Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/030201

On Asymptotically Ideal ϕ-Statistical Equivalent Sequences of Fuzzy Real Numbers

Bipan Hazarika[1] *and Ayhan Esi*[2,∗]

[1] Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791112, Arunachal Pradesh, India
[2] Adiyaman University, Science and Art Faculty, Department of Mathematics,Adiyaman-02040, Turkey

Received: 28 May 2015, Revised: 3 Jun. 2015, Accepted: 4 Jun. 2015
Published online: 1 Jul. 2015

Abstract: In this paper we introduce some definitions which are the natural combination of the definition of asymptotic equivalence, statistical convergence, ϕ-statistical convergence of fuzzy real numbers and ideal. In addition, we introduce asymptotically ideal equivalent sequences of fuzzy real numbers and established some relations related to this concept. Finally we introduce the notion of Cesàro Orlicz asymptotically equivalent sequences of fuzzy real numbers and establish their relationship with other classes.

Keywords: Asymptotic equivalence, statistical convergence, ϕ-sequence, ideal convergence, fuzzy number, Orlicz function

1 Introduction

Pobyvancts [33] introduced the concepts of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. Marouf [26] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [31] extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş [32] introduced the concepts of an asymptotically lacunary statistical equivalent sequences of real numbers. Braha [1] extend the definitions presented in [32] to Δ^m-lacunary statistical equivalent real sequences. Savaş [38] introduced the concepts of asymptotically generalized statistical equivalent sequences of fuzzy numbers. Kumar and Sharma [25] was introduced the generalized equivalent sequences of real numbers using ideals and studied some basic properties of this notion.

Actually the idea of statistical convergence was formerly given under the name ”almost convergence” by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [53]. The concept was formally introduced by Fast [10], Steinhaus [41] and later was reintroduced by Schoenberg [40] and also independently by Buck [2]. A lot of developments have been made in this areas after the works of Šalát [35] and Fridy [12]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory.

Kostyrko et. al [21] introduced the notion of I-convergence with the help of an admissible ideal I denotes the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Quite recently, Das et al.[6], unified these two approaches to indroduce new concepts I-statistical convergence, I-lacunary statistical convergence and investigated some of its consequences. More applications of ideals we refer to [3, 7, 9, 13, 14, 16, 17, 18, 19, 20, 23, 36, 37, 43, 44, 45, 46].

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A,B \in I$, we have $A\cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathscr{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin \mathscr{F}$, for each $A,B \in \mathscr{F}$, we have $A\cap B \in \mathscr{F}$ and each $A \in \mathscr{F}$ and each $B \supset A$, we have $B \in \mathscr{F}$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathscr{F} = \mathscr{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of $P(\mathbb{N})$ can be found

∗ Corresponding author e-mail: aesi23@hotmail.com

© 2015 NSP
Natural Sciences Publishing Cor.

in Kostyrko, et.al [21]. Recall that a sequence $x=(x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\}\in I$ for every $\varepsilon>0$ ([21]). In this case we write $I-\lim x_k=\ell$. If we take $I=I_f=\{A\subseteq\mathbb{N}:A$ is a finite subset $\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I=I_\delta=\{A\subseteq\mathbb{N}:\delta(A)=0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [52] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [27] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [5], Mursaleen and Basarir [29], Mursaleen [28], Nanda [30], Çanak [4], Savas [39], Roy and Sen [34], Tripathy and Dutta [48,49], Tripathy and Baruah [50], Tripathy et al [47], Tripathy and Sen [51] and many others.

Let $C(\mathbb{R}^n)=\{A\subset\mathbb{R}^n:A \text{ is compact and convex set}\}$. The space $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A+B=\{a+b:a\in A,b\in B\}$$

and

$$\gamma A=\{\gamma a:a\in A\} \text{ for } A,B\in C(\mathbb{R}^n) \text{ and } \gamma\in\mathbb{R}.$$

The Hausdorff distance between A and B in $C(\mathbb{R}^n)$ is defined by

$$\delta_\infty(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||,\sup_{b\in B}\inf_{a\in A}||a-b||\right\}.$$

It is well-known that $(C(\mathbb{R}^n),\delta_\infty)$ is a complete metric space.

A fuzzy number is a function X from $\mathbb{R}^n$ to $[0,1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of $\{X\in\mathbb{R}^n:X(x)>0\}$ is compact. These properties imply that for each $0<\alpha\leq 1$, the α-level set

$$X^\alpha=\{X\in\mathbb{R}^n:X(x)>\alpha\}$$

is a non-empty compact, convex subset of $\mathbb{R}^n$ with support $X^{0.}$

If $\mathbb{R}^n$ is replaced by $\mathbb{R}$, then obviously the set $C(\mathbb{R}^n)$ is reduced to the set of all closed bounded intervals $A=[\underline{A},\overline{A}]$ on R, and also

$$\delta_\infty(A,B)=\max(|\underline{A}-\underline{B}|,|\overline{A}-\overline{B}|).$$

Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces the addition $X+Y$ and the scalar multiplication λX in terms of α-level sets, by

$$[X+Y]^\alpha=[X]^\alpha+[Y]^\alpha$$

and

$$[\lambda X]^\alpha=\lambda[X]^\alpha \text{ for each } 0\leq\alpha\leq 1.$$

The set $\mathbb{R}$ of real numbers can be embedded in $L(\mathbb{R})$ if we define $\overline{r}\in L(\mathbb{R})$ by

$$\overline{r}(t)=\begin{cases}1, & \text{if } t=r;\\ 0, & \text{if } t\neq r\end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows:

$$rX(t)=\begin{cases}X(r^{-1}t), & \text{if } r\neq 0;\\ 0, & \text{if } r=0\end{cases}$$

Define a map $d:L(\mathbb{R})\times L(\mathbb{R})\to\mathbb{R}$ by

$$d(X,Y)=\sup_{0\leq\alpha\leq 1}\delta_\infty(X^\alpha,Y^\alpha).$$

For $X,Y\in L(\mathbb{R})$ define $X\leq Y$ if and only if $X^\alpha\leq Y^\alpha$ for any $\alpha\in[0,1]$. It is known that $(L(\mathbb{R}),d)$ is complete metric space (see [27]).

A sequence $u=(u_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k\in\mathbb{N}$ (see [27]). We denote by w^F the set of all sequences $u=(u_k)$ of fuzzy numbers.

Definition 1.[27] *A sequence $u=(u_k)$ of fuzzy numbers is said to be bounded if the set $\{u_k:k\in\mathbb{N}\}$ of fuzzy numbers is bounded.*

We denote by ℓ_∞^F the set of all bounded sequences $u=(u_k)$ of fuzzy numbers.

Definition 2.[27] *A sequence $u=(u_k)$ of fuzzy numbers is said to be convergent to a fuzzy number u_0 if for every $\varepsilon>0$ there is a positive integer k_0 such that $d(u_k,u_0)<\varepsilon$ for $k>k_0$.*

We denote by c^F the set of all convergent sequences $u=(u_k)$ of fuzzy numbers. It is straightforward to see that $c^F\subset\ell_\infty^F\subset w^F$.

Throughout the paper, we denote I is an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

Now we recall the following difinitions.

© 2015 NSP
Natural Sciences Publishing Cor.

Definition 3. *A real or complex number sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

Definition 4. *[26] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 5. *[31] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{S^L}{\sim} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 6. *[25] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically I-equivalent of multiple L provided that for each $\varepsilon > 0$*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \in I$$

denoted by $x \overset{I-[C_1]^L}{\sim} y$ and simply strongly asymptotically I-equivalent if $L = 1$.

Definition 7. *[24] A sequence $u = (u_k)$ of fuzzy numbers is said to be I-convergent to a fuzzy number u_0 if for each $\varepsilon > 0$ the set*

$$A = \{k \in \mathbb{N} : d(u_k, u_0) \ge \varepsilon\} \in I.$$

In this paper we define asymptotically I-statistical equivalent, asymptotically I-ϕ-statistical equivalent sequences of fuzzy real numbers and establish some basic results regarding the notions. In last section we introduce the concepts of Cesaro Orlicz asymptotically I-equivalent,Orlicz asymptotically ϕ-statistical equivalent sequences of fuzzy real numbers and establish some relationships with other classes.

2 Asymptotically statistical equivalent sequences using ideals

In this section, we define I-statistical convergence, asymptotically I-equivalent and asymptotically I-statistical equivalent sequences of fuzzy real numbers and obtain some analogous results from these new definitions point of views.

Definition 8. *[38] Two non-negative sequences $u = (u_k)$ and $v = (v_k)$ of fuzzy real numbers are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : d\left(\frac{x_k}{y_k}, L\right) \ge \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{\mathscr{S}^L}{\sim} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 9. *A sequence (u_k) of fuzzy real numbers is said to be I-statistically convergent to a fuzzy real number u_0 for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \le n : d(u_k, u_0) \ge \varepsilon\}| \ge \delta \right\} \in I.$$

In this case we write $I(\mathscr{S}) - \lim u_k = u_0$.

Definition 10. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be asymptotically I-equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\left\{ k \in \mathbb{N} : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon \right\} \in I,$$

denoted by $(u_k) \overset{I^L}{\sim} (v_k)$ and simply asymptotically I-equivalent if $L = 1$.

Lemma 1. *Let $I \subset P(\mathbb{N})$ be an admissible ideal. Let $(u_k), (v_k) \in \ell_\infty^F$ with $I - \lim_k u_k = \bar{0} = I - \lim_k v_k$ such that $(u_k) \overset{I^L}{\sim} (v_k)$. Then there exists a sequence $(w_k) \in \ell_\infty^F$ with $I - \lim_k w_k = \bar{0}$ such that $(u_k) \overset{I^L}{\sim} (w_k) \overset{I^L}{\sim} (v_k)$.*

Definition 11. *Two non-negative sequences (u_k) and (v_k) of fuzzy numbers are said to be asymptotically I-statistical equivalent of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \le n : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon \right\} \right| \ge \delta \right\} \in I,$$

denoted by $(u_k) \overset{I(\mathscr{S})^L}{\sim} (v_k)$ and simply asymptotically I-statistical equivalent if $L = 1$.

Definition 12. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be strongly asymptotically Cesaro I-equivalent (or $I - [C_1]$-equivalent) of multiple L provided that for every $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \ge \delta \right\} \in I$$

denoted by $(u_k) \overset{I-[C_1]^L}{\sim} (v_k)$ and simply strongly asymptotically Cesàro I-equivalent if $L = 1$.

© 2015 NSP
Natural Sciences Publishing Cor.

Theorem 1.*Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers. If $(u_k)\in \ell_\infty^F$ and $(u_k)\overset{I(\mathscr{S})^L}{\sim}(v_k)$. Then $(u_k)\overset{I-[C_1]^L}{\sim}(v_k)$.*

*Proof.*Suppsoe that $(u_k)\in \ell_\infty^F$ and $(u_k)\overset{I(\mathscr{S})^L}{\sim}(v_k)$. Then we can assume that

$$\sup_k d\left(\frac{u_k}{v_k},L\right)\le K \text{ for all } k.$$

Let $\varepsilon>0$. Then we have

$$\left|\frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k},L\right)\right|\le \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k},L\right)$$

$$\le \frac{1}{n}\sum_{\substack{k=1\\ d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon}}^{n} d\left(\frac{u_k}{v_k},L\right)+\frac{1}{n}\sum_{\substack{k=1\\ d\left(\frac{u_k}{v_k},L\right)<\varepsilon}}^{n} d\left(\frac{u_k}{v_k},L\right)$$

$$\le K.\frac{1}{n}\left|\left\{k\le n:\ d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|+\frac{1}{n}.n.\varepsilon.$$

Consequently for any $\delta>\varepsilon>0$, δ and ε are independent, we have

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k},L\right)\ge\delta\right\}$$

$$\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\ge\frac{\delta_1}{K}\right\}\in I$$

where $\delta_1=\delta-\varepsilon>0$. This shows that $(u_k)\overset{I-[C_1]^L}{\sim}(v_k)$.

Theorem 2.*Let I be a non-trivial admissible ideal. Suppose for given $\delta>0$ and every $\varepsilon>0$*

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|<\delta\right\}\in\mathscr{F}$$

then $(u_k)\overset{I(\mathscr{S})^L}{\sim}(v_k)$.

*Proof.*Let $\delta>0$ be given. For every $\varepsilon>0$, choose n_1 such that

$$\frac{1}{n}\left|\left\{0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|<\frac{\delta}{2},\ \text{for all } n\ge n_1. \tag{1}$$

It is sufficient to show that there exists n_2 such that for $n\ge n_2$

$$\frac{1}{n}\left|\left\{0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|<\frac{\delta}{2}.$$

Let $n_0=\max\{n_1,n_2\}$. The the relation (1) will be true for $n>n_0$. If m_0 chosen fixed, then we get

$$\left|\left\{0\le k\le m_0-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|=M.$$

Now for $n>m_0$ we have

$$\frac{1}{n}\left|\left\{0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|$$

$$\le\frac{1}{n}\left|\left\{0\le k\le m_0-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|$$

$$+\frac{1}{n}\left|\left\{m_0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|$$

$$\le\frac{M}{n}+\frac{1}{n}\left|\left\{m_0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\le\frac{M}{n}+\frac{\delta}{2}.$$

Thus for sufficiently large n

$$\frac{1}{n}\left|\left\{m_0\le k\le n-1: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\le\frac{M}{n}+\frac{\delta}{2}<\delta.$$

This established the result.

3 Cesàro Orlicz asymptotically ideal ϕ-statistical equivalent sequences

In this section we define the notion of Cesàro Orlicz asymptotically ϕ-statistical equivalent and Orlicz asymptotically ideal ϕ-statistical equivalent sequences of fuzzy real numbers and established some interesting relationship between these notions.

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma)=1$ for $n\in\sigma$ and $p_n(\sigma)=0$ otherwise. Further

$$P_s=\left\{\sigma\in P:\sum_{n=1}^{\infty}p_n(\sigma)\le s\right\},$$

i.e. P_s is the set of those σ whose support has cardinality at most s, and we get

$$\Phi=\{\phi=(\phi_n):0<\phi_1\le\phi_n\le\phi_{n+1} \text{ and } n\phi_{n+1}\le(n+1)\phi_n\}.$$

We define

$$\tau_s=\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}u_k.$$

Now we give the following definitions.

Definition 13.*A sequence $u=(u_k)$ of fuzzy real numbers is said to be ϕ-summable to ℓ if $\lim_s\tau_s=\ell$.*

Definition 14.*A sequence $u=(u_k)$ of fuzzy real numbers is said to be strongly ϕ-summable to ℓ if*

$$\lim_{s\to\infty}\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}d(u_k,\ell)=0.$$

In this case we write $u_k\overset{[\phi]}{\to}\ell$ and $[\phi]$ denote the set of all strongly ϕ-summable sequences.

© 2015 NSP
Natural Sciences Publishing Cor.

Definition 15. *Let $E \subseteq \mathbb{N}$. The number*

$$\delta_{\phi}(E) = \lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : k \in E\}|$$

is said to be the ϕ-density of E. It is clear that $\delta_{\phi}(E) \leq \delta(E)$.

Definition 16. *A sequence $u = (u_k)$ of fuzzy real numbers is said to be ϕ-statistical convergent to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$*

$$\lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : d(u_k, \ell) \geq \varepsilon\}| = 0.$$

In this case we write $\mathscr{S}_{\phi} - \lim_k u_k = \ell$ or $u_k \overset{\mathscr{S}_{\phi}}{\to} \ell$ and $\mathscr{S}_{\phi}$ denote the set of all ϕ-statisticallly convergent sequences.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M is said to satisfy the $\Delta_2 - condition$ for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$,then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$(see [22]).

Now we define the following asymptotic ϕ-statistical equivalence sequences of fuzzy real numbers.

Definition 17. *Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be Cesàro Orlicz asymptotically equivalent of multiple L provided that*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$ and simply Cesàro Orlicz asymptotically equivalent if $L = 1$.

Definition 18. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be Cesàro Orlicz asymptotically I-equivalent of multiple L provided that for every $\delta > 0$*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I-[C_1]^L(M)}{\sim} (v_k)$ and simply Cesàro Orlicz asymptotically I-equivalent if $L = 1$.

Definition 19. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be Orlicz asymptotically ϕ-equivalent of multiple L provided that*

$$\lim_s \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$ and simply Orlicz asymptotically ϕ-equivalent if $L = 1$.

Definition 20. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be Orlicz asymptotically ideal ϕ-equivalent (or $I - [\phi]$-equivalent) of multiple L provided that for every $\delta > 0$*

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I-[\phi]^L(M)}{\sim} (v_k)$ and simply Orlicz asymptotically I-$[\phi]$-equivalent if $L = 1$.

Definition 21. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be asymptotically ϕ-statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_s \frac{1}{\phi_s} \left|\left\{k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathscr{S}_{\phi}^L}{\sim} (v_k)$ and simply asymptotically ϕ-statistical equivalent if $L = 1$.

Definition 22. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be asymptotically I-ϕ-statistical equivalent of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$*

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s} \left|\left\{k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I-\mathscr{S}_{\phi}^L}{\sim} (v_k)$ and simply asymptotically I-ϕ-statistical equivalent if $L = 1$.

Definition 23. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be Orlicz asymptotically ϕ-statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_s \frac{1}{\phi_s} \left|\left\{k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathscr{S}_{\phi}^L(M)}{\sim} (v_k)$ and simply Orlicz asymptotically ϕ-statistical equivalent if $L = 1$.

Definition 24. *Two non-negative sequences (u_k) and (v_k) of fuzzy real numbers are said to be Orlicz asymptotically I-ϕ-statistical equivalent of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$*

$$\left\{s \in \mathbb{N} : \frac{1}{\phi_s} \left|\left\{k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I-\mathscr{S}_{\phi}^L(M)}{\sim} (v_k)$ and simply Orlicz asymptotically I-ϕ-statistical equivalent if $L = 1$.

© 2015 NSP
Natural Sciences Publishing Cor.

Theorem 3. *Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers and M be an Orlicz function. Then*

(a) $(u_k) \overset{I-[C_1]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathscr{S})^L}{\sim} (v_k)$.
(b) $(u_k) \overset{I(\mathscr{S})^L}{\sim} (v_k)$ *implies* $(u_k) \overset{I-[C_1]^L(M)}{\sim} (v_k)$, *if M is finite.*

Proof.(a) Suppose that $(u_k) \overset{I-[C_1]^L(M)}{\sim} (v_k)$ and let $\varepsilon > 0$ be given, then we can write

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right) \geq \frac{1}{n}\sum_{\substack{k=1\\ d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$\geq \frac{M(\varepsilon)}{n}\left|\left\{k\leq n : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|.$$

Consequently for any $\eta > 0$, we have

$$\left\{n\in\mathbb{N} : \frac{1}{n}\left|\left\{k\leq n : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right| \geq \frac{\eta}{M(\varepsilon)}\right\}$$

$$\subseteq \left\{n\in\mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq\eta\right\}\in I.$$

Hence $(u_k) \overset{I(\mathscr{S})^L}{\sim} (v_k)$.

(b) Suppose that M is finite and $(u_k) \overset{I(\mathscr{S})^L}{\sim} (v_k)$. Since M is finite then there exists a real number $K > 0$ such that $\sup_t M(t) \leq K$. Moreover for any $\varepsilon > 0$ we can write

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right) = \frac{1}{n}\Bigg[\sum_{\substack{k=1\\ d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$+\sum_{\substack{k=1\\ d\left(\frac{u_k}{v_k},L\right)<\varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\Bigg]$$

$$\leq \frac{K}{n}\left|\left\{k\leq n : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right| + M(\varepsilon).$$

Now applying $\varepsilon\to 0$, then the result follows.

Theorem 4. *Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers and (ϕ_s) be a nondecreasing sequence of positive real numbers such that $\phi_s\to\infty$ as $s\to\infty$ and $\phi_s\leq s$ for every $s\in\mathbb{N}$. Then*

$$(u_k) \overset{\mathscr{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathscr{S}_\phi^L}{\sim} (v_k).$$

*Proof.*By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists a $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathscr{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\frac{1}{\phi_s}\left|\left\{k\in\sigma,\sigma\in P_s : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|$$

$$=\frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k\leq s : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|$$

$$-\frac{1}{\phi_s}\left|\left\{k\in\{1,2,...s\}-\sigma,\sigma\in P_s : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|$$

$$\leq\frac{1+a}{a}\frac{1}{s}\left|\left\{k\leq s : d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|$$

$$-\frac{1}{\phi_s}\left|\left\{k_0\in\{1,2,...s\}-\sigma,\sigma\in P_s : d\left(\frac{u_{k_0}}{v_{k_0}},L\right)\geq\varepsilon\right\}\right|.$$

This completes the proof of the theorem.

Theorem 5. *Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers and let M be an Orlicz function satifies the Δ_2-conditions. Then* $(u_k) \overset{\mathscr{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathscr{S}_\phi^L(M)}{\sim} (v_k)$.

*Proof.*By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists a $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathscr{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\frac{1}{\phi_s}\left|\left\{k\in\sigma,\sigma\in P_s : M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq\varepsilon\right\}\right|$$

$$=\frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k\leq s : M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq\varepsilon\right\}\right|$$

$$-\frac{1}{\phi_s}\left|\left\{k\in\{1,2,...s\}-\sigma,\sigma\in P_s : M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq\varepsilon\right\}\right|$$

$$\leq\frac{1+a}{a}\frac{1}{s}\left|\left\{k\leq s : M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq\varepsilon\right\}\right|$$

$$-\frac{1}{\phi_s}\left|\left\{k_0\in\{1,2,...s\}-\sigma,\sigma\in P_s : M\left(d\left(\frac{u_{k_0}}{v_{k_0}},L\right)\right)\geq\varepsilon\right\}\right|. \tag{2}$$

Since M satisfies the Δ_2-conditions, it follows that

$$M\left(d\left(\frac{u_k}{v_k},L\right)\right)\leq K.d\left(\frac{u_k}{v_k},L\right)$$

for some constant $K > 0$ in both the cases where $d\left(\frac{u_k}{v_k},L\right)\leq 1$ and $d\left(\frac{u_k}{v_k},L\right)\geq 1$.

© 2015 NSP
Natural Sciences Publishing Cor.

In first case it follows from the definition of Orlicz function and for the second case we have

$$d\left(\frac{u_k}{v_k},L\right)=2.L^{(1)}=2^2.L^{(2)}=...=2^s.L^{(s)}$$

such that $L^{(s)}\leq 1$. Using the Δ_2-conditions of Orlicz functions we get the following estimation

$$M\left(d\left(\frac{u_k}{v_k},L\right)\right)\leq T.L^{(s)}.M(1)=K.d\left(\frac{u_k}{v_k},L\right), \qquad (3)$$

where K and T are constants. The proof of the theorem follows from the relations (2) and (3).

Theorem 6.*Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers. Let M be an Orlicz function and $k\in\mathbb{Z}$ such that $\phi_s\leq[\phi_s]+k$, $\sup_s\frac{[\phi_s]+k}{\phi_{s-1}}<\infty$. Then $(u_k)\overset{\mathscr{S}_\phi^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{\mathscr{S}^L}{\sim}(v_k)$.*

*Proof.*If $\sup_s\frac{[\phi_s]+k}{\phi_{s-1}}<\infty$, then there exists $K>0$ such that $\frac{[\phi_s]+k}{\phi_{s-1}}<K$ for all $s\geq 1$. Let n be an integer such that $\phi_{s-1}<n\leq\phi_s$. Then for every $\varepsilon>0$, we have

$$\frac{1}{n}\left|\left\{k\leq n: d\left(\frac{u_k}{v_k},L\right)\geq\varepsilon\right\}\right|$$

$$\leq\frac{1}{n}\left|\left\{k\leq n: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|$$

$$\leq\frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k\leq\phi_s: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|$$

$$\leq\frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k\in\sigma,\sigma\in P_{[\phi_s]+k}: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|$$

$$\leq\frac{K}{[\phi_s]+k}\left|\left\{k\in\sigma,\sigma\in P_{[\phi_s]+k}: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|.$$

This established the result.

Theorem 7.*Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a)$(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$.

(b)$\sup_s\frac{\phi_s}{\phi_{s-1}}<\infty$ for every $s\in\mathbb{N}$, then $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$.

Proof.(a) From definition of the sequence (ϕ_s) it follows that $\inf_s\frac{s}{s-\phi_s}\geq 1$. Then there exists $a>0$ such that

$$\frac{s}{\phi_s}\leq\frac{1+a}{a}.$$

Then we get the following relation

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$=\frac{s}{\phi_s}.\frac{1}{s}\sum_{k=1}^{n}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$-\frac{1}{\phi_s}\sum_{k\in\{1,2,...s\}-\sigma,\sigma\in P_s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$\leq\frac{1+a}{a}\frac{1}{s}\sum_{k=1}^{s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$-\frac{1}{\phi_s}\sum_{k_0\in\{1,2,...s\}-\sigma,\sigma\in P_s}M\left(d\left(\frac{u_{k_0}}{v_{k_0}},L\right)\right).$$

Since $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$ and M is continuous, letting $s\to\infty$ on the last relation we get

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

Hence $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$.

(b) Suppose that $\sup_s\frac{\phi_s}{\phi_{s-1}}<\infty$ then there exists $A>0$ such that $\frac{\phi_s}{\phi_{s-1}}<A$ for all $s\geq 1$. Suppose $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ there exists $R>0$ such that for every $s\geq R$

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)<\varepsilon.$$

We can also find a constant $K>0$ such that

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s}M\left(d\left(\frac{u_k}{v_k},L\right)\right)<K \text{ for all } s\in\mathbb{N}.$$

Let n be any integer with $\phi_{s-1}<n\leq[\phi_s]$ for every $s>R$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\leq\frac{1}{\phi_{s-1}}\sum_{k=1}^{[\phi_s]}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$=\frac{1}{\phi_{s-1}}\left(\sum_{k=1}^{[\phi_1]}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+\sum_{[\phi_1]}^{[\phi_2]}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$+...+\sum_{[\phi_{s-1}]}^{[\phi_s]}M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$

$$\leq\frac{\phi_1}{\phi_{s-1}}\left(\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in P^{(1)}}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)$$

$$+\frac{\phi_2}{\phi_{s-1}}\left(\frac{1}{\phi_2}\sum_{k\in\sigma,\sigma\in P^{(2)}}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+$$

$$...+\frac{\phi_R}{\phi_{s-1}}\left(\frac{1}{\phi_R}\sum_{k\in\sigma,\sigma\in P^{(R)}}M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)$$

© 2015 NSP
Natural Sciences Publishing Cor.

$$+\ldots+\frac{\phi_s}{\phi_{s-1}}\left(\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P^{(s)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right),$$

where $P^{(t)}$ are sets of integer which have more than $[\phi_t]$ elements for $t\in\{1,2,\ldots s\}$. By taking limit as $n\to\infty$ on the last relation we get

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

It follows that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$.

Theorem 8. *Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{\mathscr{S}^L}{\sim}(v_k)$.
(b)If M satisfies the Δ_2-condition and $(u_k)\in\ell_\infty^F(M)$ such that $(u_k)\overset{\mathscr{S}^L}{\sim}(v_k)$ then $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$.
(c)If M satisfies the Δ_2-condition, then $[C_1]^L(M)\cap\ell_\infty^F(M)=\mathscr{S}^L\cap\ell_\infty^F(M)$, where $\ell_\infty^F(M)=\{(u_k)\in w^F: M(u_k)\in\ell_\infty^F\}$.

Proof.(a) Suppose that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ we have

$$\frac{1}{n}\left|\left\{k\le n: d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|$$

$$\frac{1}{n}\left|\left\{k\le n: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\ge M(\varepsilon)\right\}\right|$$

$$\le\frac{1}{n}\sum_{\substack{k=1\\ M\left(d\left(\frac{u_k}{v_k},L\right)\right)\ge M(\varepsilon)}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\le\frac{1}{n}\sum_{k=1}^{n}M\left(d\left(\frac{u_k}{v_k},L\right)\right).$$

This established the result.
(b) Proof of this part follows from the same techniques used in the Theorem 3 and Theorem 7.

(c) It follows from (a) and (b).

Theorem 9. *Let $(u_k),(v_k)$ be two non-negative sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{\mathscr{S}_\phi^L}{\sim}(v_k)$.
(b)If M satisfies the Δ_2-condition and $(u_k)\in\ell_\infty^F(M)$ such that $(u_k)\overset{\mathscr{S}_\phi^L}{\sim}(v_k)$ then $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$.
(c)If M satisfies the Δ_2-condition, then $[\phi]^L(M)\cap\ell_\infty^F(M)=\mathscr{S}_\phi^L\cap\ell_\infty^F(M)$.

*Proof.*Proof of this theorem follows from the same techniques used in the Theorem 3 and Theorem 8.

Acknowledgement

The authors are grateful to the anonymous referee for a careful reading this article and for helpful comments that improved the presentation of paper.

References

[1] N. L. Braha, On asymptotically Δ^m-lacunary statistical equivalent sequences, Appl. Math. Comput., 219(1)(2012), 280-288.
[2] R.C. Buck, Generalized asymptotic density, Amer. J. Math. 75(1953), 335-346.
[3] H. Çakalli, B. Hazarika, Ideal quasi-Cauchy sequences, Journal of Inequalities and Applications, 2012(2012) pages 11, doi:10.1186/1029-242X-2012-234
[4] Ibrahim Çanak, On the Riesz mean of sequences of fuzzy real numbers, Jour. Intel. Fuzzy Systems, 26(6)(2014), DOI 10.3233/IFS-130938
[5] P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, Fuzzy Sets and Systems, 35(1990), 241-249.
[6] P. Das, E. Savas, S. Ghosal, On generalization of certain summability methods using ideal, Appl. Math. Letters, 24(2011), 1509-1514.
[7] A. J. Dutta, B. C. Tripathy, On I-acceleration convergence of sequences of fuzzy real numbers, Math. Modell. Anal., 17(4)(2012), 549-557.
[8] A. Esi, B. Hazarika, Lacunary Summable Sequence Spaces of Fuzzy Numbers Defined By Ideal Convergence and an Orlicz Function, Afrika Matematika, DOI: 10.1007/s13370-012-0117-3.
[9] A. Esi, B. Hazarika, λ-ideal convergence in intuitionistic fuzzy 2-normed linear space, Journal of Intelligent and Fuzzy Systems, 24(4)(2013), 725-732, DOI: 10.3233/IFS-2012-0592
[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2(1951) 241-244.
[11] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesàro-type summability spaces, Proc. London Math. Soc., 37(3)(1978), 508-520.
[12] J. A. Fridy, On statistical convergence, Analysis, 5(1985) 301-313.
[13] B. Hazarika, Lacunary I-convergent sequence of fuzzy real numbers, The Pacific Jour. Sci. Techno., 10(2)(2009),203-206.
[14] B. Hazarika, Fuzzy real valued lacunary I-convergent sequences, Applied Math. Letters, 25(3)(2012), 466-470.
[15] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, Journal of Intelligent and Fuzzy Systems, 25(1)(2013), 157-166, DOI: 10.3233/IFS-2012-0622.
[16] B. Hazarika, On fuzzy real valued generalized difference I-convergent sequence spaces defined by Musielak-Orlicz function, Journal of Intelligent and Fuzzy Systems, 25(1)(2013), 9-15, DOI: 10.3233/IFS-2012-0609.
[17] B. Hazarika, On σ-uniform density and ideal convergent sequences of fuzzy real numbers, Journal of Intelligent and Fuzzy Systems, 26(2)(2014), 793-799, doi 10.3233/IFS-130769.
[18] B. Hazarika, On λ-ideal convergent interval valued difference classes defined by Musielak-Orlicz function, Acta Mathematica Vietnamica, 38(4)(2013), 627-639.
[19] B. Hazarika, E. Savaş, Some I-convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, Math. Comp. Modell., 54(2011) 2986-2998.

© 2015 NSP
Natural Sciences Publishing Cor.

[20] B. Hazarika, V. Kumar, Bernardo Lafuerza-Guillén, Generalized ideal convergence in intuitionistic fuzzy normed linear spaces, Filomat, 27(5)(2013), 811-820.

[21] P. Kostyrko, T. Šalát, and W. Wilczyński, I-convergence, Real Analysis Exchange, 26(2) (2000-2001), 669-686.

[22] M. A. Krasnoselskii, Y. B. Rutitsky, Convex function and Orlicz spaces, Groningen, Netherland, 1961.

[23] V. Kumar, A. Sharma, Asymptotically lacunary equivalent sequences defined by ideals and modulus function, Mathematical Sciences 2012, 6:23 doi:10.1186/2251-7456-6-23.

[24] V. Kumar , K. Kumar, On the ideal convergence of sequences of fuzzy numbers, Inform. Sci., 178(2008), 4670-4678.

[25] V. Kumar, A. Sharma, On Asymptotically generalized equivalent sequences via ideals, Tamkang Jour. Math., 43(3)(2012) 469-478.

[26] M. S. Marouf, Asymptotic equivalence and summability, Internat. J. Math. Math. Sci., 16(4)(1993) 755-762.

[27] M. Matloka, Sequences of fuzzy numbers , *BUSEFAL*, **28**(1986), 28-37.

[28] M.Mursaleen, Statistical summability $(C,1)$ for sequences of fuzzy real numbers and a Tauberian theorem, Journal of Intelligent and Fuzzy Systems, 21(6)(2010) 379-384.

[29] M. Mursaleen, M. Basarir, On some new sequence spaces of fuzzy numbers, Indian J. Pure Appl. Math., 34(9)(2003), 1351-1357.

[30] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems, 33(1989), 123-126.

[31] R. F. Patterson, On asymptotically statistically equivalent sequences, Demonstratio Math., 36(1) (2003) 149-153.

[32] R.F. Patterson, E. Savaş, On asymptotically lacunary statistically equivalent sequences. Thai Journal of Mathematics, 4(2006), 267-272.

[33] I. P. Pobyvancts, Asymptotic equivalence of some linear transformation defined by a nonnegative matrix and reduced to generalized equivalence in the sense of Cesaro and Abel, Mat. Fiz. 28(1980) 83-87.

[34] S. Roy and M. Sen Some I-convergent multiplier double classes of sequences of fuzzy numbers defined by Orlicz functions, Jour. Intell. Fuzzy Systems, DOI 10.3233/IFS-130832

[35] T. Šalát, On statistical convergence of real numbers, Math. Slovaca, 30(1980), 139-150.

[36] T. Šalát, B.C. Tripathy and M. Ziman, On some properties of I-convergence, Tatra Mt. Math. Publ., 28(2004), 279-286.

[37] T. Šalát, B.C. Tripathy and M. Ziman, On I-convergence field, Indian Jour. Pure Applied Math., 17 (2005),45-54.

[38] E. Savaş, On asymptotically λ-statistical equivalent sequences of fuzzy numbers, New Math. Natural Computation, 3(3)(2007), 301-306.

[39] E. Savas, Some I-convergent double sequence spaces of fuzzy numbers defined by Orlicz function, Jour. Intell. Fuzzy Systems, DOI 10.3233/IFS-130866

[40] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66(1959) 361-375.

[41] H. Steinhaus, Sr la convergence ordinate et la convergence asymptotique, Colloq. Math., 2 (1951) 73-84.

[42] B. C. Tripathy, B. Hazarika, Some I-convergent sequence spaces defined by Orlicz functions, Acta Math. Appl. Sinica, 27(1) (2011) 149-154.

[43] B. C. Tripathy, B. Hazarika, I-monotonic and I-convergent sequences, Kyungpook Math. J. 51(2011), 233-239, DOI 10.5666-KMJ.2011.51.2.233

[44] B. C. Tripathy, B. Hazarika, B. Choudhary, Lacunary I-convergent sequences, Kyungpook Math. J. 52(4)(2012) 473-482.

[45] B. C. Tripathy, M. Sen, S. Nath, I-convergence in probabilistic n-normed space, Soft Compt., 16(2012), 1021-1027.

[46] B. C. Tripathy, S. Mahanta, On I-acceleration convergence of sequences, Jour. Franklin Institute, 347(2010), 1031-1037.

[47] B. C. Tripathy, N.L. Braha and A.J. Dutta, A new class of fuzzy sequences related to the l_p space defined by orlicz function, Jour. Intell. Fuzzy Systems, DOI 10.3233/IFS-130813.

[48] B. C. Tripathy, A. J. Dutta, Bounded variation double sequence space of fuzzy real numbers, Comput. Math. Appl., 59(2)(2010), 1031–1037.

[49] B. C. Tripathy, A. J. Dutta, Lacunary bounded variation sequence of fuzzy real numbers. Journal of Intelligent and Fuzzy Systems 24(1)(2013), 185-189.

[50] B. C. Tripathy, A. Baruah, Nörlund and Riesz mean of sequences of fuzzy real numbers. Appl. Math. Lett. 23(5)(2010) 651-655.

[51] B. C. Tripathy, M. Sen, On fuzzy I-convergent difference sequence spaces, Journal of Intelligent and Fuzzy Systems 25(3)(2013) 643-647.

[52] L. A. Zadeh, Fuzzy sets, Inform. Control, 8(1965), 338-353.

[53] A. Zygmund, Trigonometric Series, Cambridge Univ. Press, Cambridge, UK, (1979).

Bipan Hazarika is an Associate Professor Department of Mathematics at Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh, India. He received his PhD degree in Mathematics from Gauhati University, Guwahati, Assam, India. His main research interests are field of sequences spaces, summability theory, fuzzy analysis and Functional Analysis. He has published more than 50 research papers in several international journals. He is referee and editor of mathematical journals.

© 2015 NSP
Natural Sciences Publishing Cor.

Ayhan Esi was born in Istanbul, Turkey, on March 5, 1965. Ayhan Esi got his B.Sc. from Inonu University in 1987 and M. Sc. and Ph.D. degree in pure mathematics from Elazig University, Turkey in 1990 and 1995, respectively. The field of interests are Summability Theory, Sequences and Series in Analysis and Functional Analysis. In 2000, Esi was appointed to Education Faculty in Gaziantep University. In 2002, Esi was appointed as the head of Department of Mathematics in Science and Art Faculty in Adiyaman of the Inonu University. In 2006, Esi joined the Department of Mathematics of Adiyaman University. He has published more than 100 research papers in several international journals. He is referee and editor of mathematical journals.

© 2015 NSP
Natural Sciences Publishing Cor.