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On Generalized $I-$ Convergent Paranormed Spaces

Kuldip Raj and Seema Jamwal*

School of Mathematics, Shri Mata Vaishno Devi University Katra - 182320, J&K, India

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Abstract: In the present paper we introduce some generalized $I-$convergent sequence spaces and study some topological and algebraic properties of these spaces. We also make an effort to study some inclusion relations between these spaces.

Keywords: double sequence, $\sigma-$mean, $\sigma-$bounded variation, ideal convergence, paranorm

1 Introduction and Preliminaries

Let w denote the space of all real or complex sequences. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\varepsilon > 0$ there exists some $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \varepsilon, \quad \forall i, j \in N.$$

The study of double sequence spaces was initiated by Bromwich [2] and further generalized and studied by Hardy [6], Moricz [15], Moricz and Rhoades [16], Tripathy ([27], [28]), Başarir and Sonalcan [4] and many others. Quite recently, Zeltser [31] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. For more details about double sequence spaces (see [20], [17],[18]) and references therein. Let l_∞ and c denote the Banach spaces of bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let V denote the space of sequences of bounded variation that is,

$$V = \Big\{ x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, \;\; x_{-1} = 0 \Big\},$$

where V is a Banach space normed by

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|, \quad \text{(see [19])}.$$

Let σ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if
(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k;
(ii) $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$;
(ii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.
In case σ is the translation mapping $n \to n+1$, a $\sigma-$mean is often called a Banach limit (see [3]) and V_σ the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences (see [14]). If $x = (x_k)$, then $Tx = (Tx_k) = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \Big\{ x = (x_k) : \sum_{m=1}^{\infty} t_{m,k}(x) = L \;\text{ uniformally in } k \;\; L = \sigma - \lim x \Big\} \tag{1}$$

where $m \geq 0, \; k > 0$. Consider

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + x_{\sigma^2(k)} + ... + x_{\sigma^m(k)}}{m+1}, \; t_{-1,k} = 0,$$

where $\sigma^m(k)$ denote the mth iterate of $\sigma(k)$ at k. The special case of (1) in which $\sigma(n) = n+1$ was given by Lorentz [[14], Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c.
A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits that is to say, if and only if, for all $k \geq 0, \; j \geq 1$, (see [19])

$$\sigma^j(k) \neq k.$$

* Corresponding author e-mail: kuldipraj68@gmail.com

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x),$$

assuming that $t_{-1,k} = 0$. A straight forward calculation shows (see [21]) that

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_{\sigma^j(k)} - x_{\sigma^{j-1}(k)}), & (m \geq 1), \\ x_k, & (m = 0). \end{cases}$$

For any sequence x, y and scalar λ, we have

$$\phi_{m,k}(x+y) = \phi_{m,k}(x) + \phi_{m,k}(y),$$

$$\phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x).$$

A sequence $x \in l_\infty$ is of σ-bounded variations if and only if

(i) $\sum_{k=0}^{\infty} |\phi_{m,k}(x)|$converges uniformly in m;

(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variations (see [8]):

$$BV_\sigma = \Big\{ x \in l_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{ uniformaly in } k \Big\}.$$

BV_σ is a Banach space normed by

$$\|x\| = \sup_k \sum_{k=0}^{\infty} |\phi_{m,k}(x)| \quad \text{(see [22])}.$$

Subsequently, invariant mean have been studied by Ahmad and Mursaleen [1], Mursaleen et al. ([19],[21]), Raimi [23], Vakeel et al. ([9], [10], [11]), and many others. For the first time, $I-$convergence was studied by Kostyrko et al. [13]. Later on, it was studied by Salat et al. [26], Tripathy and Hazarika [29] and many others.

The notion of difference sequence spaces was introduced by Kızmaz [7], who defined the sequence spaces

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\} \text{ for } Z = c, c_0 \text{ and } l_\infty$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. The notion was further generalized by Et and Çolak [5] by introducing the spaces. Let r be a non-negative integer, then

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\} \text{ for } Z = c, c_0 \text{ and } l_\infty$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. The generalized difference sequence has the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}.$$

Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (Power set of $\mathbb{N}$) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if $\Phi \notin \pounds(I)$ for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

An ideal $I \subseteq 2^{\mathbb{N}}$ is called non trivial if $I \neq 2^{\mathbb{N}}$. A non trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.1. A double sequence $(x_{ij}) \in w$ is said to be I-convergent to a number L if for every $\varepsilon > 0$, the set $\{i, j \in \mathbb{N} : |x_{ij} - L| \geq \varepsilon\} \in I$. In this case we write $I - \lim x_{ij} = L$.

Definition 1.2. A double sequence $(x_{ij}) \in w$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_{ij} = 0$.

Definition 1.3. A double sequence $(x_{ij}) \in w$ is said to be I-Cauchy if for every $\varepsilon > 0$, there exist a number $a = a(\varepsilon)$ and $b = b(\varepsilon)$ such that $\{i, j \in \mathbb{N} : |x_{ij} - x_{ab}| \geq \varepsilon\} \in I$.

Definition 1.4. A double sequence $(x_{ij}) \in w$ is said to be I-bounded if there exist $M > 0$ such that $\{i, j \in \mathbb{N} : |x_{ij}| > M\} \in I$.

Definition 1.5. A double-sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $i, j \in \mathbb{N}$.

Definition 1.6. Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$;
2. $p(-x) = p(x)$ for all $x \in X$;
3. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm ([Theorem 10.4.2, pp. 183] see [30]). For more details about sequence spaces see ([24], [25]) and references therein.

Let $p = (p_{ij})$ be any double bounded sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. In this paper we define the following sequence space:

$$\begin{aligned} &{}_2BV_\sigma^I(u, p, \Delta^r) \\ &= \Big\{ x = (x_{ij}) \in w : \Big\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(u_{ij} \Delta^r x) - L|^{p_{ij}} \geq \varepsilon \Big\} \in I, \\ &\qquad \text{for some } L \in \mathbb{C} \Big\}. \end{aligned}$$

If we take $u = (u_{ij}) = 1$, $p = (p_{ij}) = 1$, for all i, j and $r = 0$ then we get the sequence space defined by Vakeel and Nazneen [12].

The main purpose of this paper is to introduce the sequence space ${}_2BV_\sigma^I(u, p, \Delta^r)$. We have also make an attempt to study some topological, algebraic properties and inclusion relations between the sequence spaces ${}_2BV_\sigma^I(u,p,\Delta^r)$.

2 Main Results

Theorem 2.1. Let $p=(p_{ij})$ be a double bounded sequence of positive real numbers and $u=(u_{ij})$ be a double sequence of strictly positive real numbers. Then the space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is a linear space over the complex field $\mathbb{C}$.

Proof. Let $x=(x_{ij}),\ y=(y_{ij})\in {}_2BV_\sigma^I(u,p,\Delta^r)$ and $\alpha,\ \beta\in\mathbb{C}$. Then for a given $\varepsilon>0$, we have
$\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-L_1|^{p_{ij}}\geq\frac{\varepsilon}{2}\right\}\in I,$
for some$L_1\in\mathbb{C}$, $\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r y)-L_2|^{p_{ij}}\geq\frac{\varepsilon}{2}\right\}\in I,$
for some$L_2\in\mathbb{C}$. Now let
$A_1=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-L_1|^{p_{ij}}\geq\frac{\varepsilon}{2}\right\}\in I,$
for some$L_1\in\mathbb{C}$, $A_2=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r y)-L_2|^{p_{ij}}\geq\frac{\varepsilon}{2}\right\}\in I,$
for some$L_2\in\mathbb{C}$ be such that $A_1^c,A_2^c\in I$. Now consider

$$
\begin{aligned}
&|\phi_{mn,ij}(u_{ij}\Delta^r(\alpha x+\beta y))-(\alpha L_1+\beta L_2)|^{p_{ij}}\\
&=|\phi_{mn,ij}(\alpha u_{ij}\Delta^r x)+\phi_{mn,ij}(\beta u_{ij}\Delta^r y)-\alpha L_1-\beta L_2|^{p_{ij}}\\
&=|\phi_{mn,ij}(\alpha u_{ij}\Delta^r x)-\alpha L_1+\phi_{mn,ij}(\beta u_{ij}\Delta^r y)-\beta L_2|^{p_{ij}}\\
&\leq|\phi_{mn,ij}(\alpha u_{ij}\Delta^r x)-\alpha L_1|^{p_{ij}}+|\phi_{mn,ij}(\beta u_{ij}\Delta^r y)-\beta L_2|^{p_{ij}}\\
&=|\alpha||\phi_{mn,ij}(u_{ij}\Delta^r x)-L_1|^{p_{ij}}+|\beta||\phi_{mn,ij}(u_{ij}\Delta^r y)-L_2|^{p_{ij}}\\
&\leq|\alpha|\frac{\varepsilon}{2}+|\beta|\frac{\varepsilon}{2}\\
&=(|\alpha|+|\beta|)\frac{\varepsilon}{2}\\
&\leq\varepsilon'\ \text{(say)}.
\end{aligned}
$$

This implies that the sequence space
$A_3=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r(\alpha x+\beta y))-(\alpha L_1+\beta L_2)|^{p_{ij}}<\varepsilon'\right\}\in I,$ for some $L_1,\ L_2\in\mathbb{C}$. Hence $(\alpha x+\beta y)\in {}_2BV_\sigma^I(u,p,\Delta^r)$. Therefore ${}_2BV_\sigma^I(u,p,\Delta^r)$ is a linear space over the complex field $\mathbb{C}$. This completes the proof.

Theorem 2.2. Let $p=(p_{ij})$ be a double bounded sequence of positive real numbers and $u=(u_{ij})$ be a double sequence of strictly positive real numbers. Then the space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is a paranormed space, paranormed by

$$g(x_{ij})=\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}.$$

Proof. For $x=(x_{ij})=0,\ \ g(x_{ij})=0$ is trivial. For $x=(x_{ij})\neq0,\ \ g(x_{ij})\neq0$, we have
(i) $g(x)=\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}\geq0,$ for all $x\in {}_2BV_\sigma^I(u,p,\Delta^r)$.
(ii) $g(-x)=\sup_{ij}|\phi_{mn,ij}(-u_{ij}\Delta^r x)|^{p_{ij}}=\sup_{ij}|-\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}=\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}=g(x)$, for all $x\in {}_2BV_\sigma^I(u,p,\Delta^r)$.
(iii) $g(x+y)=\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x+u_{ij}\Delta^r y)|^{p_{ij}}\leq\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}+\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r y)|^{p_{ij}}=g(x)+g(y)$.
(iv) Let λ_{ij} be a sequence of scalars with $\lambda_{ij}\to\lambda$ as $(ij\to\infty)$ and $x\in {}_2BV_\sigma^I(u,p,\Delta^r)$ such that

$$\phi_{mn,ij}(u_{ij}\Delta^r x)\to L\ \text{ as }\ (ij\to\infty)$$

in the sense that

$$g(\phi_{mn,ij}(u_{ij}\Delta^r x)-L)^{p_{ij}}\to0\ \text{ as }\ (ij\to\infty).$$

Therefore

$$
\begin{aligned}
&g(\lambda_{ij}\phi_{mn,ij}(u_{ij}\Delta^r x)-\lambda L)^{p_{ij}}\\
&\leq g(\lambda_{ij}\phi_{mn,ij}(u_{ij}\Delta^r x))^{p_{ij}}-g(\lambda L)^{p_{ij}}\\
&=\lambda_{ij}g(\phi_{mn,ij}(u_{ij}\Delta^r x))^{p_{ij}}-\lambda g(L)^{p_{ij}}\\
&\to0\ \text{ as }\ ij\to\infty.
\end{aligned}
$$

Hence ${}_2BV_\sigma^I(u,p,\Delta^r)$ is a paranormed space. This completes the proof.

Theorem 2.3. The space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is solid and monotone.

Proof. Let $x=(x_{ij})\in {}_2BV_\sigma^I(u,p,\Delta^r)$ and (α_{ij}) be a sequence of scalars with $|\alpha_{ij}|\leq1$, for all $i,j\in\mathbb{N}$. Then we have

$$
\begin{aligned}
|\alpha_{ij}\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}&\leq|\alpha_{ij}||\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}\\
&\leq|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}},\ \forall\, i,j\in\mathbb{N}.
\end{aligned}
$$

The space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is solid follows from the following inclusion relation:
$\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}\geq\varepsilon\right\}$
$\supseteq\left\{i,j\in\mathbb{N}:|\alpha_{ij}\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}\geq\varepsilon\right\}$. Also a sequence is solid implies monotone. Hence the space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is monotone. This completes the proof.

Theorem 2.4. ${}_2BV_\sigma^I(u,p,\Delta^r)$ is a closed subspace of ${}_2l_\infty^I(u,p,\Delta^r)$.

Proof. Let $(x_{ij}^{(bd)})$ be a Cauchy sequence in ${}_2BV_\sigma^I(u,p,\Delta^r)$ such that $x^{(bd)}\to x$. We show that $x\in {}_2BV_\sigma^I(u,p,\Delta^r)$. Since $(x_{ij}^{(bd)})\in {}_2BV_\sigma^I(u,p,\Delta^r)$, then there exist a_{bd} such that

$$\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x^{(bd)})-a_{bd}|^{p_{ij}}\geq\varepsilon\right\}\in I.$$

We need to show that
(i) (a_{bd}) converges to a.

(ii) If $U=\left\{i,j\in\mathbb{N}:|x_{ij}-a|<\varepsilon\right\}$, then $U^c\in I$.

Since $(x_{ij}^{(bd)})$ is a Cauchy sequence in ${}_2BV_\sigma^I(u,p,\Delta^r)$. Then for a given $\varepsilon>0$, their exists $k_0\in\mathbb{N}$ such that

$$\sup_{ij}|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(bd)})-\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(ef)})|^{p_{ij}}<\frac{\varepsilon}{3},$$

$\forall b,d,e,f\ \geq k_0$. For a given $\varepsilon>0$, we have

$B_{bdef}=\Big\{i,j\in\mathbb{N}:$
$|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(bd)})-\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(ef)})|^{p_{ij}}<\frac{\varepsilon}{3}\Big\}$,

$$B_{bd}=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(bd)})-a_{bd}|^{p_{ij}}<\frac{\varepsilon}{3}\right\},$$

$$B_{ef}=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(ef)})-a_{ef}|^{p_{ij}}<\frac{\varepsilon}{3}\right\}.$$

Then B_{bdef}^c, B_{bd}^c and $B_{ef}^c\in I$. Let $B^c=B_{bdef}^c\cap B_{bd}^c\cap B_{ef}^c$, where $B=\{i,j\in\mathbb{N}:|a_{bd}-a_{ef}|<\varepsilon\}$. Then $B^c\in I$. We choose $k_0\in B^c$, then for each $b,d,e,f\geq k_0$, we have

$\{i,j\in\mathbb{N}:|a_{bd}-a_{ef}|<\varepsilon\}\supseteq\Big\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(bd)})-a_{bd}|^{p_{ij}}<\frac{\varepsilon}{3}\Big\}$

$$\cap\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(bd)})-\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(ef)})|^{p_{ij}}<\frac{\varepsilon}{3}\right\}$$

$$\cap\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(ef)})-a_{ef}|^{p_{ij}}<\frac{\varepsilon}{3}\right\}.$$

Then (a_{bd}) is a Cauchy sequence of scalars in $\mathbb{N}$, so their exists a scalar $a\in\mathbb{C}$ such that $(a_{bd})\to a$ as $b,d\to\infty$.

For the next step, let $0<\delta<1$ be given. Then, we show that if $U=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-a|^{p_{ij}}<\delta\right\}$, then $U^c\in I$. Since $\phi_{mn,ij}(u_{ij}\Delta^r x^{(bd)})\to\phi_{mn,ij}(u_{ij}\Delta^r x)$, then their exist a scalar $b_0d_0\in\mathbb{N}$ such that

$$P=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(b_0d_0)})-\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}<\frac{\delta}{3}\right\} \quad (2)$$

which implies that $P^c\in I$. The number b_0d_0 can be so chosen together with , we have

$$Q=\left\{i,j\in\mathbb{N}:|a_{b_0d_0}-a|^{p_{ij}}<\frac{\delta}{3}\right\}$$

such that $Q^c\in I$. Since $\Big\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(b_0d_0)})-a_{b_0d_0}|^{p_{ij}}\geq\delta\Big\}\in I$, then we have a subset S of $\mathbb{N}$ such that $S^c\in I$, where

$$S=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(b_0d_0)})-a_{b_0d_0}|^{p_{ij}}<\frac{\delta}{3}\right\}.$$

Let $U^c=P^c\cap Q^c\cap S^c$, where $U=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-a|^{p_{ij}}<\delta\right\}$, therefore for each $i,j\in U^c$, we have

$\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-a|^{p_{ij}}<\delta\right\}$

$$\supseteq\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(b_0d_0)})-\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}<\frac{\delta}{3}\right\}$$

$$\cap\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x_{ij}^{(b_0d_0)})-a_{b_0d_0}|^{p_{ij}}<\frac{\delta}{3}\right\}$$

$\cap\left\{i,j\in\mathbb{N}:|a_{b_0d_0}-a|^{p_{ij}}<\frac{\delta}{3}\right\}$.

Hence the result ${}_2BV_\sigma^I(u,p,\Delta^r,)\subset{}_2l_\infty^I(u,p,\Delta^r)$ follows. This completes the proof.

Theorem 2.5. The space ${}_2BV_\sigma^I(u,p,\Delta^r)$ is nowhere dense subset of ${}_2l_\infty^I(u,p,\Delta^r)$.

Proof. Proof of the result follows from the previous theorem.

Theorem 2.6. The inclusions ${}_2C_0^I(u,p,\Delta^r)\subset{}_2BV_\sigma^I(u,p,\Delta^r,)\subset{}_2l_\infty^I(u,p,\Delta^r)$ are proper.

Proof. Let $x=(x_{ij})\in{}_2C_0^I(u,p,\Delta^r)$. Then, we have $\Big\{i,j\in\mathbb{N}:|u_{ij}\Delta^r x_{ij}|^{p_{ij}}\geq\varepsilon\Big\}\in I$. Since ${}_2C_0\subset{}_2BV_\sigma$, $x=(x_{ij})\in{}_2BV_\sigma^I(u,p,\Delta^r)$ implies

$$\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}\geq\varepsilon\right\}\in I.$$

Now let

$$A_1=\left\{i,j\in\mathbb{N}:|u_{ij}\Delta^r x_{ij}|^{p_{ij}}<\varepsilon\right\},$$

$$A_2=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)|^{p_{ij}}<\varepsilon\right\}$$

be such that A_1^c, $A_2^c\in I$. As ${}_2l_\infty^I(u,p,\Delta^r)=\left\{x=(x_{ij}):\sup_{ij}|u_{ij}\Delta^r x_{ij}|^{p_{ij}}<\infty\right\}\in I$, taking supremum over i,j we get $A_1^c\subset A_2^c$. Hence

$${}_2C_0^I(u,p,\Delta^r)\subset{}_2BV_\sigma^I(u,p,\Delta^r)\subset{}_2l_\infty^I(u,p,\Delta^r).$$

Next we show that the inclusion is proper. First for ${}_2C_0^I(u,p,\Delta^r)\subset{}_2BV_\sigma^I(u,p,\Delta^r)$. Consider $x\in{}_2BV_\sigma^I(u,p,\Delta^r)$, then by the definition

$${}_2BV_\sigma^I(u,p,\Delta^r)=\Big\{x=(x_{ij})\in w:\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}\geq\varepsilon\right\}\in I,\ \text{for some } L\in\mathbb{C},\Big\}$$

we have

$$\phi_{mn,ij}(u_{ij}\Delta^r x)=t_{mn,ij}(u_{ij}\Delta^r x)-t_{(m-1)(n-1),ij}(u_{ij}\Delta^r x),$$

where

$$t_{mn,ij}(u_{ij}\Delta^r x)=\frac{u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}+....+u_{ij}\Delta^r x_{\sigma^{mn}(ij)}}{mn}.$$

Therefore

$$t_{mn,ij}(u_{ij}\Delta^r x)-t_{(m-1)(n-1),ij}(u_{ij}\Delta^r x)$$
$$=\frac{u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}}{mn}+....+\frac{u_{ij}\Delta^r x_{\sigma^{mn}(ij)}}{mn}-\frac{u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}}{(m-1)(n-1)}+....+\frac{u_{ij}\Delta^r x_{\sigma^{(m-1)(n-1)}(ij)}}{(m-1)(n-1)}$$
$$=\frac{(m-1)(n-1)(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}}{mn(m-1)(n-1)}+....+\frac{u_{ij}\Delta^r x_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)}-\frac{mn(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}}{mn(m-1)(n-1)}+....+\frac{u_{ij}\Delta^r x_{\sigma^{(m-1)(n-1)}(ij)}}{mn(m-1)(n-1)}.$$

On solving we get

$$\phi_{mn,ij}(u_{ij}\Delta^r x)=\frac{mnu_{ij}\Delta^r x_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)}+\frac{(1-m-n)(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{\sigma(ij)}+u_{ij}\Delta^r x_{\sigma^2(ij)}}{mn(m-1)(n-1)}+....+\frac{u_{ij}\Delta^r x_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)}.$$

As σ is a translation map, that is $\sigma(n)=n+1$, we have

$$\phi_{mn,ij}(u_{ij}\Delta^r x)=\frac{mnu_{ij}\Delta^r x_{(i+m)(j+n)}}{mn(m-1)(n-1)}+\frac{(1-m-n)(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{(i+1)(j+1)}}{mn(m-1)(n-1)}\frac{+....+u_{ij}\Delta^r x_{(i+m)(j+n)}}{mn(m-1)(n-1)}.$$

taking limit $i,j\to\infty$, we have

$$\lim_{(i,j)\to\infty}\phi_{mn,ij}(u_{ij}\Delta^r x)=\lim_{(i,j)\to\infty}\Big[(mnu_{ij}\Delta^r x_{(i+m)(j+n)}+(1-m-n)(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{(i+1)(j+1)}\ +....+u_{ij}\Delta^r x_{(i+m)(j+n)})(mn(m-1)(n-1))^{-1}\Big],$$
$$L(mn(m-1)(n-1))=\lim_{(i,j)\to\infty}\Big[mnu_{ij}\Delta^r x_{(i+m)(j+n)}+(1-m-n)(u_{ij}\Delta^r x_{ij}+u_{ij}\Delta^r x_{(i+1)(j+1)}+....+u_{ij}\Delta^r x_{(i+m)(j+n)})\Big].$$

Since $m,n,L\neq 0$, therefore $\lim_{(i,j)\to\infty}\phi_{mn,ij}(u_{ij}\Delta^r x)\neq 0$ which implies that $x\notin {}_2C_0^I(u,p,\Delta^r)$. Hence we get that the inclusion is proper. For ${}_2BV_\sigma^I(u,p,\Delta^r)\subset {}_2l_\infty^I(u,p,\Delta^r)$, the result of this part follows from the proof of the Theorem (2.4). This completes the proof.

Theorem 2.7. The inclusions ${}_2C^I(u,p,\Delta^r)\subset {}_2BV_\sigma^I(u,p,\Delta^r)\subset {}_2l_\infty^I(u,p,\Delta^r)$ are proper.

Proof. Let $x=(x_{ij})\in {}_2C^I(u,p,\Delta^r)$. Then, we have $\left\{i,j\in\mathbb{N}:|u_{ij}\Delta^r x_{ij}-L|^{p_{ij}}\geq\varepsilon\right\}\in I$. Since ${}_2C_0^I(u,p,\Delta^r)\subset {}_2BV_\sigma^I(u,p,\Delta^r)\subset {}_2l_\infty^I(u,p,\Delta^r)$, which implies $x=(x_{ij})\in {}_2BV_\sigma^I(u,p,\Delta^r)$ then

$$\left\{i,j\in\mathbb{N}:|u_{ij}\Delta^r\phi_{mn,ij}(x)-L|^{p_{ij}}\geq\varepsilon\right\}\in I.$$

Now let

$$B_1=\left\{i,j\in\mathbb{N}:|u_{ij}\Delta^r x_{ij}-L|^{p_{ij}}<\varepsilon\right\},$$
$$B_2=\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}<\varepsilon\right\}$$

be such that B_1^c, $B_2^c\in I$. As ${}_2l_\infty^I(u,p,\Delta^r)=\left\{x=(x_{ij}):\sup_{ij}|u_{ij}\Delta^r x_{ij}|^{p_{ij}}<\infty\right\}\in I$, taking lim sup over i,j we get $B_1^c\subset B_2^c$. Hence ${}_2C^I(u,p,\Delta^r)\subset {}_2BV_\sigma^I(u,p,\Delta^r)\subset {}_2l_\infty^I(u,p,\Delta^r)$. Next we show that the inclusion is proper. First for ${}_2C^I(u,p,\Delta^r)\subset {}_2BV_\sigma^I(u,p,\Delta^r)$. Let $x=(x_{ij})\in {}_2BV_\sigma^I(u,p,\Delta^r)$, then by the definition

$${}_2BV_\sigma^I(u,p,\Delta^r)=\left\{x=(x_{ij})\in w:\left\{i,j\in\mathbb{N}:|\phi_{mn,ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}\geq\varepsilon\right\}\in I,\ \text{for some } L\in\mathbb{C}\right\}.$$

We have $|\phi_{mn,ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}\geq\varepsilon$. We say that the

$$I-\lim_{ij}(\phi_{mn,ij}(u_{ij}\Delta^r x))=L.$$

Now considering the case when $|\phi_{mn,ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}<\varepsilon$. Then

$$\left\{|t_{mn,ij}(u_{ij}\Delta^r x)-t_{(m-1)(n-1),ij}(u_{ij}\Delta^r x)-L|^{p_{ij}}<\varepsilon\right\}$$

when $m,n=0$, then we have

$$\phi_{mn,ij}(u_{ij}\Delta^r x)=t_{ij}(u_{ij}\Delta^r x)=u_{ij}\Delta^r x_{ij}.$$

Therefore, we get

$$|u_{ij}\Delta^r x_{ij}-L|^{p_{ij}}<\varepsilon,\ \forall\, i,j\in\mathbb{N}.$$

Hence,

$$x\notin {}_2C^I(u,p,\Delta^r)=\left\{i,j\in\mathbb{N}:|u_{ij}\Delta^r x_{ij}-L|^{p_{ij}}\geq\varepsilon\right\}\in I.$$

Hence, the inclusion is proper. For ${}_2BV_\sigma^I(u,p,\Delta^r)\subset {}_2l_\infty^I(u,p,\Delta^r)$, the result of this part follows from the proof of the Theorem (2.4). This completes the proof.

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References

[1] Z. U. Ahmad and M. Mursaleen, Proc. Amer. Math. Soc. **103,** 244-246 (1988).
[2] T. J. Bromwich, Macmillan and Co. Ltd. New York 1965.
[3] S. Banach, Theorie des Operations Lineaires, Warszawa, Poland, (1932).
[4] M. Başarir and O. Sonalcan, J. Indian Acad. Math. **21,** 193-200 (1999).
[5] M. Et and R. Çolak, Soochow. J. Math. **21** 377-386 (1995).
[6] G. H. Hardy, Proc. Camb. Phil. Soc. **19,** 86-95 (1917).
[7] H. Kizmaz, Canad. Math. Bull. **24,** 169-176 (1981).
[8] V. A. Khan, Comm. Fac. Sci. Univ. Ankara Sér. A, **57,** 2533 (2008).
[9] V. A. Khan, K. Ebadullah and S. Suantai, Analysis, **32,** 199-208 (2012).
[10] V. A. Khan and K. Ebadullah, Theory and Applications of Mathematics and Computer Science, **1,** 22-30 (2011).
[11] V. A. Khan and K. Ebadullah, J. Math. Comput. Sci. **2,** 265-273 (2012).
[12] V. A. Khan and Nazneen Khan, Int. J. Anal. (2013) 7 pages.
[13] P. Kostyrko, T. Salat and W. Wilczynski, Real Analysis Exchange, **26,** 669-685 (2000).
[14] G. G. Lorentz, Acta Mathematica, **80,** 167-190 (1948).
[15] F. Moricz, Acta Math. Hungarica, **57,** 129-136 (1991) .
[16] F. Moricz and B. E. Rhoades, Math. Proc. Camb. Phil. Soc. **104,** 283-294 (1988).
[17] M. Mursaleen, J. Math. Anal. Appl. **293,** 523-531 (2004).
[18] M. Mursaleen and O. H. H. Edely, J. Math. Anal. Appl. **293,** 532-540 (2004).
[19] M. Mursaleen, Quarterly J. Math. **34,** 77-86 (1983).
[20] M. Mursaleen and O. H. H. Edely, J. Math. Anal. Appl. **288,** 223-231 (2003).
[21] M. Mursaleen, Houston J. Math. **9,** 505-509 (1983).
[22] M. Mursaleen and S. A. Mohiuddine, Glas. Mat. **45,** 139-153 (2010) .
[23] R. A. Raimi, Duke Math. J. **30** 81-94 (1963).
[24] K. Raj, A. K. Sharma and S. K. Sharma, Int J. Pure Appl. Math. **67,** 475-484 (2011).
[25] K. Raj and S. K. Sharma, Acta Univ. Palacki. Olomuc. Fac. rer. nat. **51,** 89-100 (2012).
[26] T. Salat, B. C. Tripathy and M. Ziman, Tatra Mt. Math. Publ. **28,** 279-286 (2004).
[27] B. C. Tripathy, Soochow J. Math. **30,** 431-446 (2004).
[28] B. C. Tripathy, Tamkang J. Math. **34,** 231-237 (2003).
[29] B. C. Tripathy and B. Hazarika, **59,** 485-494 (2009).
[30] A. Wilansky, Summability through Functional Analysis, North- Holland Math. Stud. (1984).
[31] M. Zeltser, Diss. Math. Univ. Tartu, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu **25** (2001).

Kuldip Raj got his B.Sc(1990), M.Sc(1992) and Ph.D(1999) from University of Jammu, Jammu, India. His research interests are in the areas of Functional Analysis, Operator Theory, Sequences, Series and Summability Theory. In 1998, Kuldip Raj had been appointed in Education Department of Jammu and Kashmir Govt. In 2007, Kuldip Raj appointed as Assistant Professor in the School of Mathematics, Shri Mata Vaishno Devi University, Katra J&K India. He has been Published more than 120 papers in reputed International Journals of Mathematics. Presently he has been working in school of Mathematics Shri Mata Vaishno Devi University Katra J&K India.

Seema Jamwal is a Research Scholar in the School of Mathematics Shri Mata Vaishno Devi University Katra J&K India.