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A New Generalization of Linear Exponential Distribution: Theory and Application

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Abstract: The linear exponential distribution is a very well-known distribution for modeling lifetime data in reliability and medical studies. We introduce in this paper a new four-parameter generalized version of the linear exponential distribution which is called Kumaraswamy linear exponential distribution. We provide a comprehensive account of the mathematical properties of the new distributions. In particular, a closed-form expressions for the density, cumulative distribution and hazard rate function of the distribution is given. Also, the r_{th} order moment and moment generating function are derived. The maximum likelihood estimation of the unknown parameters is discussed.

Keywords: Kumaraswamy distribution, Hazard function, Linear failure rate distribution, Maximum likelihood estimation, Moments.

1 Introduction and Motivation

The Linear Exponential distribution LED has many applications in applied statistics and reliability analysis. Broadbent [3], uses the (LED) to describe the service of milk bottles that are filled in a dairy, circulated to customers, and returned empty to the dairy. The Linear exponential model was also used by Carbone et al. [4] to study the survival pattern of patients with plasmacytic myeloma. The type-2 censored data is used by Bain [2] to discuss the least square estimates of the parameters α and β and by Pandey et al. [23] to study the Bayes estimators of (α,β).

The linear exponential distribution is also known as the Linear Failure Rate distribution, having exponential and Rayleigh distributions as special cases, is a very well-known distribution for modeling lifetime data in reliability and medical studies. It is also models phenomena with increasing failure rate. However, the LE distribution does not provide a reasonable parametric fit for modeling phenomenon with decreasing, non linear increasing, or non-monotone failure rates such as the bathtub shape, which are common in firm ware reliability modeling, biological studies, see Lai et al. [15] and Zhang et al. [28].

A random variable X is said to have the linear exponential distribution with two parameters λ and θ, if it has the cumulative distribution function

$$F(x,\lambda,\theta) = 1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}, \quad x>0, \lambda,\theta>0, \tag{1}$$

and the corresponding probability density function (pdf) is given by

$$f(x,\lambda,\theta) = (\lambda+\theta x)e^{-(\lambda x + \frac{\theta}{2}x^2)}, \quad x>0, \lambda,\theta>0. \tag{2}$$

The distribution introduced by Kumaraswamy [14], also refereed to as the minimax distribution, is not very common among statisticians and has been little explored in the literature, nor its relative interchangeability with the beta distribution has been widely appreciated. We use the term K distribution to denote the Kumaraswamy distribution. Its cdf is given by

$$F(x,a,b) = 1-(1-x^a)^b, \quad 0<x<1, \tag{3}$$

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where $a > 0$ and $b > 0$ are shape parameters, and the probability density function

$$f(x,a,b) = abx^{a-1}(1-x^a)^{b-1}, \tag{4}$$

which can be unimodal, increasing, decreasing or constant, depending on the parameter values. In a very recent paper, Jones [13] explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and K distributions. However, the beta distribution has the following advantages over the K distribution: simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, we combine the works of Kumaraswamy [14] and Cordeiro and de Castro [8] to derive some mathematical properties of a new model, called the Kumaraswamy linear exponential (KLE) distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x) = 1 - [1 - G(x)^a]^b \tag{5}$$

where $a > 0$ and $b > 0$ are two additional shape parameters. The $K-G$ distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(x) = abg(x)G(x)^{a-1}[1-G(x)^a]^{b-1} \tag{6}$$

The density family (6) has many of the same properties of the class of beta-G distributions (see Eugene et al. [9]), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special $K-G$ distributions can be generated as follows: the K_w-normal distribution is obtained by taking $G(x)$ in (4) to be the normal cumulative function. Analogously, the $K-$Weibull (Cordeiro et al.[7]), general results for the Kumaraswamy-G distribution (Nadarajah et al.[20]). K_w-generalized gamma (Pascoa et al.[22]), Kw$-$Birnbaum-Saunders (Saulo et al. [25]) Beta-Linear Failure Rate Distribution and its Applications (see Jafari et al.[12]) and K_w-Gumbel (Cordeiro et al. [8]) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new $Kw-G$ distribution can be generated from a specified G distribution.

A physical interpretation of the $K-G$ distribution given by (5) and (6) (for a and b positive integers) is as follows. Suppose a system is made of b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, X_{j2}, \ldots, X_{ja}$ denote the life times of the subcomponents with in the j_{th} component, $j = 1, \ldots,$ b with common (cdf) G. Let X_j denote the lifetime of the j_{th} component, $j = 1, \ldots, b$ and let X denote the lifetime of the entire system.Then the (cdf) of X is giveb by

$$\begin{aligned}
P(X \le x) &= 1 - P(X_1 > x, X_2 > x, ..., X_b > x) \\
&= 1 - [P(X_1 > x)]^b = 1 - \{1 - P(X_1 \le x\}^b \\
&= 1 - \{1 - P(X_{11} \le x, X_{12} \le x, \ldots, X_{1a} \le x\}^b \\
&= 1 - \{1 - P[X_{11} \le x]^a\}^b = 1 - \{1 - G^a(x)\}^b.
\end{aligned}$$

So, it follows that the $K-G$ distribution given by (5) and (6) is precisely the time to failure distribution of the entire system.

The rest of the article is organized as follows. In Section 2, we define the cumulative, density and hazard functions of the KLE distribution and some special cases. Quantile function , median, moments, moment generating function discussed in Section 3. Section 4 included the order statistics . Finally, maximum likelihood estimation is performed and the observed information matrix is determined in Section 6. Section 7 provides applications to real data sets. Section 8 ends with some conclusions.

2 Kumaraswamy Linear Exponential Distribution

In this section,we propose the Kumaraswamy Linear Exponential (KLE) distribution and provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research.The linear exponential distribution represents only a special case of the Kumaraswamy linear exponential distribution. By taking the cdf

$$G(x,\lambda,\theta) = 1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}, \quad x > 0, \lambda, \theta > 0,$$

of linear exponential, the cdf and pdf of the (KLE) distribution are obtained from Eqs.(5) and (6) as

$$F_{KLE}(x,a,b,\lambda,\theta) = 1 - \left[1 - \left(1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right)^a\right]^b, \tag{7}$$

and

$$\begin{aligned} f_{KLE}(x,a,b,\lambda,\theta) = \Big\{ & ab(\lambda+\theta x)e^{-(\lambda x + \frac{\theta}{2}x^2)}\left(1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right)^{a-1} \\ & \times \left[1 - \left(1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right)^a\right]^{b-1}\Big\}. \end{aligned} \tag{8}$$

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the KLE distribution for selected values of the parameters a, b, λ and θ, respectively.

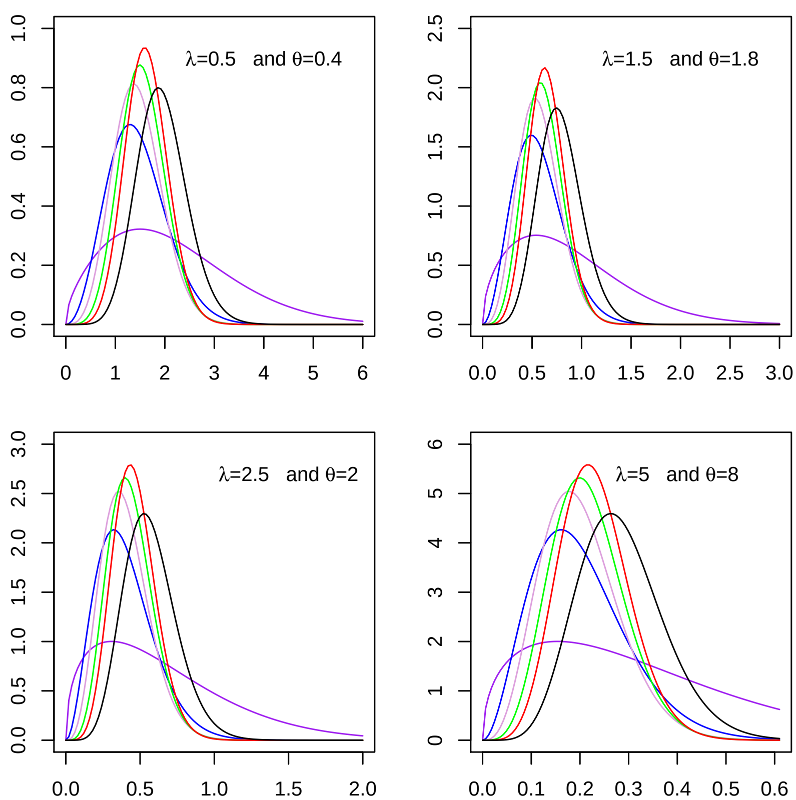

Fig. 1: The pdf's of various KLE distributions.

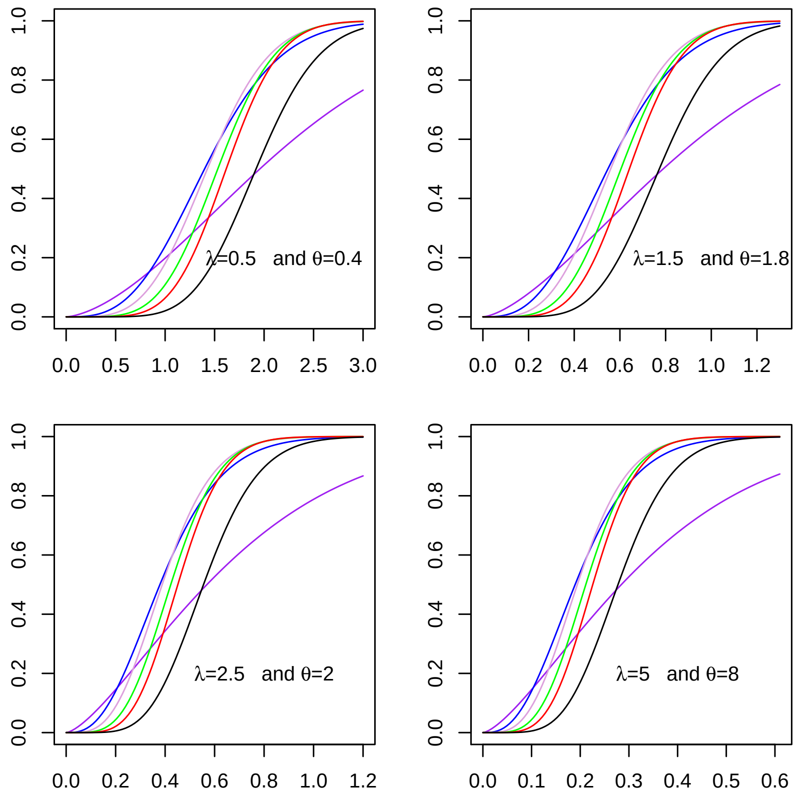

Fig. 2: The cdf's of various KLE distributions.

The associated hazard (failure) rate function (hrf) is

$$\begin{aligned} h_{KLE}(x,a,b,\lambda,\theta) &= \frac{f_{KLE}(x,a,b,\lambda,\theta)}{1-F_{KLE}(x,a,b,\lambda,\theta)} \\ &= \frac{ab(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a-1}}{1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a}}. \end{aligned} \quad (9)$$

Figure 3 illustrates some of the possible shapes of the hazard function of the KLE distribution for selected values of the parameters a,b,λ and θ, respectively.

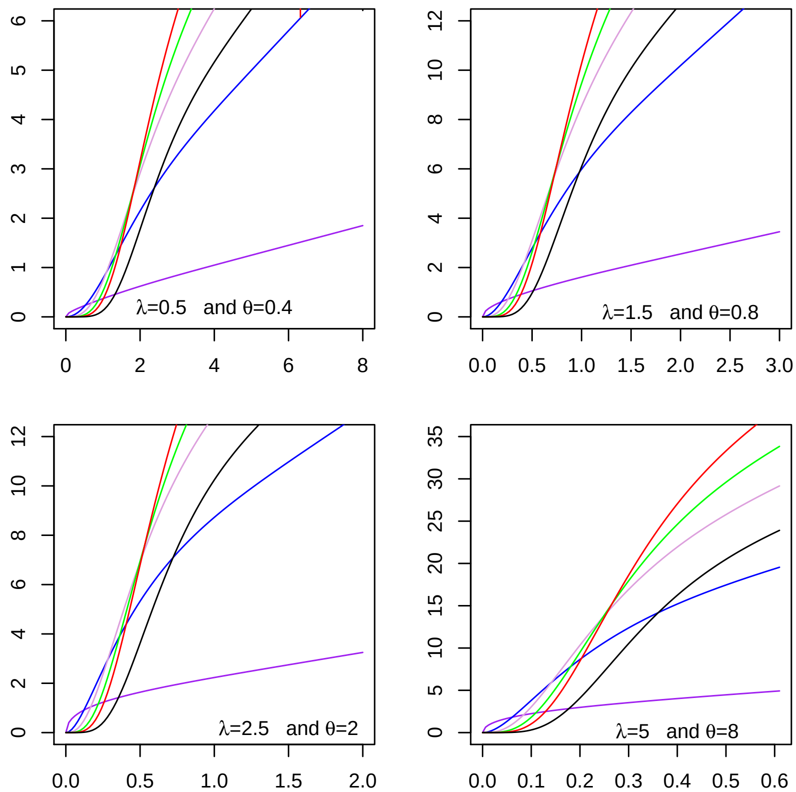

Fig. 3: The hazard function's of various KLE distributions.

The following are special cases of the $KLE\ (a,b,\lambda,\theta)$:

1.If $b=1$ we get the generalized linear failure rate cumulative distribution (see Sarhan, A., Kundu, D. [26]).
2.If $a=b=1$ we get the linear exponential (failure rate) cumulative distribution.
3.If $\theta=0$ we get the Kumaraswamy exponential cumulative distribution.
4.If $\lambda=0$ we get the Kumaraswamy generalized Rayleigh cumulative distribution.
5.If $b=1$ and $\lambda=0$ we get the Kumaraswamy Rayleigh cumulative distribution.

From the above, we see that the Kumaraswamy linear exponential distribution generalizes all the distributions mentioned above.

3 Statistical properties

This section is devoted to studying statistical properties for the kumaraswamy linear exponential, specifically Quantile function ,median, moments, moment generating function.

3.1 Quantile and Median

Starting with the well known definition of the $100q-th$ quantile, which is simply the solution of the following equation, with respect to x_q, $0<q<1$,

$$q=P(X\leq x_q)=F(x_q)=1-\left[1-\left(1-e^{-(\lambda x_q+\frac{\theta}{2}x_q^2)}\right)^a\right]^b$$

$$(1-q)^{\frac{1}{b}}=1-\left(1-e^{-(\lambda x_q+\frac{\theta}{2}x^2)}\right)^a$$

$$1-\left\{1-(1-q)^{\frac{1}{b}}\right\}^{\frac{1}{a}}=e^{-(\lambda x_q+\frac{\theta}{2}x_q^2)}$$

$$-\lambda x_q-\frac{\theta}{2}x_q^2=\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}^{\frac{1}{a}},$$

which finally produces the following equation

$$\frac{\theta}{2}x_q^2+\lambda x_q+\frac{1}{a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}=0. \tag{10}$$

Solving equation (10) with respect to x_q, we get

$$x_q=\frac{-\lambda\pm\sqrt{\lambda^2-\frac{2\theta}{a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}}}{\theta}$$

Since x_q is positive, then

$$x_q=\frac{-\lambda+\sqrt{\lambda^2-\frac{2\theta}{a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}}}{\theta}, \tag{11}$$

which completes the proof.

The median can be derived from (11) be setting $q=\frac{1}{2}$. That is, the median is given by the following relation

$$M(X)=\frac{-\lambda+\sqrt{\lambda^2-\frac{2\theta}{a}\ln\left\{1-\left[1-(\frac{1}{2})^{\frac{1}{b}}\right]\right\}}}{\theta}$$

3.2 The moments

In this subsection, we derive the r_{th} moments and moment generating function $(M_X(t))$ of the $KLE(\varphi)$ where $\varphi=(a,b,\lambda,\theta)$.

Lemma 1: If X has $KLE\ (\varphi)$, then the r_{th} moment of X, $r=1,2,\ldots$ has the following form:

$$\mu_r'=\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{j}\binom{a(j+1)-1}{k}\right.$$
$$\left.\times\left[\frac{\lambda\Gamma(2i+r+1)}{[\lambda(k+1)]^{2i+r+1}}+\frac{\theta\Gamma(2i+r+2)}{[\lambda(k+1)]^{2i+r+2}}\right]\right\}.$$

Proof:

We start with the well known definition of the r_{th} moment of the random variable X with pdf $f(x)$ given by

$$\mu_r'=E(X^r)=\int_0^{\infty}x^r f_{KLE}(x,\varphi)dx$$
$$=ab\left\{\int_0^{\infty}x^r(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a-1}\left[1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a}\right]^{b-1}\right\}$$

Since $0<e^{-(\lambda x+\frac{\theta}{2}x^2)}<1$ for $x>0$, then by using the binomial series expansion of $\left[1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a}\right]^{b-1}$ given by

$$\left[1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{a}\right]^{b-1}=\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{ja}, \tag{12}$$

we get

$$\begin{aligned}
\mu_r' &= ab \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \left\{ \int_0^{\infty} x^r (\lambda + \theta x) e^{-(\lambda x + \frac{\theta}{2} x^2)} \right. \\
&\times \left. \left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^{a-1} \left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^{ja} \right\} dx \\
&= A \int_0^{\infty} x^r (\lambda + \theta x) e^{-(\lambda x + \frac{\theta}{2} x^2)} \left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^{a(j+1)-1} dx,
\end{aligned} \tag{13}$$

where

$$A = ab \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j}, \tag{14}$$

Also

$$\left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^{a(j+1)-1} = \sum_{k=0}^{\infty} (-1)^k \binom{a(j+1)-1}{k} e^{-(\lambda x + \frac{\theta}{2} x^2) k}, \tag{15}$$

and the series expansion of $e^{-\frac{\theta}{2}(k+1)x^2)}$ is

$$e^{-\frac{\theta}{2}(k+1)x^2)} = \sum_{i=0}^{\infty} \frac{\left[\frac{\theta}{2}(k+1)x^2\right]^i}{i!} \tag{16}$$

Substituting (15) and (16) into (13), we get

$$\begin{aligned}
\mu_r' &= A^* \int_0^{\infty} (\lambda + \theta x) x^{2i+r} e^{-\lambda(k+1)x} dx \\
&= A^* \left[\lambda \int_0^{\infty} x^{2i+r} e^{-\lambda(k+1)x} dx + \theta \int_0^{\infty} x^{2i+r+1} e^{-\lambda(k+1)x} dx \right] \\
&= A^* \left[\frac{\lambda \Gamma(2i+r+1)}{[\lambda(k+1)]^{2i+r+1}} + \frac{\theta \Gamma(2i+r+2)}{[\lambda(k+1)]^{2i+r+2}} \right]
\end{aligned}$$

where

$$A^* = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{j} \binom{a(j+1)-1}{k}$$

which completes the proof .

Lemma 2: If X has $KLE\ (\varphi)$, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = A^* \left[\frac{\lambda \Gamma(2i+1)}{[\lambda(k+1)-t]^{2i+1}} + \frac{\theta \Gamma(2i+2)}{[\lambda(k+1)-t]^{2i+2}} \right]$$

Proof.

We start with the well known definition of the moment generating function given by

$$\begin{aligned}
M_X(t) = E(e^{tx}) &= \int_0^{\infty} e^{tx} f_{KLE}(x, \Phi) dx \\
&= ab \left\{ \int_0^{\infty} e^{tx} (\lambda + \theta x) e^{-(\lambda x + \frac{\theta}{2} x^2)} \right. \\
&\times \left. \left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^{a-1} \left[1 - \left(1 - e^{-(\lambda x + \frac{\theta}{2} x^2)}\right)^a\right]^{b-1} \right\}
\end{aligned} \tag{17}$$

Substituting (14) , (15) and (16) into (17)), we get

$$M_X(t) = A^* \left[\lambda \int_0^\infty x^{2i} e^{-(\lambda(k+1)-t)x} dx + \theta \int_0^\infty x^{2i+1} e^{-(\lambda(k+1)-t)x} dx \right]$$
$$= A^* \left[\frac{\lambda \Gamma(2i+1)}{[\lambda(k+1)-t]^{2i+1}} + \frac{\theta \Gamma(2i+2)}{[\lambda(k+1)-t]^{2i+2}} \right].$$

which completes the proof.

4 Order statistics

Moments of order statistics play an important role in quality control testing and reliability,where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density function of the r_{th} order statistic $X_{r:n}$, say $f_{r:n}(x)$, in a random sample of size n from the KLE distribution. To prove the r_{th} order statistic $X_{r:n}$ we need the following Lemma.

Lemma 3:

The probability density function of $X_{r:n}$, $r = 1,2,...,n$ of K_w-LE distributio is

$$f_{r:n}(x) = \sum_{j=0}^{n-r} d_j(n,r) f_{KLE}(x, a_{r+j}, b_{r+j}, \lambda, \theta) \tag{18}$$

where

$$a_i = ai \text{ and } d_j(n,r) = \frac{n(-1)^j \binom{n-1}{r-1} \binom{n-r}{j}}{r+j}. \tag{19}$$

Proof:

The pdf of $X_{r:n}$, $r = 1,2,...,n$ is given by, David (1981)

$$f_{r:n}(x) = \frac{1}{\beta(r, n-r+1)} [F(x,\varphi)]^{r-1} [1 - F(x,\Phi)]^{n-r} f(x,\Phi)$$

where $F(x,\varphi)$ and $f(x,\varphi)$ are CDF and pdf given by (7) and (8), respectively. since $0 < F(x,\Phi) < 1$ for $x > 0$, by using the binomial series expansion of $[1 - F(x,\varphi)]^{n-r}$, given by

$$[1 - F(x,\varphi)]^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x,\varphi)]^j$$

we have

$$f_{r:n}(x) = \frac{1}{\beta(r, n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x)]^{r+j-1} f(x) \tag{20}$$

Substituting (7) and (8) into (20), we get

$$f_{r:n}(x) = \sum_{j=0}^{n-r} d_j(n,r) f_{KLE}(x, a_{r+j}, b_{r+j}, \lambda, \theta). \tag{21}$$

The coefficients $d_j(n,r)$, $j = 1,2,...,n-r$ do not depend on a, b, λ, θ. Thus $f_{r:n}(x)$ is the weighted average of the KLE distribution with different shape parameters.

Theorem (4.1):

The k_{th} moment of order statistic $X_{r:n}$ is

$$\mu_{r:n}^{(k)} = \sum_{i=0}^\infty \sum_{l=0}^\infty \sum_{k=0}^\infty \sum_{j=0}^{n-r} d_j(n,r)(-1)^{i+j+k} \binom{b_{r+j}-1}{l} \binom{a_{r+j}(l+1)-1}{k}$$
$$\times \left[\frac{\lambda \Gamma(2i+k+1)}{[\lambda(j+1)]^{2i+k+1}} + \frac{\theta \Gamma(2i+k+2)}{[\lambda(k+1)]^{2i+k+2}} \right]. \tag{22}$$

Proof: The general definition of the k_{th} moment of order statistic $X_{r:n}$ is

$$\mu_{r:n}^{(k)} = \int_0^\infty x^k f_{r:n}(x,a,b,\lambda,\theta)dx. \tag{23}$$

Substituting from (21) into (23), one gets

$$\mu_{r:n}^{(k)} = \sum_{j=0}^{n-r} d_j(n,r) \int_0^\infty x^k f(x,a_{r+j},b_{r+j},\lambda,\theta)dx. \tag{24}$$

Since the integral in (24) is the k_{th} moment of K_w-$LE(\lambda,\theta,a_{r+j},b_{r+j})$, then from (24) with the Lemma (3) we get (22) which completes the proof.

5 Estimation and Inference

In this section, we derive the maximum likelihood estimates of the unknown parameters $\varphi = (a,b,\lambda,\theta)$ of KLE distribution based on a complete sample. Let us assume that we have a simple random sample $X_1,X_2,...,X_n$ from $KLE(a,b,\lambda,\theta)$. The likelihood function of this sample is

$$L = \prod_{i=1}^n f(x_i,a,b,\lambda,\theta). \tag{25}$$

Substituting from (8) into (25), we get

$$\begin{aligned} L &= \prod_{i=1}^n \left\{ ab(\lambda+\theta x_i) e^{-(\lambda x_i + \frac{\theta}{2}x_i^2)} \left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right)^{a-1} \right. \\ &\quad \left. \times \left[1-\left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right)^a\right]^{b-1} \right\} \\ &= (ab)^n \prod_{i=1}^n (\lambda+\theta x_i) e^{-\sum\limits_{i=1}^n(\lambda x_i+\frac{\theta}{2}x_i^2)} \Pi_{i=1}^n \left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right)^{a-1} \\ &\quad \times \prod_{i=1}^n \left[1-\left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^2)}\right)^a\right]^{b-1}. \end{aligned} \tag{26}$$

The log-likelihood function for the vector of parameters $\varphi = (a,b,\lambda,\theta)$ can be written as

$$\begin{aligned} \ell = \log\ L &= n\log a + n\log b + \sum_{i=1}^n \log(\lambda+\theta x_i) + \sum_{i=1}^n z_i \\ &\quad + (a-1)\sum_{i=1}^n \log(1-e^{z_i}) \\ &\quad + (b-1)\sum_{i=1}^n \log[1-(1-e^{z_i})^a] \end{aligned} \tag{27}$$

where

$$z_i = -(\lambda x_i + \frac{\theta}{2}x_i^2)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (27). The components of the score vector $W\ (\Phi)$ are given by

$$\begin{aligned} \ell_a = \frac{\partial \log\ L}{\partial a} &= \frac{n}{a} + \sum_{i=1}^n \log(1-e^{z_i}) \\ &\quad - (b-1)\sum_{i=1}^n \frac{(1-e^{z_i})^a \log(1-e^{z_i})}{1-(1-e^{z_i})^a}, \end{aligned} \tag{28}$$

$$\ell_b = \frac{\partial \log\, L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left[1-(1-e^{z_i})^{a}\right], \tag{29}$$

$$\begin{aligned}\ell_\lambda(\Phi) = \frac{\partial \log\, L}{\partial \lambda} &= \sum_{i=1}^{n} \frac{1}{(\lambda+\theta x_i)} - \sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1-e^{z_i}} \\ &+ (b-1)\sum_{i=1}^{n} \frac{a x_i (1-e^{z_i})^{a-1}}{1-(1-e^{z_i})^{a}},\end{aligned} \tag{30}$$

and

$$\begin{aligned}\ell_\theta = \frac{\partial \log\, L}{\partial \theta} &= \sum_{i=1}^{n} \frac{x_i}{(\lambda+\theta x_i)} - \sum_{i=1}^{n} \frac{x_i^2}{2} \\ &+ (a-1)\sum_{i=1}^{n} \frac{x_i^2 (1-e^{z_i})^{a-1}}{2(1-e^{z_i})} + \frac{a(b-1)}{2}\sum_{i=1}^{n} \frac{x_i^2 (1-e^{z_i})^{a-1}}{1-(1-e^{z_i})^{a}}.\end{aligned} \tag{31}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (28)- (31) to zero and solve them simultaneously.

The Hessian matrix, second partial derivatives of the log-likelihood, is given by

$$\begin{pmatrix} \ell_{aa} & \ell_{ab} & \ell_{a\lambda} & \ell_{a\theta} \\ \ell_{ba} & \ell_{bb} & \ell_{b\lambda} & \ell_{b\theta} \\ \ell_{\lambda a} & \ell_{\lambda b} & \ell_{\lambda\lambda} & \ell_{\lambda\theta} \\ \ell_{\theta a} & \ell_{\theta b} & \ell_{\theta\lambda} & \ell_{\theta\theta} \end{pmatrix}$$

$$\ell_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n} \frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta x_i^2}\right)^{a}\left(\ln\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)\right)^2}{\left(-1+\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}\right)^2}$$

$$\ell_{ab} = -\sum_{i=1}^{n} \frac{\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}\ln\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)}{1-\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}}$$

$$\ell_{a\lambda} = \sum_{i=1}^{n} \frac{x_i \mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}}{1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}} + (b-1)\sum_{i=1}^{n}(A_i+B_i)$$

$$A_i = -\frac{\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a} x_i \mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\left(a\ln\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)+1\right)}{\left(-1+\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)\left(-1+\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}\right)}$$

$$B_i = -\frac{\left(\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}\right)^2 \ln\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right) a x_i \mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}}{\left(1-\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)^{a}\right)^2\left(1-\mathrm{e}^{-\lambda x_i-1/2\,\theta x_i^2}\right)}$$

$$\ell_{a\theta} = \sum_{i=1}^{n} 1/2\,\frac{{x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}} + (b-1)\sum_{i=1}^{n}(C_i + D_i)$$

$$C_i = -1/2\,\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a {x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\left(a\ln\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)+1\right)}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)\left(-1+\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}$$

$$D_i = -1/2\,\frac{\left(\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)^2 \ln\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right) a{x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(1-\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)^2\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)}$$

$$\ell_{bb} = -\frac{n}{b^2}$$

$$\ell_{b\lambda} = -\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a a x_i \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)\left(1-\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}$$

$$\ell_{b\theta} = \sum_{i=1}^{n} -1/2\,\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a a{x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)\left(1-\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}$$

$$\ell_{\lambda\lambda} = -\sum_{i=1}^{n}(\lambda+\theta x_i)^{-2} - (a-1)\sum_{i=1}^{n}\frac{{x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2}$$
$$-(b-1)\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a a{x_i}^2 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\left(a\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2} - 1 + \left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2\left(-1+\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)^2}$$

$$\ell_{\lambda\theta} = -\sum_{i=1}^{n}\frac{x_i}{(\lambda+\theta x_i)^2} - (a-1)\sum_{i=1}^{n} -1/2\,\frac{{x_i}^3 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2}$$
$$-(b-1)\sum_{i=1}^{n} 1/2\,\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a a{x_i}^3 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\left(a\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2} - 1 + \left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2\left(-1+\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)^2}$$

$$\ell_{\theta\theta} = -\sum_{i=1}^{n}\frac{{x_i}^2}{(\lambda+\theta x_i)^2} - (a-1)\sum_{i=1}^{n} 1/4\,\frac{{x_i}^4 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2}$$
$$-(b-1)\sum_{i=1}^{n} 1/4\,\frac{\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a a{x_i}^4 \mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\left(a\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2} - 1 + \left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)}{\left(-1+\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^2\left(-1+\left(1-\mathrm{e}^{-\lambda x_i - 1/2\,\theta {x_i}^2}\right)^a\right)^2}$$

We can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing on some the KLE sub-models. For example, we can use the LR test statistic to check whether the KLE distribution for a given data set is statistically *superior* to the LE distribution. In any case, hypothesis tests of the type $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$ can be performed using a LR test. In this case, the LR test statistic for testing H_0 versus H_1 is $\omega = 2(\ell(\hat{\theta}; x) - \ell(\hat{\theta}_0; x))$, where $\hat{\theta}$ and $\hat{\theta}_0$ are the MLEs under H_1 and H_0, respectively. The statistic ω is asymptotically (as $n \to \infty$) distributed as χ^2_k, where k is the length of the parameter vector θ of interest. The LR test rejects H_0 if $\omega > \chi^2_{k;\gamma}$, where $\chi^2_{k;\gamma}$ denotes the upper $100\gamma\%$ quantile of the χ^2_k distribution.

6 Aplication

In this section, we provide a data analysis to see how the new model works in practice. This data set is studied by Abuammoh et al. [1], which represent the lifetime in days of 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia.

Table 1: (Leukemia data) Lifetimes of 40 patients suffering from Leukemia

115	181	255	418	441	461	516	739	743	789
807	865	924	983	1024	1062	1063	1165	1191	1222
1222	1251	1277	1290	1357	1369	1408	1455	1478	1549
1578	1578	1599	1603	1605	1696	1735	1799	1815	1852

Table 2: The estimated parameters,AIC, AICC, BIC and K-S of the models based on data set

Model	a	b	λ	θ	-LL	AIC	AICC	BIC	K-S
LE.	-	-	$9.499\cdot 10^{-4}$	$2\cdot 10^{-6}$	335.627	675.254	675.578	678.631	0.512
KLE	0.083	0.050	0.005	$1.4\cdot 10^{-5}$	313.905	635.81	636.952	642.565	0.198

In order to compare the two distribution models, we consider criteria like $-\ell$, AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller -2ℓ, AIC,AICC and BIC values:‘

$$\mathrm{AIC} = 2k - 2\ell\,, \mathrm{AICC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1} \text{and}, \mathrm{BIC} = 2\ell + k * \log(n)$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model.

The LR test statistic to test the hypotheses $H_0 : a = b = c = 1$ versus $H_1 : a \neq 1 \vee b \neq 1 \vee c \neq 1$ is $\omega = 9.574 > 7.815 = \chi^2_{3;0.05}$, so we reject the null hypothesis.

Table 2 shows parameter MLEs to each one of the three fitted distributions for data set and the values of $-2\log(L)$, AIC and AICC values. The values in Table 2, indicate that the KLE is a strong competitor to other distribution used here for fitting data set.

A P-P plot compares the fitted cdf of the models with the empirical cdf of the observed data (Fig. 5).

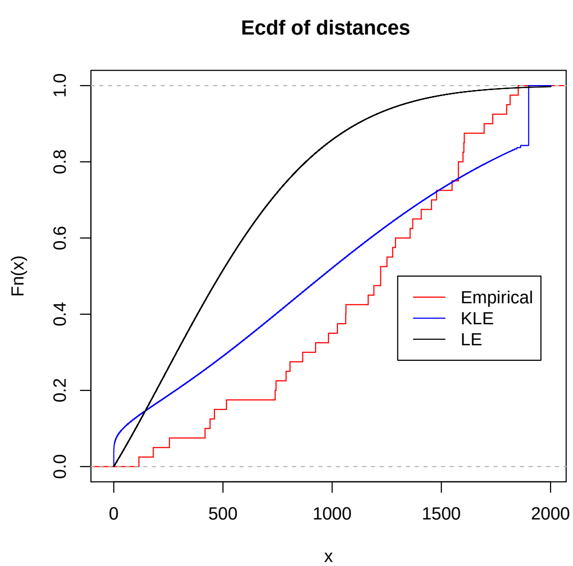

Fig. 4: Empirical, fitted KLE and LE cdf of the data set.

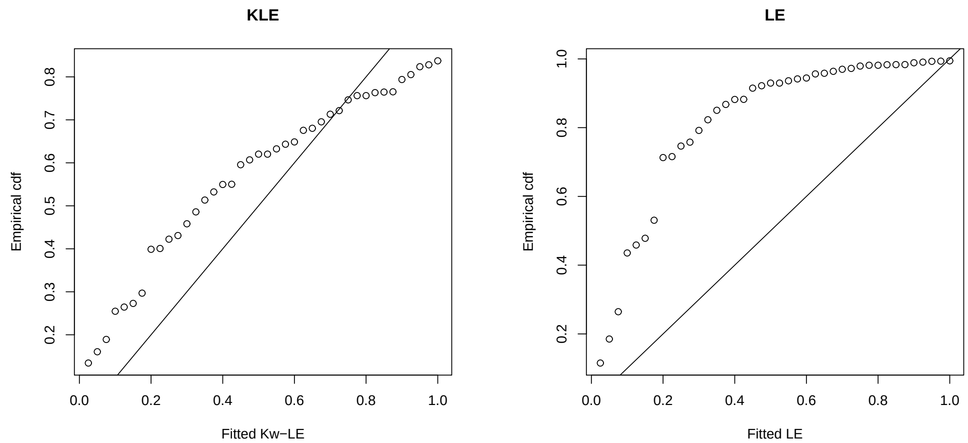

Fig. 5: PP plots for fitted KLE and the LE distributions.

7 Conclusion

Here, we propose a new model, the so-called the Kumaraswamy linear exponential distribution distribution which extends the linear exponential distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modelling real data. We derive expansions for the moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the KLE distribution to real data show that the new distribution can be used quite effectively to provide better fits than the LE distribution.

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