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He's Semi-Inverse Method and Ansatz Approach to look for Topological and Non-Topological Solutions Generalized Nonlinear Schrödinger Equation

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Abstract: This paper obtains the exact 1-soliton solutions to generalized nonlinear Schrödinger equation. Nonlinear Schrödinger equation has been widely applied in many branches of nonlinear sciences such as nonlinear optics, nonlinear optical fibers and quantum mechanics. So, finding exact solutions of such equations are very helpful in the theories and numerical studies. In this paper, the He's semi-inverse method and the ansatz method are used to establish new exact solutions of generalized nonlinear Schrödinger equation. The results reveal that these methods are very effective and powerful tool for solving nonlinear fractional differential equations arising in mathematical physics.

Keywords: He's semi-inverse method; Ansatz method; Generalized nonlinear Schrödinger equation.

1 INTRODUCTION

In this paper, we consider generalized nonlinear Schrödinger (GNLS) equation [1,2]

$$iq_t + aq_{xx} + bq|q|^2 + icq_{xxx} + id\left(q|q|^2\right)_x = ke^{i(\chi(\xi)-wt)}, \tag{1}$$

where $\xi = \alpha(x - vt)$ is a real function and $a,\ b,\ c,\ d,\ w,\ \alpha,\ v$ are non-zero constants and q is a complex-valued function of two real variables $x,\ t$. The nonlinear Schrödinger's equation describes numerous nonlinear physical phenomena in the field of applied sciences such as solitons in nonlinear optical fibers, solitons in the mean-field theory of Bose-Einstein condensates, rogue waves in oceanography, etc.

In recent years, many powerful methods have been developed to construct exact solutions of nonlinear evolution equations. One of the most effective direct methods to develop the solitary wave solutions of nonlinear evolution equations is the He's variational principle. He's semi-inverse variational principle, which is a direct and effective algebraic method for the computation of soliton solutions, was first proposed by He [3]. This method was further developed by many authors [4,5,6,7,8,9,10,11,12,13]. Biswas et. al., [5,6,7, 8,9,10,11] obtained optical solitons and soliton solutions with higher order dispersion by using the He's variational principle. Using extended tanh-coth, sine-cosine, exp-function and first integral methods [1,2], exact solutions of GNLS equation have been obtained. The aim of this paper is to find new exact solutions of GNLS equation by using the He's semi-inverse variational principle method and the ansatz method [14,15].

2 THE SEMI-INVERSE VARIATIONAL PRINCIPLE METHOD

Let us consider a general nonlinear PDE in the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \frac{\partial^2 u}{\partial t^2}, \ldots\right) = 0, \tag{2}$$

where P is a polynomial in its arguments. Jabbari et. al., [13] have been written the He's semi-inverse method in the following steps:

Step 1: Seek solitary wave solutions of Eq. (2) by taking $u(x,t) = U(\xi),\ \ \xi = x - ct$, and transform Eq. (2) to the

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ordinary differential equation (ODE)

$$Q\left(U,\ \frac{dU}{d\xi},\ \frac{d^2U}{d\xi^2},\ ...\right)=0. \tag{3}$$

Step 2: If possible, integrating Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3: According to He's semi-inverse method, we construct the following trial-functional, we construct the following trial-functional

$$J(U)=\int L\,d\xi, \tag{4}$$

where L is an unknown function of U and its derivatives.
Step 4: By the Ritz method, we can obtain different forms of solitary wave solutions , such as

$$U(\xi)=A\,\text{sech}\,(B\xi)\,,\qquad U(\xi)=A\,\tanh\,(B\xi) \tag{5}$$

and so on. For example in this paper, we search a solitary wave solution in the form

$$U(\xi)=A\,\text{sech}\,(B\xi)\,, \tag{6}$$

where A and B are constants to be further determined. Substituting Eq. (6) into Eq. (4) and making J stationary with respect to A and B results in

$$\frac{\partial J}{\partial A}=0 \tag{7}$$

$$\frac{\partial J}{\partial B}=0 \tag{8}$$

Solving Eqs. (7) and (8), we obtain A and B. Hence, the solitary wave solution (6) is well determined.

2.1 APPLICATION TO THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION

In order to solve Eq. (1), we use the following wave transformation

$$\begin{aligned}&q(x,t)=e^{i(\chi(\xi)-wt)}U(\xi),\\&\chi(\xi)=\beta\xi+x_0,\\&\xi=\alpha\,(x-vt)\,,\end{aligned} \tag{9}$$

where α, w, v, β, x_0 are constants and $U(\xi)$ is real function.
By replacing Eq. (9) into Eq. (1) and separating the real and imaginary parts of the result, we obtain the two following ordinary differential equations:

$$\begin{aligned}&c\alpha^3U'''(\xi)+\left(-\alpha v+2a\beta\alpha^2-3c\alpha^3\beta^2\right)U'(\xi)\\&+3d\alpha U^2(\xi)U'(\xi)=0.\end{aligned} \tag{10}$$

$$\begin{aligned}&\left(a\alpha^2-3c\alpha^3\beta\right)U''(\xi)\\&+\left(\alpha\beta v+w-a\alpha^2\beta^2+c\alpha^3\beta^3\right)U(\xi)\\&+(b-d\alpha\beta)U^3(\xi)-k=0.\end{aligned} \tag{11}$$

Integrating Eq. (10) once, with respect to ξ, yields:

$$\begin{aligned}&c\alpha^2U''(\xi)+(-v+2a\beta\alpha-3c\alpha^2\beta^2)U(\xi)\\&+d\,U^3(\xi)=0,\end{aligned} \tag{12}$$

where we neglect the integration constant.
Since the same function $U(\xi)$ satisfies two Eqs. (11) and (12) , we obtain the following constraint condition:

$$\begin{aligned}&\frac{(a\alpha^2-3c\alpha^3\beta)}{c\alpha^2}\\&=\frac{(\alpha\beta v+w-a\alpha^2\beta^2+c\alpha^3\beta^3)}{(-v+2a\beta\alpha-3c\alpha^2\beta^2)}\\&=\frac{(b-d\alpha\beta)}{d}.\end{aligned} \tag{13}$$

By He's semi-inverse principle [3,4], we can obtain the following variational formulation

$$\begin{aligned}J=&\int_0^\infty[-\frac{c\alpha^2}{2}\left(U'\right)^2\\&+\frac{(-v+2a\beta\alpha-3c\alpha^2\beta^2)}{2}U^2+\frac{d}{4}U^4]d\xi.\end{aligned} \tag{14}$$

By a Ritz-like method, we search a solitary wave solution in the form

$$U(\xi)=A\,\text{sech}\,(B\xi)\,, \tag{15}$$

where A and B are unknown constants to be further determined. Substituting Eq. (15) into Eq. (14), we have

$$\begin{aligned}J=&\int_0^\infty[-\frac{A^2B^2c\alpha^2}{2}\text{sech}^2\,(B\xi)\tanh^2\,(B\xi)\\&+\frac{\left(-v+2a\beta\alpha-3c\alpha^2\beta^2\right)A^2}{2}\text{sech}^2\,(B\xi)\\&+\frac{dA^4}{4}\text{sech}^4\,(B\xi)]d\xi=-\frac{A^2Bc\alpha^2}{6}\\&+\frac{\left(-v+2a\beta\alpha-3c\alpha^2\beta^2\right)A^2}{2B}+\frac{dA^4}{6B}.\end{aligned} \tag{16}$$

Making J stationary with A and B yields

$$\begin{aligned}\frac{\partial J}{\partial A}=&-\frac{ABc\alpha^2}{3}+\frac{\left(-v+2a\beta\alpha-3c\alpha^2\beta^2\right)A}{B}\\&+\frac{2dA^3}{3B}=0,\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\partial J}{\partial B}=&-\frac{A^2c\alpha^2}{6}-\frac{\left(-v+2a\beta\alpha-3c\alpha^2\beta^2\right)A^2}{2B^2}\\&-\frac{dA^4}{6B^2}=0.\end{aligned} \tag{18}$$

From Eqs. (17) and (18), we have

$$\begin{aligned}&A=\pm\sqrt{\frac{2\,(v+3c\alpha^2\beta^2-2a\beta\alpha)}{d}},\\&B=\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c\alpha^2}}.\end{aligned} \tag{19}$$

Using the travelling wave transformation (9), we have the following bright soliton solutions of the Eq. (1):

$$q(x,t)=\pm\sqrt{\frac{2\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right)}{d}}\,\text{sech}\left[\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c}}\,(x-vt)\right]\times e^{i(\beta\alpha(x-vt)-wt+x_0)}. \tag{20}$$

3 ANSATZ APPROACH

This section will utilize the ansatz method to solve GNLS equation. The bright soliton, dark soliton and singular soliton solutions to Eq. (1) will be obtained by the aid of ansatz method. In order to solve Eq. (1) by the ansatz method, we use the following wave transformation

$$q(x,t)=e^{i(\chi(\tau)-wt)}U(\tau),\quad \chi(\tau)=\beta\tau+x_0,\quad \tau=\alpha\,(x-vt). \tag{21}$$

By replacing Eq. (21) into Eq. (1), we have

$$c\alpha^2U''(\tau)+(-v+2a\beta\alpha-3c\alpha^2\beta^2)U(\tau)+d\,U^3(\tau)=0. \tag{22}$$

3.1 BRIGHT SOLITON SOLUTION

For bright soliton, the hypothesis is

$$U(\tau)=A\,\text{sech}^pB\tau, \tag{23}$$

where

$$\tau=\alpha\,(x-vt) \tag{24}$$

The value of the unknown exponent p will fall out during the course of derivation of the soliton solutions. Also A and B are free parameters, while v is the speed of the soliton. Thus from (23), we have

$$\frac{d^2U(\tau)}{d\tau^2}=AB^2p^2\,\text{sech}^pB\tau-AB^2p(p+1)\,\text{sech}^{p+2}B\tau \tag{25}$$

and

$$U^3(\tau)=A^3\,\text{sech}^{3p}B\tau. \tag{26}$$

Substitution of (23) into Eq. (22) leads to

$$+\,c\alpha^2\left\{AB^2p^2\,\text{sech}^pB\tau-AB^2p(p+1)\,\text{sech}^{p+2}B\tau\right\}+(-v+2a\beta\alpha-3c\alpha^2\beta^2)A\,\text{sech}^pB\tau+d\,A^3\,\text{sech}^{3p}B\tau=0. \tag{27}$$

By virtue of balancing principle, on equating the exponents $3p$ and $p+2$, from (27), gives

$$p=1. \tag{28}$$

Next, from (27) setting the coefficients of the linearly independent functions to zero implies

sech^1 coeff.:

$$c\alpha^2AB^2+(-v+2a\beta\alpha-3c\alpha^2\beta^2)A=0, \tag{29}$$

sech^3 coeff.:

$$dA^3-2c\alpha^2AB^2=0.$$

Solving the above equations yields

$$A=\pm\sqrt{\frac{2\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right)}{d}}, \tag{30}$$

and

$$B=\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c\alpha^2}}. \tag{31}$$

Equations (30) and (31) prompts the constraints

$$d\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right)>0, \tag{32}$$

and

$$c\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right)>0, \tag{33}$$

respectively. Thus, the bright 1-soliton solution to Eq. (1) is given by

$$q(x,t)=\pm\sqrt{\frac{2\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right)}{d}}\,\text{sech}\left[\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c}}\,(x-vt)\right]\times e^{i(\beta\alpha(x-vt)-wt+x_0)}, \tag{34}$$

with the constraints (32) and (33).

3.2 TOPOLOGICAL (DARK) SOLITON SOLUTION

The starting hypothesis for dark 1-soliton solution to Eq. (22) is

$$U(\tau)=A\,\tanh^pB\tau, \tag{35}$$

where τ is the same as (24). However, for dark solitons the parameters A and B are indeed free soliton parameters, although v still represents the velocity of the dark soliton. Thus from (35), we have

$$\frac{d^2U(\tau)}{d\tau^2}=AB^2p(p-1)\,\tanh^{p-2}B\tau-2AB^2p^2\tanh^pB\tau+AB^2p(p+1)\tanh^{p+2}B\tau \tag{36}$$

and

$$U^3(\tau) = A^3 \tanh^{3p} B\tau. \quad (37)$$

In this case, substituting this hypothesis (35) into Eq. (22) leads to

$$c\alpha^2\{AB^2 p(p-1)\tanh^{p-2} B\tau$$
$$-2AB^2 p^2 \tanh^p B\tau$$
$$+AB^2 p(p+1)\tanh^{p+2} B\tau\}$$
$$+(-v+2a\beta\alpha-3c\alpha^2\beta^2)A\tanh^p B\tau$$
$$+dA^3\tanh^{3p} B\tau = 0. \quad (38)$$

By balancing the power of $\tanh^{p+2}$ and $\tanh^{3p}$ in Eq. (38) we have:

$$p = 1. \quad (39)$$

Now, from (38), setting the coefficients of the linearly independent functions $\tanh^{(p+j)}\tau$ to zero, where $j = 0,\ 2$, gives

$\tanh^1$ coeff.:

$$-2c\alpha^2 AB + (-v+2a\beta\alpha-3c\alpha^2\beta^2)A = 0, \quad (40)$$

$\tanh^3$ coeff.:

$$2c\alpha^2 AB^2 + dA^3 = 0.$$

Solving the above equations yields

$$A = \pm\sqrt{\frac{v+3c\alpha^2\beta^2-2a\beta\alpha}{d}}, \quad (41)$$

and

$$B = \pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{2c\alpha^2}}. \quad (42)$$

Equations (41) and (42) prompts the constraints

$$d\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right) > 0, \quad (43)$$

and

$$c\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right) > 0, \quad (44)$$

respectively. Thus, the topological 1-soliton solution to Eq. (1) is given by

$$q(x,t) = \pm\sqrt{\frac{v+3c\alpha^2\beta^2-2a\beta\alpha}{d}}$$
$$\tanh\left[\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{2c}}(x-vt)\right]$$
$$\times e^{i(\beta\alpha(x-vt)-wt+x_0)}, \quad (45)$$

with the constraints (43) and (44).

Remark: In this case, comparing our results with Taghizadeh's results [2], it can be seen that our solutions are same.

3.3 SINGULAR SOLITON SOLUTION

For singular soliton, the hypothesis is

$$U(\tau) = A\,\mathrm{csch}^p \tau, \quad (46)$$

where τ is the same as (24). The value of the unknown exponent p will fall out during the course of derivation of the soliton solutions. Also A and B are free parameters, while λ is the speed of the soliton. Substitution of (46) into Eq. (22) leads to

$$c\alpha^2\left\{AB^2p^2\,\mathrm{csch}^p\tau + AB^2p(p+1)\,\mathrm{csch}^{p+2}\tau\right\}$$
$$+(-v+2a\beta\alpha-3c\alpha^2\beta^2)A\,\mathrm{csch}^p B\tau$$
$$+d\,A^3\,\mathrm{csch}^{3p}B\tau = 0. \quad (47)$$

From (47), the balancing principle yields

$$p = 1. \quad (48)$$

Next, from (47) setting the coefficients of the linearly independent functions to zero implies

$$A = \pm\sqrt{\frac{2\left(2a\beta\alpha-v-3c\alpha^2\beta^2\right)}{d}}, \quad (49)$$

and

$$B = \pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c\alpha^2}}. \quad (50)$$

Equations (49) and (50) prompts the constraints

$$d\left(2a\beta\alpha-v-3c\alpha^2\beta^2\right) > 0, \quad (51)$$

and

$$c\left(v+3c\alpha^2\beta^2-2a\beta\alpha\right) > 0, \quad (52)$$

respectively. Thus, the singular 1-soliton solution to Eq. (1) is given by

$$q(x,t) = \pm\sqrt{\frac{2\left(2a\beta\alpha-v-3c\alpha^2\beta^2\right)}{d}}$$
$$\mathrm{csch}\left[\pm\sqrt{\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{c}}(x-vt)\right]$$
$$\times e^{i(\beta\alpha(x-vt)-wt+x_0)}, \quad (53)$$

with the constraints (51) and (52).

4 CONCLUSIONS

In this paper, the He's semi-inverse variational principle method and the ansatz method have been applied to obtain the new exact solutions of generalized nonlinear Schrödinger equation. The results show that these methods are powerful tool for obtaining the exact solutions of complex nonlinear partial differential equations. We have predicted that the He's semi-inverse variational principle method and the ansatz method can be extended to solve many systems of complex nonlinear partial differential equations in mathematical and physical sciences.

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