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Majorization for Certain Subclasses of Meromorphic Functions Defined by Linear Operator

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Abstract: In this paper, majorization problem is studied for certain subclasses of meromorphic functions in the punctured unit disk having a pole of order p at the origin. The subclasses under investigation is the meromorphic analogue of the operator defined by Prajapat (2012) on the p-valent analytic function. Several corollaries and consequences of the main results are also considered.

Keywords: Meromorphic function, subordination, univalent function, majorization problem

1 Introduction and Definition

Let $\sum_p$ denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p} \tag{1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^*=\{z\in\mathbb{C}:0<|z|<1\}=\mathbb{U}\setminus\{0\}$$

having a pole of order p at origin. In particular for $p=1$, we write $\sum_1=\sum$.

Let $f(z)$ and $g(z)$ be analytic in the open unit disk $\mathbb{U}$. Then we say that f is majorized by g in $\mathbb{U}$ (see [9]) and we write

$$f(z)\prec\prec g(z)\quad(z\in\mathbb{U}),$$

if there exists a function $\phi(z)$ analytic in $\mathbb{U}$ such that $|\phi(z)|\leq 1$ and

$$f(z)=\phi(z)g(z)\quad(z\in\mathbb{U}). \tag{2}$$

The majorization (2) is closely related to the concept of quasi subordination between analytic functions in $\mathbb{U}$ (see [1]).

For two analytic functions f and g, we say $f(z)$ is subordinate to $g(z)$, written as $f(z)\prec g(z)$ $(z\in\mathbb{U})$, if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$ $(z\in\mathbb{U})$. It follows from this definition that

$$f(z)\prec g(z)\Longrightarrow f(0)=g(0)\text{ and }f(\mathbb{U})\subset g(\mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [10]).

$$f(z)\prec g(z)\ \ (z\in\mathbb{U})\Longleftrightarrow f(0)=g(0)\text{ and }f(\mathbb{U})\subset g(\mathbb{U}).$$

For $f\in\sum_p$, $f^{(q)}$ denote qth order ordinary differential operator given by

$$f^{(q)}(z)=(-1)^q\frac{(p+q-1)!}{(p-1)!}z^{-p-q}+\sum_{k=1}^{\infty}\frac{(k-p)!}{(k-p-q)!}a_{k-p}z^{k-p-q}$$
$$(p\in\mathbb{N},\ q\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ z\in\mathbb{U}^*). \tag{3}$$

Analogue to the operator defined by Prajapat (see [12]) on the p-valent analytic function, we introduce a generalized multiplier transformation operator $\mathscr{I}_p^m(\lambda,l)$ as follows. For $z\in\mathbb{U}^*$,

$$\mathscr{I}_p^{-m}(\lambda,l)f(z)=\tfrac{p+l}{\lambda}z^{-p-\frac{p+l}{\lambda}}\int_0^z t^{\frac{p+l}{\lambda}+p-1}\mathscr{I}_p^{-(m-1)}(\lambda,l)f(t)dt,$$

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$$\vdots$$

$$\mathscr{I}_p^{-2}(\lambda,l)f(z) = \frac{p+l}{\lambda} z^{-p-\frac{p+l}{\lambda}} \int_0^z t^{\frac{p+l}{\lambda}+p-1} \mathscr{I}_p^{-1}(\lambda,l)f(t)dt,$$

$$\mathscr{I}_p^{-1}(\lambda,l)f(z) = \frac{p+l}{\lambda} z^{-p-\frac{p+l}{\lambda}} \int_0^z t^{\frac{p+l}{\lambda}+p-1} f(t)dt,$$

$$\mathscr{I}_p^{0}(\lambda,l)f(z) = f(z),$$

$$\mathscr{I}_p^{1}(\lambda,l)f(z) = \frac{\lambda}{p+l} z^{1-p-\frac{p+l}{\lambda}} \left(z^{\frac{p+l}{\lambda}+p} f(z)\right)'$$

$$\mathscr{I}_p^{2}(\lambda,l)f(z) = \frac{\lambda}{p+l} z^{1-p-\frac{p+l}{\lambda}} \left(z^{\frac{p+l}{\lambda}+p} \mathscr{I}_p^{1}(\lambda,l)f(z)\right)',$$

$$\vdots$$

$$\mathscr{I}_p^{m}(\lambda,l)f(z) = \frac{\lambda}{p+l} z^{1-p-\frac{p+l}{\lambda}} \left(z^{\frac{p+l}{\lambda}+p} \mathscr{I}_p^{m-1}(\lambda,l)f(z)\right)'.$$

Thus for $f \in \sum_p$, we have

$$\mathscr{I}_p^{m}(\lambda,l)f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{\lambda k+p+l}{p+l}\right)^m a_{k-p} z^{k-p}, \quad (4)$$

$(\lambda > 0,\ l > -p,\ p \in \mathbb{N},\ m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\},\ z \in \mathbb{U}^*)$.

It is easily verified from (4) that

$$\lambda z(\mathscr{I}_p^{m}(\lambda,l)f(z))' = (p+l)\mathscr{I}_p^{m+1}(\lambda,l)f(z) \\ -(l+p+\lambda p)\mathscr{I}_p^{m}(\lambda,l)f(z) \quad (\lambda > 0). \quad (5)$$

By using the operator $\mathscr{I}_p^{m}(\lambda,l)$ given by (4), we now introduce a new class of meromorphically p-valent analytic functions defined as follows.

Definition 1. A function $f \in \sum_p$ is said to be in the class $\mathscr{R}_p^{m,q}(\lambda,l,\gamma;A,B)$ $(-1 \le B < A \le 1)$ of meromorphic functions of complex order $\gamma \neq 0$ in $\mathbb{U}^*$ if and only if

$$1 - \frac{1}{\gamma}\left(\frac{z\left(\mathscr{I}_p^{m,q}(\lambda,l)f(z)\right)'}{\mathscr{I}_p^{m,q}(\lambda,l)f(z)} + p + q\right) \prec \frac{1+Az}{1+Bz},$$

$$(q \in \mathbb{N}_0,\ \gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}, p \in \mathbb{N},\ l > -p;\ z \in \mathbb{U}), \quad (6)$$

where $\mathscr{I}_p^{m,q}(\lambda,l)f := \left(\mathscr{I}_p^{m}(\lambda,l)f\right)^{(q)}$ represents the q times derivative of $\mathscr{I}_p^{m}(\lambda,l)f$.

In particular, for $A = 1$ and $B = -1$, we have

$$\mathscr{R}_p^{m,q}(\lambda,l,\gamma,1,-1) = \mathscr{R}_p^{m,q}(\lambda,l,\gamma)$$

$$= \Re\left\{1 - \frac{1}{\gamma}\left(\frac{z(\mathscr{I}_p^{m,q}(\lambda,l)f(z))'}{\mathscr{I}_p^{m,q}(\lambda,l)f(z)} + p + q\right)\right\} > 0.$$

We note that, by specializing the parameters p, m, q and γ, we obtain the following subclasses studied by various authors.

(i) For $m = 0$ and $q = 0$, $\mathscr{R}_p^{0,0}(\lambda,l,\gamma)$ is the class of p-valent meromorphic starlike function of order γ in $\mathbb{U}^*$;

(ii) for $m = 0$ and $q = 1$, $\mathscr{R}_p^{0,1}(\lambda,l,\gamma)$ is the class of p-valent meromorphic convex function of order γ in $\mathbb{U}^*$;

(iii) for $m = 0$, $q = 0$ and $p = 1$, $\mathscr{R}_1^{0,0}(\lambda,l,\gamma) = S(\gamma)$ $(\gamma \in \mathbb{C}^*)$, the class of meromorphic starlike univalent function of order γ;

(iv) for $m = 0$, $q = 1$ and $p = 1$, $\mathscr{R}_1^{0,1}(\lambda,l,\gamma) = K(\gamma)$ $(\gamma \in \mathbb{C}^*)$, the class of meromorphic convex function of order γ;

(v) for $m = 0,\ q = 0,\ p = 1$ and $\gamma = 1-\eta$ $(0 \le \eta < 1)$, $\mathscr{R}_1^{0,0}(\lambda,l,1-\eta) = \sum^*(\eta)$, the class of meromorphic starlike function of order η has been studied in [8];

(vi) for $m = 0,\ q = 1,\ p = 1$ and $\gamma = 1-\eta$ $(0 \le \eta < 1)$, $\mathscr{R}_1^{0,1}(\lambda,l,1-\eta) = \sum_k(\eta)$, the class of meromorphic convex function of order η has been studied in [8].

There are good amount of literature about majorization problems for normalized univalent function and p-valent analytic functions defined by various researchers for different classes. For instance, a majorization problem for the normalized classes of starlike functions have been investigated by Altinas et al. [2] and MacGregor [9]. Goswami and Wang [3], Goyal and Goswami [6] generalized these results for the class of multivalent functions using fractional derivatives. For recent expository work on majorization problem see ([4, 5,7]).

Motivated by the aforementioned works, in this paper the author investigates the majorization problem for the class of meromorphic functions using generalized multiplier transformation operator $\mathscr{I}_p^{m}(\lambda,l)$ which is analogue to the operator defined by Prajapat (see [12]) on the p-valent analytic function.

2 Majorization problem for the class $\mathscr{R}_p^{m,q}(\lambda,l,\gamma;A,B)$

We state and prove the following results.

Theorem 1. Let the function $f \in \sum_p$ and suppose that $g \in \mathscr{R}_p^{m,q}(\lambda,l,\gamma;A,B)$. If $\mathscr{I}_p^{m,q}(\lambda,l)f(z)$ is majorized by $\mathscr{I}_p^{m,q}(\lambda,l)g(z)$ in $\mathbb{U}^*$, then

$$\left|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)\right| \le \left|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)\right| \quad (7)$$

for $|z| < r_1$, where $r_1 = r_1(p,\lambda,l,\gamma;A,B)$ is the smallest positive root of the equation

$$|(A-B)\lambda\gamma - (p+l)B|r^3 - (p+l+2\lambda|B|)r^2 \\ -(|(A-B)\lambda\gamma - (p+l)B| + 2\lambda)r + (p+l) = 0. \quad (8)$$

Proof. Define

$$h(z) = 1 - \frac{1}{\gamma}\left(\frac{z\left(\mathscr{I}_p^{m,q}(\lambda,l)g(z)\right)'}{\mathscr{I}_p^{m,q}(\lambda,l)g(z)} + p + q\right). \quad (9)$$

Since $g \in \mathscr{R}_p^{m,q}(\lambda,l,\gamma;A,B)$, hence by Definition 1 we have

$$h(z) = \frac{1+Aw(z)}{1+Bw(z)} \quad (10)$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ and $w \in \mathscr{P}$, $\mathscr{P}$ denote the well-known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the condition (see [11])

$$w(0) = 0 \ \text{ and } \ |w(z)| \le |z| \quad (z \in \mathbb{U}).$$

From (9) and (10) we have

$$\frac{z\left(\mathscr{I}_p^{m,q}(\lambda,l)g(z)\right)'}{\mathscr{I}_p^{m,q}(\lambda,l)g(z)} = -\frac{p+q+[(A-B)\gamma+(p+q)B]w(z)}{1+Bw(z)}. \tag{11}$$

An application of principle of mathematical induction on (5) gives

$$\lambda z\left(\mathscr{I}_p^{m,q}(\lambda,l)g(z)\right)' = (p+l)\mathscr{I}_p^{m+1,q}(\lambda,l)g(z) - (l+p+\lambda p+\lambda q)\mathscr{I}_p^{m,q}(\lambda,l)g(z). \tag{12}$$

Now using (12) in (11), we get

$$\mathscr{I}_p^{m,q}(\lambda,l)g(z) = \frac{1}{(p+l)-[(A-B)\lambda\gamma-(p+l)B]w(z)} [(p+l)(1+Bw(z))\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)]. \tag{13}$$

Since $|w(z)| \le |z| \ \ (z \in \mathbb{U})$, the equation (13) gives

$$|\mathscr{I}_p^{m,q}(\lambda,l)g(z)| \le \frac{1}{(p+l)-|(A-B)\lambda\gamma-(p+l)B||z|} \left[(p+l)(1+|B||z|)\left|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)\right|\right]. \tag{14}$$

Since $\mathscr{I}_p^{m,q}(\lambda,l)f(z)$ is majorized by $\mathscr{I}_p^{m,q}(\lambda,l)g(z)$ in the punctured unit disk $\mathbb{U}^*$, hence from (2) we have

$$\mathscr{I}_p^{m,q}(\lambda,l)f(z) = \phi(z)\mathscr{I}_p^{m,q}(\lambda,l)g(z). \tag{15}$$

Differentiating both sides of (15) with respect to z and simplifying, we get

$$z\left(\mathscr{I}_p^{m,q}(\lambda,l)f(z)\right)' = \phi(z)z\left(\mathscr{I}_p^{m,q}(\lambda,l)g(z)\right) + z\phi'(z)\mathscr{I}_p^{m,q}(\lambda,l)g(z). \tag{16}$$

Using (12) and (15) in (16) yields

$$\mathscr{I}_p^{m+1,q}(\lambda,l)f(z) = \frac{\lambda}{p+l} z\phi'(z)\mathscr{I}_p^{m,q}(\lambda,l)g(z) + \phi(z)\mathscr{I}_p^{m+1,q}(\lambda,l)g(z). \tag{17}$$

Since $\phi \in \mathscr{P}$, it follows that (see [11])

$$|\phi'(z)| \le \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in \mathbb{U}). \tag{18}$$

Making use of (14) and (18) in (17), we get

$$|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)| \le \left(|\phi(z)| + \frac{\lambda(1-|\phi(z)|^2)}{1-|z|^2} \frac{|z|(1+|B||z|)}{[(p+l)-|(A-B)\lambda\gamma-(p+l)B||z|]}\right)|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)|.$$

Upon setting

$$|z| = r \text{ and } |\phi(z)| = \rho \quad (0 \le \rho \le 1),$$

leads to the inequality

$$|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)| \le \frac{\psi(\rho)}{(1-r^2)(p+l-|(A-B)\lambda\gamma-(p+l)B|r)} |\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)|,$$

where

$$\psi(\rho) = -\lambda r(1+|B|r)\rho^2 + (1-r^2)\left[(p+l)-|(A-B)\lambda\gamma - (p+l)B|r\right]\rho + \lambda r(1+|B|r) \tag{19}$$

takes its maximum value at $\rho = 1$, with $r_1 = r_1(p,\lambda,l,\gamma;A,B)$, where r_1 is the smallest positive root of the equation (8). Furthermore, if $0 \le \sigma \le r_1$, then the function $\chi(\rho)$ defined by

$$\chi(\rho) = -\lambda\sigma(1+|B|\sigma)\rho^2 + (1-\sigma^2)(p+l - |(A-B)\lambda\gamma-(p+l)B|\sigma)\rho + \lambda\sigma(1+|B|\sigma) \tag{20}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\chi(\rho) \le \chi(1) = (1-\sigma^2)\left[p+l-|(A-B)\lambda\gamma-(p+l)B|\sigma\right] \quad (0 \le \rho \le 1, 0 \le \sigma \le r_1).$$

Hence, upon setting $\rho = 1$ in (20), we conclude that (7) of Theorem 1 holds true for $|z| \le |r_1| = r_1(p,\lambda,l,\gamma;A,B)$ where r_1 is the smallest positive root of the equation (8). This complete the proof of Theorem 1. □

Letting $A = 1$ and $B = -1$ in Theorem 1, we have

Corollary 1. Let the function $f \in \sum_p$ and suppose that $g \in \mathscr{R}_p^{m,q}(\lambda,l,\gamma)$. If $\mathscr{I}_p^{m,q}(\lambda,l)f(z)$ is majorized by $\mathscr{I}_p^{m,q}(\lambda,l)g(z)$ in $\mathbb{U}^*$, then

$$|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)| \le |\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)| \quad \text{for} \quad |z| < r_2,$$

where $r_2 = r_2(p,\lambda,l,\gamma)$ is the smallest positive root of the equation

$$|2\lambda\gamma+p+l|r^3 - (p+l+2\lambda)r^2 - (|2\lambda\gamma+p+l|+2\lambda)r + p+l = 0.$$

Putting $m = 0$ in Corollary 1, we get

Corollary 2. Let the function $f \in \sum_p$ and suppose that $g \in \mathscr{R}_p^{0,q}(\lambda,l,\gamma)$. If $f^{(q)}(z) \prec\prec g^{(q)}(z)$ in $\mathbb{U}^*$, then

$$|\mathscr{I}_p^{1,q}(\lambda,l)f(z)| \le |\mathscr{I}_p^{1,q}(\lambda,l)g(z)| \quad \text{for} \quad |z| \le r_3,$$

where

$$r_3 = \frac{k_1 - \sqrt{k_1^2 - 4(p+l)|2\lambda\gamma+p+l|}}{2|2\lambda\gamma+p+l|}$$

and

$$k_1 = |2\lambda\gamma+p+l| + p+l+2\lambda \quad (q \in \mathbb{N}_0, \gamma \in \mathbb{C}^*, \lambda > 0).$$

Further setting $q=0$ and $\lambda=1$ in the above result yields

Corollary 3. Let the function $f\in\sum_p$ and suppose that $g\in\mathscr{R}_p^{0,0}(1,l,\gamma)$. If $f(z)\prec\prec g(z)$ in $\mathbb{U}^*$, then

$$|(2p+l)f(z)+zf'(z)|\leq|(2p+l)g(z)+zg'(z)|\text{ for }|z|\leq r_4$$

where

$$r_4=\frac{k_2-\sqrt{k_2^2-4(p+l)|2\gamma+p+l|}}{2|2\gamma+p+l|}$$

and

$$k_2=|2\gamma+p+l|+2+p+l.$$

Taking $\gamma=1,\ p=0$ and $l=0$, the above corollary reduces to the following.

Corollary 4. Let the function $f\in\sum$ and suppose that $g\in\mathscr{R}_1^{0,0}(1,0,1)$. If $f(z)\prec\prec g(z)$ in $\mathbb{U}^*$, then

$$\left|f(z)+\frac{zf'(z)}{2}\right|\leq\left|g(z)+\frac{zg'(z)}{2}\right|\quad\text{for }|z|\leq r_5$$

where

$$r_5=\frac{3-\sqrt{6}}{3}.$$

3 Majorization problem for the class $\mathscr{R}(\alpha,\gamma)$

Let $\mathscr{R}(\alpha,\gamma)$ be the class of functions $h(z)$ of the form

$$h(z)=1-\sum_{k=1}^{\infty}c_kz^k\quad(c_k\geq0),\tag{21}$$

that are analytic in $\mathbb{U}$ satisfying the inequality

$$|h(z)+\alpha zh'(z)-1|<|\gamma|\quad(z\in\mathbb{U};\ \Re(\alpha)\geq0,\ \gamma\in\mathbb{C}^*).\tag{22}$$

For $\gamma=1-\beta\ \ (0\leq\beta<1)$, the class $\Re(\alpha,\gamma)=\Re(\alpha,1-\beta)$ was considered by Altintas and Owa [1].

We need the following lemma to prove our result:

Lemma 1. (see [2]) If the function $h(z)$ defined by (21) is in the class $\mathscr{R}(\alpha,\gamma)$, then

$$1-\frac{|\gamma|}{1+\Re(\alpha)}|z|\leq|h(z)|\leq1+\frac{|\gamma|}{1+\Re(\alpha)}|z|\quad(z\in\mathbb{U}).\tag{23}$$

Theorem 2.

Let the function $f(z)\in\sum_p$ and $g(z)\in\mathscr{R}(\alpha,\gamma)$ be analytic in $\mathbb{U}$ and suppose that the function $g(z)$ is so normalized that it also satisfies the following inclusion property

$$\frac{\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)}{\mathscr{I}_p^{m,q}(\lambda,l)g(z)}\in\mathscr{R}(\alpha,\gamma).\tag{24}$$

If $\mathscr{I}_p^{m,q}(\lambda,l)f(z)$ is majorized by $\mathscr{I}_p^{m,q}(\lambda,l)g(z)$ in $\mathbb{U}^*$, then

$$|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)|\leq|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)|\quad(|z|<r_6)\tag{25}$$

where $r_6=r_6(p,l,\alpha,\lambda,\gamma)$ is the smallest positive root of the cubic equation

$$(p+l)|\gamma|r^3-(p+l)[1+\Re(\alpha)]r^2-[2\lambda+(p+l)|\gamma|+2\lambda\Re(\alpha)]r+[1+\Re(\alpha)](p+l)=0.\tag{26}$$

Proof.

For appropriately normalized analytic function $g(z)$ satisfying the inclusion property (24), we find from (23) of Lemma 1 that

$$\left|\frac{\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)}{\mathscr{I}_p^{m,q}(\lambda,l)g(z)}\right|\geq1-\frac{|\gamma|}{1+\Re(\alpha)}r\quad(|z|=r,\ 0<r<1),\tag{27}$$

which implies

$$|\mathscr{I}_p^{m,q}(\lambda,l)g(z)|\leq\frac{1+\Re(\alpha)}{1+\Re(\alpha)-|\gamma|r}|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)|\quad(|z|=r,\ 0<r<1).\tag{28}$$

Since $\mathscr{I}_p^{m,q}(\lambda,l)f(z)\ll\mathscr{I}_p^{m,q}(\lambda,l)g(z)\ \ (z\in\mathbb{U}^*)$, there exists an analytic function w with $|w(z)|<1$ such that

$$\mathscr{I}_p^{m,q}(\lambda,l)f(z)=w(z)\mathscr{I}_p^{m,q}(\lambda,l)g(z).\tag{29}$$

Therefore, in view of (28) and proceeding as in the proof of Theorem 1, we have

$$|w'(z)|\leq\frac{1-|w(z)|^2}{1-|z|^2}\quad(z\in\mathbb{U}),\tag{30}$$

and

$$\left|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)\right|\leq\left[|w(z)|+\frac{\lambda}{(p+l)}\frac{(1-|w(z)|^2)(1+\Re(\alpha))r}{(1-r^2)(1+\Re(\alpha)-|\gamma|r)}\right]\left|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)\right|.\tag{31}$$

Taking $|w(z)|=\rho$ in (31), we have

$$|\mathscr{I}_p^{m+1,q}(\lambda,l)f(z)|\leq\frac{\theta(\rho)}{(p+l)(1-r^2)(1+\Re(\alpha)-|\gamma|r)}\left|\mathscr{I}_p^{m+1,q}(\lambda,l)g(z)\right|,\tag{32}$$

where

$$\theta(\rho)=(p+l)(1-r^2)(1+\Re(\alpha)-|\gamma|r)\rho+\lambda r(1+\Re(\alpha))-\lambda r(1+\Re(\alpha))\rho^2\quad(0\leq\rho\leq1),$$

takes on its maximum value at $\rho=1$ with $r_6=r_6(p,l,\alpha,\lambda,\gamma)$ given by (26). Moreover, if $0\leq\eta\leq r_6(p,l,\alpha,\lambda,\gamma)$ where $r_6(p,l,\alpha,\lambda,\gamma)$ is the root of the cubic equation (26) such that $0<r_6(p,l,\alpha,\lambda,\gamma)<1$, then the function $H(\rho)$ defined by

$$H(\rho)=(p+l)(1-\eta^2)(1+\Re(\alpha)-|\gamma|\eta)\rho+\lambda\eta(1+\Re(\alpha))-\lambda\eta(1+\Re(\alpha))\rho^2\quad(0\leq\rho\leq1)\tag{33}$$

is seen to be an increasing function on the interval $0 \leq \rho \leq 1$ so that

$$H(\rho) \leq H(1) = (p+l)(1-\eta^2)(1+\Re(\alpha) - |\gamma|\eta) \quad (0 \leq \rho \leq 1, 0 \leq \eta \leq r_6(p,l,\alpha,\lambda,\gamma)). \tag{34}$$

Therefore, upon setting $\rho = 1$ in (32), we complete the proof of Theorem 2.□

4 Open Problem

In the present paper, we have investigated the majorization problems for the class of p-valent meromorphic function. If we define a class $f \in \mathscr{A}_p$ such that

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (z \in \mathbb{U}),$$

then we need to modify the generalized operator $\mathscr{I}_p^{m,q}(\lambda, l)$ for the class of p-valent analytic function. Further using this modified operator we have to find the new majorization conditions for the modified operator.

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