the H-function were established by Mathai and Saxena [7].

The generalized M-Series [18] is defined as

$$ {}_pM_q^{\alpha,\beta}(z) = {}_pM_q^{\alpha,\beta}(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ z) = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{3}$$

where $z,\alpha,\beta \in C,\ \Re(\alpha) > 0;\ (a_i)_n\ (i=\overline{1,p})$ and $(b_j)_n\ (j=\overline{1,q})$ are the Pochhammer symbols. The series (3) is defined when none of the parameters $(b_j)_n\ \left(j=\overline{1,q}\right)$, is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (3) is convergent for all z if $p \le q$, it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = q+1$ and divergent, if $p > q+1$. When $p = q+1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters. Further detailed account of the M-Series can be found in the paper [18].
The fractional integral operator involving various special functions, have been found of significant importance and applications in various sub-fields of application mathematical analysis.
The generalized M-series can be represented as a special case of the Wright generalized hypergeometric function

* Corresponding author e-mail: dinesh_dino03@yahoo.com

${}_p\psi_q(z)$ and of the Fox H-function [7] as shown below.

$$ {}_pM_q^{\alpha,\beta}(a_1,...,a_p;\, b_1,...,b_q;\, z) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\, {}_{p+1}\psi_{q+1}\left[z\left|\begin{matrix}(a_1,1),...,(a_p,1),(1,1)\\(b_1,1),...,(b_q,1),(\beta,\alpha)\end{matrix}\right.\right] \tag{4}$$

$$ = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} H^{1,p+1}_{p+1,q+2}\left[-z\left|\begin{matrix}(1-\alpha_j,1;1)_{1,p},(0,1)\\(0,1),(1-\beta_j,1)_{1,q},(1-\beta,\alpha)\end{matrix}\right.\right]. \tag{5}$$

Furthermore, if we set $p=q=1, b=1$ and $a=\gamma$ where $\gamma \in C$ in (3), then we obtain the generalized Mittag-Leffler function, as given below.

$$ E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{(1)_n}\frac{z^n}{\Gamma(\alpha n+\beta)} = {}_1M_1^{\alpha,\beta}(\gamma;1;z). \tag{6}$$

Generalized fractional calculus operators:
Let $\alpha,\alpha',\beta,\beta',\gamma \in C$, $x>0$, then the left-sided $(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma})$ and right-sided $(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma})$ generalized fractional integral operators of a function $f(x)$ for $Re(\gamma)>0$ is defined by Saigo and Maeda [12], in the following form:

$$ \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\mathrm{d}t, \tag{7}$$

$$ \left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\mathrm{d}t, \tag{8}$$

These operators reduce to the Saigo fractional integral operators [11, 14] due to the following relations:

$$ I_{0+}^{\alpha,0,\beta,\beta',\gamma}f(x) = I_{0+}^{\gamma,\alpha-\gamma,-\beta}f(x) \qquad (\gamma \in C), \tag{9}$$

and

$$ I_{-}^{\alpha,0,\beta,\beta',\gamma}f(x) = I_{-}^{\gamma,\alpha-\gamma,-\beta}f(x) \qquad (\gamma \in C). \tag{10}$$

Let $\alpha,\alpha',\beta,\beta',\gamma \in C$, and $x \in R_+$, then the generalized fractional differentiation operators [12] involving the Appell function F_3 as a kernel are defined by the following equations:

$$ \left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{11}$$

$$ = \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}f\right)(x),\ (Re(\gamma)>0; n=[Re(\gamma)]+1), \tag{12}$$

$$ \left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{13}$$

$$ = (-1)^n\frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}f\right)(x),\ (Re(\gamma)>0; n=[Re(\gamma)]+1) \tag{14}$$

These operators reduce to the Saigo fractional derivative operators [11, 16] as

$$ \left(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x), \quad (Re(\gamma)>0); \tag{15}$$

$$ \left(D_{-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x), \quad (Re(\gamma)>0). \tag{16}$$

Further [[12], p. 394, Eqns. (4.18) and (4.19)] we also have

$$ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1} = \Gamma\left[\begin{matrix}\rho,\ \rho+\gamma-\alpha-\alpha'-\beta,\ \rho+\beta'-\alpha'\\ \rho+\gamma-\alpha-\alpha',\ \rho+\gamma-\alpha'-\beta,\ \rho+\beta'\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{17}$$

where
$Re(\gamma)>0, Re(\rho)>\max\left[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')\right]$,
and

$$ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1} = \Gamma\left[\begin{matrix}1+\alpha+\alpha'-\gamma-\rho,\ 1+\alpha+\beta'-\gamma-\rho,\ 1-\beta-\rho\\ 1-\rho,\ 1+\alpha+\alpha'+\beta'-\gamma-\rho,\ 1+\alpha-\beta-\rho\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{18}$$

where
$Re(\gamma)>0, Re(\rho)<1+\min\left[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\right]$.
Here, we have used the symbol $\Gamma\left[\begin{matrix}\cdots\\ \cdots\end{matrix}\right]$ representing the fraction of many Gamma functions.
Recently, Srivastava and Saxena [19] have discussed the operators of fractional integration and their applications. Similarly, generalized fractional calculus formulae of the Aleph-function associated with the Appell function F_3 is given by Saxena et al. [16], and Ram & Kumar [8].

2 Generalized fractional integration of the generalized *M*-Series

In this section we derive the left and right-sided generalized fractional integration formulas of the generalized *M*-Series.

Theorem 1 *Let* $\alpha,\alpha',\beta,\beta',\gamma,\xi,\eta,\sigma \in C$, $x>0$, $\mu>0, z\in\Re$; $Re(\xi)>0$, $Re(\gamma)>0$ *and* $a_j, b_j \in C$, $(i=1,...,p; j=1,...,q)$, *then we have the following relation:*

$$ \left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}(zt^{\mu})\right)\right\}(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} \times {}_{p+4}\psi_{q+4}\left[zx^{\mu}\left|\begin{matrix}(a_1,1),...,(a_p,1),(\sigma,\mu),(\sigma+\gamma-\alpha-\alpha'-\beta,\mu),(\sigma+\beta'-\alpha',\mu),(1,1)\\(b_1,1),...,(b_q,1),(\sigma+\gamma-\alpha-\alpha',\mu),(\sigma+\gamma-\alpha'-\beta,\mu),(\sigma+\beta',\mu),(\eta,\xi)\end{matrix}\right.\right]. \tag{19}$$

provided each member of the equation exists.

Proof. Following the definition of left-sided Saigo-Maeda fractional integral as given in (7), we have the following relation:

$$ \left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}(zt^{\mu})\right)\right\}(x)$$

$$= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'+\sigma-1} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) {}_pM_q^{\xi,\eta}\left(zt^{\mu}\right) dt.$$

By virtue of (3) and (17); and interchanging the order of integration and summations, evaluating the inner integral with the help of Beta function and using Gauss summation theorem, it becomes

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)\Gamma(\mu n+\sigma)\Gamma(\mu n+\sigma+\gamma-\alpha-\alpha'-\beta)}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)\Gamma(\mu n+\sigma+\gamma-\alpha-\alpha')\Gamma(\mu n+\sigma+\gamma-\alpha'-\beta)}$$

$$\times \frac{\Gamma(\mu n+\sigma+\beta'-\alpha')\Gamma(n+1)}{\Gamma(\mu n+\sigma+\beta')\Gamma(\xi n+\eta)} \frac{(zx^{\mu})^n}{n!}$$

Following the definition of the Wright generalized hypergeometric function as given in (1), we obtain (19). This completes the proof of the Theorem 1.

On setting $p=q=1$; $a=\lambda\in C$; and $b=1$ in (19), then we obtained the following interesting result:

Corollary 1*Let* $\alpha,\alpha',\beta,\beta',\gamma,\xi,\eta,\sigma \in C$, $x>0$, $\mu>0$, $z\in\Re$; $Re(\xi)>0$, $Re(\gamma)>0$ *and* $a_j,b_j\in C$, $(i=1,\ldots,p; j=1,\ldots,q)$, *then there holds the following formula:*

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}E_{\xi,\eta}^{\lambda}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\lambda)}$$

$$\times\, {}_4\psi_4\left[zx^{\mu}\left|\begin{matrix}(\lambda,1)(\sigma,\mu),(\sigma+\gamma-\alpha-\alpha'-\beta,\mu),(\sigma+\beta'-\alpha',\mu),(1,1)\\(\sigma+\gamma-\alpha-\alpha',\mu),(\sigma+\gamma-\alpha'-\beta,\mu),(\sigma+\beta',\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (20)$$

If we set $\sigma=\eta$ and $\mu=\xi$ in Theorem 1, then we obtain an interesting result as given following:

Corollary 2*Let* $\alpha,\alpha',\beta,\beta',\gamma,\xi,\eta\in C$, $x>0$, $z\in\Re$; $Re(\xi)>0$, $Re(\gamma)>0$ *and* $a_j,b_j\in C$, $(i=1,\ldots,p; j=1,\ldots,q)$, *then we have the following relation:*

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\eta-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\xi}\right)\right)\right\}(x) = \frac{x^{\eta-\alpha-\alpha'+\gamma-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} \times$$

$${}_{p+3}\psi_{q+3}\left[zx^{\xi}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(\eta+\gamma-\alpha-\alpha'-\beta,\xi),(\eta+\beta'-\alpha',\xi),(1,1)\\(b_1,1),\ldots,(b_q,1),(\eta+\gamma-\alpha-\alpha',\xi),(\eta+\gamma-\alpha'-\beta,\xi),(\eta+\beta',\xi)\end{matrix}\right.\right]. \quad (21)$$

In view of the relation (9), then we arrive at the following corollary concerning left-sided Saigo fractional integral operator discussed by Sharma [17].

Corollary 3*Let* $\alpha,\beta,\gamma,\sigma,\xi,\eta\in C$, $\mu>0, x>0$, $z\in\Re$, *and* $Re(\alpha)>0, Re(\xi)>0$, *then we have the following result:*

$$\left\{I_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\beta-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\, {}_{p+3}\psi_{q+3}\left[zx^{\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(\sigma,\mu),(\sigma+\gamma-\beta,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(\sigma-\beta,\mu),(\sigma+\alpha+\gamma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (22)$$

Theorem 2*Let* $\alpha,\alpha',\beta,\beta',\gamma,\xi,\eta,\sigma\in C$, $x>0$, $\mu>0, z\in\Re$; $Re(\xi)>0$, $Re(\gamma)>0$ *and* $a_j,b_j\in C$, $(i=1,\ldots,p; j=1,\ldots,q)$, *then we have the following relation:*

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\, {}_{p+4}\psi_{q+4}\left[zx^{-\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(1+\alpha+\alpha'-\gamma-\sigma,\mu),(1+\alpha+\beta'-\gamma-\sigma,\mu),(1-\beta-\sigma,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(1-\sigma,\mu),(1+\alpha+\alpha'+\beta'-\gamma-\sigma,\mu),(1+\alpha-\beta-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (23)$$

provided each member of the equation exists.

*Proof.*Following the definition of right-sided Saigo-Maeda fractional integral as given in (8), we have the following relation:

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x)$$

$$= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) {}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right) dt.$$

By virtue of (3); and following similar lines to that of Theorem 1, we obtain (23).

By putting $p=q=1$; $a=\lambda\in C$; and $b=1$ in (23), then we obtained the following interesting result:

Corollary 4*Let* $\alpha,\alpha',\beta,\beta',\gamma,\xi,\eta,\sigma\in C$, $x>0$, $\mu>0$, $z\in\Re$; $Re(\xi)>0$, $Re(\gamma)>0$ *and* $a_j,b_j\in C$, $(i=1,\ldots,p; j=1,\ldots,q)$, *then there holds the following formula:*

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}E_{\xi,\eta}^{\lambda}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\lambda)}$$

$$\times\, {}_4\psi_4\left[zx^{-\mu}\left|\begin{matrix}(\lambda,1),(1+\alpha+\alpha'-\gamma-\sigma,\mu),(1+\alpha+\beta'-\gamma-\sigma,\mu),(1-\beta-\sigma,\mu),(1,1)\\(1-\sigma,\mu),(1+\alpha+\alpha'+\beta'-\gamma-\sigma,\mu),(1+\alpha-\beta-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (24)$$

In view of the relation (10), then we arrive at the following corollary concerning right-sided Saigo fractional integral operator discussed by Sharma [17].

Corollary 5*Let* $\alpha,\beta,\gamma,\sigma,\xi,\eta\in C, x>0,\mu>0,z\in\Re$ and $Re(\xi)>0, Re(\alpha)>0$, *then we get the following:*

$$\left\{I_{-}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma-\beta-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\, {}_{p+3}\psi_{q+3}\left[zx^{-\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(1+\beta-\sigma,\mu),(1+\gamma-\sigma,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(1-\sigma,\mu),(1+\alpha+\beta+\gamma-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (25)$$

Remark 1*We can also obtain results concerning Riemann-Liouville and Erdélyi-Kober fractional integral operators [11, 13, 14] by putting* $\beta=-\alpha$ *and* $\beta=0$ *respectively in Corollary 3 and 5.*

3 Generalized fractional differentiation of the generalized M-Series

In this section we derive the left and right-sided generalized fractional differentiation formulas of the generalized M-Series.

Theorem 3*Let* $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\xi,\eta,\sigma \in C$, $\mu > 0$, $x > 0$, $z \in \Re$; $Re(\xi) > 0$, $Re(\gamma) > 0$ *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$, *then we have the following formula:*

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\alpha+\alpha'-\gamma-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\ {}_{p+4}\psi_{q+4}\left[zx^{\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(\sigma,\mu),(\sigma+\alpha+\alpha'+\beta'-\gamma,\mu),(\sigma+\alpha-\beta,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(\sigma+\alpha+\alpha'-\gamma,\mu),(\sigma+\alpha+\beta'-\gamma,\mu),(\sigma-\beta,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (26)$$

provided each member of the equation exists.

*Proof.*Following the definition of left-sided Saigo-Maeda fractional differentiation as given in (12), we get

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) =$$

$$\frac{d^k}{dx^k}\left\{I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x),$$

where $k = [Re(\gamma)+1]$. Interchanging the order of integration and summations, using (12) and (17) and making some simplification, we arrive at

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{d^k}{dx^k}\frac{x^{\sigma+\alpha+\alpha'-\gamma+k-1}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}$$

$$\times\ \sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)\Gamma(\mu n+\sigma)\Gamma(\mu n+\sigma+\alpha+\alpha'+\beta'-\gamma)}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)\Gamma(\mu n+\sigma+\alpha+\alpha'-\gamma+k)\Gamma(\mu n+\sigma+\alpha+\beta'-\gamma)}$$

$$\frac{\Gamma(\mu n+\sigma+\alpha-\beta)\Gamma(n+1)}{\Gamma(\mu n+\sigma-\beta)\Gamma(\xi n+\eta)}\frac{(zx^{\mu})^n}{n!}.$$

By using $\frac{d^k}{dx^k}x^m = \frac{\Gamma(m+1)}{\Gamma(m-k+1)}x^{m-k}$, where $m \geq k$ in the above expression, and following the definition of the Wright generalized hypergeometric function as given in (1), we obtain (26). This completes the proof of the Theorem 3.

On setting $p = q = 1$; $a = \lambda \in C$; and $b = 1$ in (26), then we obtained the following interesting result:

Corollary 6*Let* $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\xi,\eta,\sigma \in C$, $x > 0$, $\mu > 0$, $z \in \Re$; $Re(\xi) > 0$, $Re(\gamma) > 0$ *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$, *then there holds the following formula:*

$$\left\{D_{0+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}\left(t^{\sigma-1}\,E_{\xi,\eta}^{\lambda}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\alpha+\alpha^{'}-\gamma-1}}{\Gamma(\lambda)}$$

$$\times\ {}_4\psi_4\left[zx^{\mu}\left|\begin{matrix}(\lambda,1),(\sigma,\mu),(\sigma+\alpha+\alpha'+\beta'-\gamma,\mu),(\sigma+\alpha-\beta,\mu),(1,1)\\(\sigma+\alpha+\alpha'-\gamma,\mu),(\sigma+\alpha+\beta'-\gamma,\mu),(\sigma-\beta,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (27)$$

If we set $\sigma = \eta$ and $\mu = \xi$ in Theorem 3, then we obtain an interesting result as given below.

Corollary 7*Let* $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\xi,\eta, \in C$, $x > 0$, $z \in \Re$; $Re(\xi) > 0$, $Re(\gamma) > 0$, *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$, *then we have the following result:*

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\eta-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\xi}\right)\right)\right\}(x) = \frac{x^{\eta+\alpha+\alpha'-\gamma-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\ \times$$

$${}_{p+3}\psi_{q+3}\left[zx^{\xi}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(\eta+\alpha+\alpha'+\beta'-\gamma,\xi),(\eta+\alpha-\beta,\xi),(1,1)\\(b_1,1),\ldots,(b_q,1),(\eta+\alpha+\alpha'-\gamma,\xi),(\eta+\alpha+\beta'-\gamma,\xi),(\eta-\beta,\xi)\end{matrix}\right.\right]. \quad (28)$$

In view of the relation (15), then we arrive at the following corollary concerning left-sided Saigo fractional differentiation operator.

Corollary 8*Let* $\alpha,\beta,\gamma,\sigma,\eta,\xi \in C$, $\mu > 0$, $x > 0$, $z \in \Re$; $Re(\alpha) > 0$, $Re(\xi) > 0$ *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$, *then we have the following:*

$$\left\{D_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\beta-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\ {}_{p+3}\psi_{q+3}\left[zx^{\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(\sigma,\mu),(\sigma+\alpha+\beta+\gamma,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(\sigma+\beta,\mu),(\sigma+\gamma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (29)$$

Theorem 4*Let* $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\xi,\eta,\sigma \in C$, $x > 0$, $\mu > 0, z \in \Re$; $Re(\xi) > 0$, $Re(\gamma) > 0$ *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$ *then we have the following relation:*

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\alpha+\alpha'-\gamma-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times\ {}_{p+4}\psi_{q+4}\left[zx^{-\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(1-\alpha-\alpha'+\gamma-\sigma,\mu),(1-\alpha'-\beta+\gamma-\sigma,\mu),(1+\beta'-\sigma,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(1-\sigma,\mu),(1-\alpha-\alpha'-\beta+\gamma-\sigma,\mu),(1-\alpha'+\beta'-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (30)$$

provided each member of the equation exists.

*Proof.*Following the definition of right-sided Saigo-Maeda fractional differentiation as given in (14), we get

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x)$$

$$= (-1)^k\frac{d^k}{dx^k}\left\{I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x),$$

where $k = [Re(\gamma)+1]$.

Using (14) and (18) and making some simplification, we obtain (30). This completes the proof of the Theorem 4.

By setting $p = q = 1$; $a = \lambda \in C$; and $b = 1$ in (30), then we obtained the following interesting result:

Corollary 9*Let* $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\xi,\eta,\sigma \in C$, $x > 0$, $\mu > 0$, $z \in \Re$; $Re(\xi) > 0$, $Re(\gamma) > 0$ *and* $a_j, b_j \in C$, $(i = 1,\ldots,p; j = 1,\ldots,q)$, *then there holds the following formula:*

$$\left\{D_{-}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}\left(t^{\sigma-1}\,E_{\xi,\eta}^{\lambda}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\alpha+\alpha^{'}-\gamma-1}}{\Gamma(\lambda)}$$

$$\times\ {}_4\psi_4\left[zx^{-\mu}\left|\begin{matrix}(\lambda,1),(1-\alpha-\alpha'+\gamma-\sigma,\mu),(1-\alpha'-\beta+\gamma-\sigma,\mu),(1+\beta'-\sigma,\mu),(1,1)\\(1-\sigma,\mu),(1-\alpha-\alpha'-\beta+\gamma-\sigma,\mu),(1-\alpha'+\beta'-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \quad (31)$$

In view of the relation (16), then we arrive at the following corollary concerning right-sided Saigo fractional differentiation operator.

Corollary 10 *Let* $\alpha,\beta,\gamma,\sigma,\xi,\eta \in C,\ \mu > 0, x > 0, z \in \Re$; *and* $Re(\xi) > 0, Re(\alpha) > 0$, *then we get the following:*

$$\left\{D_{-}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\,{}_pM_q^{\xi,\eta}\left(zt^{-\mu}\right)\right)\right\}(x) = \frac{x^{\sigma+\beta-1}\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$
$$\times\ {}_{p+3}\psi_{q+3}\left[zx^{-\mu}\left|\begin{matrix}(a_1,1),\ldots,(a_p,1),(1-\beta-\sigma,\mu),(1+\alpha+\gamma-\sigma,\mu),(1,1)\\(b_1,1),\ldots,(b_q,1),(1-\sigma,\mu),(1-\beta+\gamma-\sigma,\mu),(\eta,\xi)\end{matrix}\right.\right]. \tag{32}$$

Remark 2 *We can also obtain results concerning Riemann-Liouville and Erdélyi-Kober fractional derivative operators [11] by putting* $\beta = -\alpha$ *and* $\beta = 0$ *respectively in Corollary 8 and 10.*

4 Conclusion

In the present paper we investigate the generalized fractional calculus involving F_3 hypergeometric function of the M-Series. We can also obtain the number of special functions as the special cases of our main results, which are related with M-Series and Wright generalized hypergeometric function.

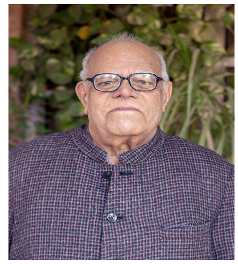
Acknowledgments

The authors wish to thank the referees for their useful suggestions towards the improvement of the paper.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of Integral Transforms*, McGraw-Hill, New York, **1** (1954).
[2] H.J. Haubold, A.M. Mathai, and R.K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math. (Article ID 298628) (2011), 1-51.
[3] A.A. Kilbas, *Fractional calculus of the generalized Wright function*, Fract. Calc. Appl. Anal., **8** (2005), 113-126.
[4] A.A. Kilbas and M. Saigo, *Fractional integrals and derivatives of Mittag-Leffler type function*, Doklady Akad. Nauk Belarusi, **39** (1995), 22-26.
[5] D. Kumar and J. Daiya, *Generalized Fractional Differentiation of* $\bar{H}$*- Function Involving General Class of Polynomials*, Int. J. Pure Appl. Sci. Technol., **16** (2013), 42-53.
[6] C.F. Lorenzo and T.T. Hartley, *Generalized function for the fractional calculus*, NASA/TP-1999-209424, (1999).
[7] A.M. Mathai and R.K. Saxena, *The H-function with Applications in Statistics and other Disciplines*, John Wiley and Sons, Inc., New York, (1978).
[8] J. Ram and D. Kumar, *Generalized fractional integration of the ℵ-function*, J. Rajasthan Acad. Phy. Sci., **10** (2011), 373-382.
[9] J. Ram and D. Kumar, *Generalized fractional integration involving Appell hypergeometric of the product of two H-functions*, Vijanana Prishad Anusandhan Patrika, **54** (2011), 33-43.
[10] E.D. Rainville, *Special Functions*, Chelsea Publishing Company, Bronx, New York, (1960).
[11] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep., College General Ed. Kyushu Univ., **11** (1978), 135-143.
[12] M. Saigo and N. Maeda, *More generalization of fractional calculus* Transform Methods and Special Functions, Varna, Bulgaria, (1996), 386-400.
[13] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives*, Theory and Applications, Gordon and Breach, Yverdon et alibi, (1993).
[14] R.K. Saxena, J. Ram, and D. Kumar, *Generalized Fractional Integration of the Product of two ℵ-Functions Associated with the Appell Function* F_3, ROMAI Journal, **9**(1) (2013), 147158.
[15] R.K. Saxena, J. Ram, and D. Kumar, *On the Two-Dimensional Saigo-Maeda fractional calculus associated with Two-Dimensional Aleph Transform*, Le Matematiche, **68** (2013), 267-281.
[16] R.K. Saxena, J. Ram, and D. Kumar, *Generalized fractional differentiation of the Aleph-Function associated with the Appell function* F_3, Acta Ciencia Indica, **38** (2012), 781-792.
[17] K. Sharma, *An introduction to the generalized fractional integration*, Bol. Soc. Paran. Mat., **30** (2012), 85-90.
[18] M. Sharma and R. Jain, *A note on a generalized M-Series as a special function of fractional calulus*, Fract. Calc. Appl. Anal., **12** (2009), 449-452.
[19] H.M. Srivastava and R.K. Saxena, *Operators of fractional integration and their applications*, Appl. Math. Comput., **118** (2001), 1-52.
[20] E.M. Wright, *The asymptotic expansion of generalized hypergeometric function*, J. London Math. Soc., **10** (1935), 286-293.

Dinesh Kumar obtained M.Sc. (2007), M.Phil. (2008) and Ph.D. (2012) degrees from Jai Narain Vyas University, Jodhpur, India. He has published more than 20 research papers in National and International peer reviewed journals so far. Currently, he is an Assistant Professor of Mathematics at the Department of Applied Sciences, Pratap University, Jaipur, India. His research interest includes Fractional Calculus, Special Functions, Integral Transforms, q-Fractional Calculus, Mathematical Physics, Differential Equation, Basic Hypergeometric functions, and Operation Research. He is an active reviewer for Kuwait Journal of Science, International Journal of Mathematics Analysis and Applications, American Association for science and Technology, USA, Mathematics and Decision Sciences, Global Journal of Science Frontier Research, Global Journals Inc. (USA), and Abstract and Applied Analysis (Hindawi Publication Corporation). He is member of editorial board for Blue Eyes Intelligence Engineering & Sciences Publication, International Journal of Mathematics Analysis and Applications, American Association for science and Technology, USA, and many more.



R. K. Saxena obtained M.Sc. (1956), Ph.D. (1962) and D.Sc. (1971) from Jai Narain Vyas University, Jodhpur, India. He is Professor (Emeritus) of Mathematics at Jai Narain Vyas University, Jodhpur, India. Before retirement he has been Professor and Head, Department of Mathematics and Statistics and Dean, Faculty of Science at Jai Narain Vyas University, Jodphur, India. He has published more than 400 research papers in international esteemed journals and he has also published 5 monographs. He is Reviewer and for Referee for Mathematical Reviews, USA (American Mathematical Society), and Zentralblatt fuer Mathematik, Germany, since 1966. He is acting as a referee for a number of reputed Indian and foreign journals. He is also the member of the editorial board for number of International and National journals. He is also a member of many learned societies. Already twenty three candidates have received Ph.D. degree under his supervision and guidance and three post doctoral fellows have completed their researches under his guidance. He has been invited by number of international universities. His research interest includes Fractional Calculus, Special Functions, Integral Transforms, Statistical distributions, Generalized functions, Reaction-diffusion models, Differential Equation, Basic Hypergeometric functions, and Fractional differential equations.