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Perishable Inventory System at Service Facilities with Multiple Server Vacations and Impatient Customers

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Abstract: This article presents a perishable inventory system under continuous review at a service facility in which a waiting area for customers is of finite size M. We assume that the replenishment of inventory is instantaneous. The items of inventory have exponential life times. It is assumed that demand for the commodity is of unit size. The arrivals of customers to the service station form a Poisson process. The server goes for a vacation of an exponentially distributed duration whenever the waiting area is zero. If the server finds the customer level is zero when he returns to the system, he immediately takes another vacation. The individual customer is issued a demanded item after a random service time, which is distributed as negative exponential. Also the waiting customer independently reneges the system after an exponentially distributed amount of time. The joint probability distribution of the number customers in the system and the inventory levels is obtained in steady state case. Some measures of system performance in the steady state are derived and the total expected cost is also considered.

Keywords: Continuous review inventory system, Perishable item, Service facility, Impatient customer, Multiple server vacation.

1 Introduction

Research on queueing systems with inventory control has captured much attention of researchers over the last decades. In this system, customers arrive at the service facility one by one and require service. In order to complete the customer service, an item from the inventory is needed. A served customer departs immediately from the system and the on - hand inventory decreases by one at the moment of service completion. This system is called a queueing - inventory system [14]. Berman and Kim [1] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [4] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [5] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [2] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [3] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. The M/M/1 queueing - inventory system with backordering was investigated by Schwarz and Daduna [13]. The authors derived the system steady state behavior under $\Pi(1)$ reorder policy which is (0, Q) policy with an additional threshold 1 for the queue length as a decision variable. Krishnamoorthy et al., [10] introduced an additional control policy (N-policy) into (s, S) inventory system with positive service time.

In many real world queueing inventory systems, server(s) may become unavailable for a random period of time when there are no customers in the waiting line at a service completion instant. This random period of server absence, often called a server vacation can represent the time of server's performing some secondary task. This has been extensively investigated (see Tian and Zhang [18], Takagi ([16], [17]) and Doshi ([6], [9]). Daniel and Ramanarayanan [8] have first introduced the concept of server vacation in inventory with two servers. In [7], they have studied an inventory system in which the server takes a rest when the level of the inventory is zero. Sivakumar [15] analyzed a retrial inventory

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system with multiple server vacations. In that paper, the author considered an (s,S) inventory system where arrivals of customers form a Poisson process and if the server finds an empty stock at the end of a vacation, he takes another vacation immediately otherwise he is ready to serve any arriving demands. Recently, Narayanan et al. [11] considered an inventory system with random positive service time. Customers arrived to the service station according to a Markovian arrival process and service times for each customers had phase-type distribution.

In this paper, we consider a $(0,S)$ queueing- inventory system at a service facility with multiple server vacations and impatient customers. The joint probability distribution of the number of customers in the waiting area and the inventory level is obtained in the steady state case. Various system performance measures are derived and the total expected cost rate is calculated. The rest of the paper is organized as follows. In the next section, we describe the mathematical model and the notations used in this paper are defined. Analysis of the model is given in section 3. The steady state solution of the model are dealt with in section 4. Some key system performance measures are derived in section 5. In the section 6, we calculate the total expected cost rate in the steady state. The last section is meant for conclusion.

2 Problem formulation

Consider a continuous review perishable inventory system at a service facility with the maximum capacity for S units. The waiting area space is limited to accommodate a maximum number M of customers including the one at the service point. The arrival of customers is assumed to form a Poisson process with parameter $\lambda(>0)$. The demand is for single item per customer. The demanded item is delivered to the customer after a random time of service. The service times of items are assumed to be independent of each other and distributed as negative exponential with parameter μ. We assume N types of services are available at service facility. The customer chooses type j service with probability $p_j,\ j=1,2,\dots,N$ and $\sum_{j=1}^{N} p_j = 1$. The life time of the commodity is assumed to be distributed as negative exponential with parameter $\gamma(>0)$. We have assumed that an item of inventory that makes it into the service process cannot perish while in service. An $(0,S)$ ordering policy is adopted with zero lead time.

When no customer in the waiting area, the server leaves for a vacation whose duration is exponentially distributed with rate $\beta(>0)$. If the server find the customer level is zero at the end of a vacation, he takes another vacation immediately (multiple vacations). If the server returns from the vacation and finds at least one customer in the waiting area then he immediately starts to serve the waiting customer. An impatient customer leaves the system independently after a random time which is distributed as negative exponential with parameter $\alpha(>0)$. Note that in this model we have assumed that the servicing customer can't impatient. The arriving customers find the waiting area full, is considered to be lost. We also assume that the inter arrival times, service times, server vacation times, impatience time and life time of each items are mutually independent random variables.

2.1 Notations:

$$\begin{aligned}
\mathbf{0} &: \text{Zero matrix}\\
[A]_{ij} &: \text{entry at } (i,j)^{th} \text{ position of a matrix A}\\
\delta_{ij} &: \begin{cases} 1, & \text{if } \quad j=i \\ 0, & \text{otherwise} \end{cases}\\
\bar{\delta}_{ij} &: 1-\delta_{ij}\\
k \in V_i^j &: k = i, i+1, \dots j\\
\underset{i=r}{\overset{k}{\Omega}} c_i &: \begin{cases} c_r c_{r-1} \cdots c_k & \text{if } r \geq k \\ 1 & \text{if } r < k \end{cases}\\
\mathbf{e}^T &: (1,1,\dots,1)
\end{aligned}$$

3 Markov Chain

Let $L(t)$ and $X(t)$ respectively, denote the inventory level and the number of customers (waiting and being served) in the waiting area at time t.

Further, let the status of the server $Y(t)$ be defined as follows:

$$Y(t):\begin{cases}0, \text{ if the server is on vacations at time t}\\ 1, \text{ if the server is busy at time t}\end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $\{(L(t),Y(t),X(t)),t\geq 0\}$ is a continuous time Markov chain with state space E, which is defined as, Here

$$E : E_1 \cup E_2, \text{with}$$
$$E_1 : \{(i_1,0,i_3) \mid i_1 = 1,2,\ldots,S,\ i_3 = 0,1,2,\ldots,M,\}$$
$$E_2 : \{(i_1,1,i_3) \mid i_1 = 1,2,\ldots,S,\ i_3 = 1,2,\ldots,M,\}$$

Define the following ordered sets:

$$\ll i_1,i_2 \gg = \begin{cases}(i_1,0,0),(i_1,0,1),\ldots,(i_1,0,M),\ i_1 = 1,2,\ldots S;\\ (i_1,1,1),(i_1,1,2),\ldots,(i_1,1,M),\ i_1 = 1,2,\ldots S;\end{cases}$$

$$\lll i_1 \ggg = \begin{cases}\ll i_1,0 \gg,\ i_1 = 1,2,\ldots S;\\ \ll i_1,1 \gg,\ i_1 = 1,2,\ldots S;\end{cases}$$

By ordering the state space $(\lll 1 \ggg, \lll 2 \ggg, \ldots, \lll S \ggg)$, the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$\Theta = \begin{array}{c} \\ \lll 1 \ggg \\ \lll 2 \ggg \\ \lll 3 \ggg \\ \vdots \\ \lll S \ggg \end{array} \begin{array}{c} \begin{array}{cccc} \lll 1 \ggg & \lll 2 \ggg & \cdots & \lll S \ggg \end{array} \\ \left(\begin{array}{cccc} \Theta_{1,1} & \Theta_{1,2} & \cdots & \Theta_{1,S} \\ \Theta_{2,1} & \Theta_{2,2} & \cdots & \Theta_{2,S} \\ \Theta_{3,1} & \Theta_{3,2} & \cdots & \Theta_{3,S} \\ \vdots & \vdots & \vdots & \vdots \\ \Theta_{S,1} & \Theta_{S,2} & \cdots & \Theta_{S,S} \end{array}\right) \end{array}$$

Due to the assumptions made on the demand and instantaneous replenishment processes, we note that

$$\text{if} \quad i_1 = 1$$
$$\Theta_{i_1,j_1} = \mathbf{0}, \quad \text{for} \quad j_1 \neq 1,S,$$
$$\text{if} \quad i_1 = 2,3,\ldots,S$$
$$\Theta_{i_1,j_1} = \mathbf{0}, \quad \text{for} \quad j_1 \neq i_1, i_1-1.$$

We first consider the case $\Theta_{1,S}$. This will occur only when the inventory level is instantaneous replenished. First we consider the inventory level is one, that is $\Theta_{1,S}$. For this, we have the following three cases occur:

Case (i) If the server is on vacation and the customer level in the buffer (waiting area) lies between zero to M

– When the item is perish, the instantaneous replenishment takes the system state from $(1,0,i_3)$ to $(S,0,i_3)$ with intensity of transition γ. The sub matrix of the transition rates from $\ll 1,0 \gg$, to $\ll S,0 \gg$, is given by

$$[C_1]_{i_3 j_3} = \begin{cases}\gamma \ \ j_3 = i_3, \quad i_3 \in V_0^M\\ 0, \text{ otherwise.}\end{cases}$$

Case (ii) If the server is busy and the customer level in the buffer is more than one

– At the time of service completion of customer, the inventory level decreases by one, in which case another customer at the head of the queue (if any) is taken up for his service. Hence the transition takes place from $(1,1,i_3)$ to $(S,1,i_3-1)$ with intensity of transition $\sum\limits_{j=1}^{N} p_j\mu$. The sub matrix of the transition rates from $\ll 1,1 \gg$, to $\ll S,1 \gg$, is given by

$$[D_1]_{i_3 j_3} = \begin{cases}\sum\limits_{j=1}^{N} p_j\mu \ \ j_3 = i_3-1,\ i_3 \in V_2^M\\ 0, \qquad \text{otherwise.}\end{cases}$$

Case (iii) If the server is busy and the number of customer in the buffer is one

–When the service completion of the customer, both the inventory level and customer level in the buffer decrease by one and the server starts his vacation. Thus a transition takes place from $(1,1,i_3)$ to $(S,1,i_3-1)$ with intensity of transition $\sum_{j=1}^{N} p_j\mu$. The sub matrix of the transition rates from $\ll 1,1 \gg$, to $\ll S,0 \gg$, is given by

$$[F]_{i_3 j_3} = \begin{cases} \sum_{j=1}^{N} p_j\mu, & j_3 = i_3-1,\ i_3 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We denote $\Theta_{1,S}$ as B_1 and the matrix B_1 is given by

$$[B_1]_{i_2 j_2} = \begin{cases} C_1 & j_2 = i_2, \quad i_2 = 0 \\ F & j_2 = 0, \quad i_2 = 1 \\ D_1 & j_2 = 2, \quad i_2 = 1 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Next, We consider the inventory level is more than one and $\Theta_{i_1,i_1-1}, i_1 = 2,\ldots,S$. This will occur only when the service completion of the customer or any one of the i_1 items perishes. For this, we have the following three cases occur:

Case (i) If the server is on vacation and the customer level in the buffer lies between zero to M

–Any one of the i_1 items perishes, the inventory level decrease by one. Hence the transition takes place from $(i_1,0,i_3)$ to $(i_1-1,0,i_3)$ with intensity of transition and the intensity of transition $i_1\gamma$.

Case (ii) If the server is busy and the number of customer in the buffer is one

–At the time of service completion of the customer, both the inventory level and customer level in the buffer decrease by one and the server starts his vacation. Hence the transition takes place from $(i_1,1,1)$ to $(i_1-1,0,0)$ with intensity of transition $\sum_{j=1}^{N} p_j\mu$.

Case (iii) If the server is busy and the number of customer in the buffer is more than one

–Any one of the i_1 items perishes, then the inventory level decrease by one with intensity of transition $(i_1-1\gamma)$. Note that in this model we have assumed that the servicing item can not perish.

–At the time of service completion of the customer, both the inventory level and customer level in the buffer decrease by one and also the server become busy with intensity of transition $\sum_{j=1}^{N} p_j\mu$. Hence $\Theta_{i_1,i_1-1}, i_1 = 2,3,\ldots,S$, is given by

$$\text{For } i_1 = 2,3,\ldots,S,$$

$$[\Theta_{i_1,i_1-1}]_{i_2 j_2} = \begin{cases} C_{i_1} & j_2 = 0, \quad i_2 = 0 \\ D_{i_1} & j_2 = 1, \quad i_2 = 1 \\ F & j_2 = 0, \quad i_2 = 1 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma & j_3 = i_3, \quad i_3 \in V_0^M \\ 0 & \text{otherwise.} \end{cases}$$

$$[D_{i_1}]_{i_3 j_3} = \begin{cases} (i_1-1)\gamma & j_3 = i_3, \quad i_3 \in V_1^M \\ \sum_{j=1}^{N} p_j\mu & j_3 = i_3-1,\ i_3 \in V_2^M \\ 0, & \text{otherwise.} \end{cases}$$

Hence we will denote Θ_{i_1,i_1-1}, denote as B_{i_1}, for $i_1 = 2,3,\ldots,S$,

Finally, we consider the case $\Theta_{i_1,i_1}, i_1 = 1,\ldots,S$. Here due to each one of the following mutually exclusive cases, a transition results:

–an arrival of a customer may occur
–a customer reneging in the buffer may occur
–server terminates his vacation may occur

When the inventory level lies between one to S, we have the following three state changes may arise:

Case(i): If the server is on vacation and no customer in the waiting area

–An arrival of a customer increases the number of customer in the waiting area increases by one and the state of the arrival process moves from $(i_1,0,0)$ to $(i_1,0,1)$, with intensity of transition λ.

Case(ii): If the server is on vacation and number of customers in the waiting area lies between one to M

–An arrival of a customer increases the number of customer in the waiting area increases by one and the state of the arrival process moves from $(i_1,0,i_3)$ to $(i_1,0,i_3+1)$, with intensity of transition λ. Note we have assumed that the customer finds the waiting area full, is considered to be lost.

–At the end of the vacation if the server finds at least one customer in the waiting area, the server terminates his vacation and the state of the process moves from $(i_1,0,i_3)$ to $(i_1,1,i_3)$, with the intensity of transition β.

–An impatient customer leaves from the system without getting service and the state of the process moves from $(i_1,0,i_3)$ to $(i_1,0,i_3-1)$, with intensity of transition $i_3\alpha$.

Case(iii): If the server is busy and number of customers in the waiting area lies between one to M

–An arrival of a customer increases the number of customer in the waiting area increases by one and the state of the arrival process moves from $(i_1,1,i_3)$ to $(i_1,1,i_3+1)$, with intensity of transition λ.

–An impatient customer leaves from the system without getting service and the state of the process moves from $(i_1,1,i_3)$ to $(i_1,1,i_3-1)$, with intensity of transition $(i_3-1)\alpha$. Note that in this model we have assumed that the servicing customer can not impatient.

Using the above arguments, we have constructed the following matrices

For $i_1 = 1,2,3,\ldots,S$,

$$[\Theta_{i_1,i_1}]_{i_2 j_2} = \begin{cases} H_{i_1} & j_2=0, \quad i_2=0 \\ G & j_2=1, \quad i_2=0 \\ K_{i_1} & j_2=1, \quad i_2=1 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[G]_{i_3 j_3} = \begin{cases} \beta & j_3=i_3, \quad i_3\in V_1^M \\ 0, & \text{otherwise.} \end{cases}$$

$$[H_{i_1}]_{i_3 j_3} = \begin{cases} \lambda & j_3=i_3+1,\ i_3\in V_0^{M-1} \\ i_3\alpha & j_3=i_3-1,\ i_3\in V_1^M \\ -(\lambda\bar{\delta}_{i_3M}+ & j_3=i_3, \quad i_3\in V_0^M \\ (i_3\alpha+\beta)\bar{\delta}_{i_30}+i_1\gamma) & \\ 0 & \text{otherwise.} \end{cases}$$

$$[K_{i_1}]_{i_3 j_3} = \begin{cases} \lambda & j_3=i_3+1,\ i_3\in V_1^{M-1} \\ (i_3-\underline{1})\alpha & j_3=i_3-1,\ i_3\in V_2^M \\ -(\lambda\bar{\delta}_{i_3M}+(i_3-1)\alpha\bar{\delta}_{i_31}+ & j_3=i_3, \quad i_3\in V_0^M \\ (i_1-1)\gamma+\sum_{j=1}^{N} p_j\mu) & \\ 0, & \text{otherwise.} \end{cases}$$

–For other transitions from (i_1,i_2,i_3) to (j_1,j_2,j_3), except $(j_1,j_2,j_3)\neq(i_1,i_2,i_3)$, the rate is zero.

–To obtain the intensity of passage, $a((i_1,i_2,i_3);(j_1,j_2,j_3))$ of state (i_1,i_2,i_3) we note that the entries in any row of this matrix add up to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly

$$a((i_1,i_2,i_3),(i_1,i_2,i_3)) = -\sum_{\substack{i_1 \\ (i_1,i_2,i_3)\neq(j_1,j_2,j_3)}}\sum_{i_2}\sum_{i_3} a((i_1,i_2,i_3),(j_1,j_2,j_3))$$

We denote Θ_{i_1,i_1}, denote as A_{i_1}, for $i_1 = 2,3,\dots,S$. Hence the matrix Θ can be written in the following form

$$[\Theta]_{i_1 j_1} = \begin{cases} A_{i_1} & j_1 = i_1,\ i_1 = 1,2,\dots,S \\ B_{i_1} & j_1 = i_1 - 1,\ i_1 = 2,\dots,S-1,S \\ B_1 & j_1 = S,\ i_1 = 1, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

More explicitly,

$$[\Theta]_{i_1 j_1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & S-1 & S \end{array} \\ \begin{pmatrix} A_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_1 \\ B_2 & A_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_3 & A_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{S-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_S & A_S \end{pmatrix} \end{array}$$

The infinitesimal generator

$$\Theta = ((a((i_1,i_2,i_3),(j_1,j_2,j_3)))), \quad (i_1,i_2,i_3),(j_1,j_2,j_3) \in E$$

of this Markov process, we can constructed the following state transitions:

$a(\ (i_1,i_2,i_3),(j_1,j_2,j_3)\ ) =$

$$\begin{cases} \lambda, & j_1 = i_1, \quad j_2 = i_2,\ j_3 = i_3 + 1, \\ & i_1 \in V_1^S, \quad i_2 = 0,\ i_3 \in V_0^{M-1}, \\ & \quad \text{or} \\ & j_1 = i_1, \quad j_2 = i_2,\ j_3 = i_3 + 1, \\ & i_1 \in V_1^S, \quad i_2 = 1,\ i_3 \in V_1^{M-1}, \\ & \quad \text{or} \\ \sum\limits_{j=1}^{N} p_j \mu & j_1 = S, \quad j_2 = 0,\ j_3 = 0, \\ & i_1 = 1, \quad i_2 = 1,\ i_3 = 1, \\ & \quad \text{or} \\ & j_1 = S, \quad j_2 = i_2,\ j_3 = i_3 - 1, \\ & i_1 = 1, \quad i_2 = 1,\ i_3 \in V_2^M, \\ & \quad \text{or} \\ & j_1 = i_1 - 1, \quad j_2 = 1,\ j_3 = i_3 - 1, \\ & i_1 \in V_2^S, \quad i_2 = 1,\ i_3 \in V_2^M, \\ & \quad \text{or} \\ & j_1 = i_1 - 1, \quad j_2 = 0,\ j_3 = 0, \\ & i_1 \in V_2^S, \quad i_2 = 1,\ i_3 = 1, \end{cases}$$

$$
\begin{cases}
\beta & j_1 = i_1, \quad j_2 = 0, \; j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = 0, \; i_3 \in V_1^M, \\
\gamma, & j_1 = S, \quad j_2 = 0, \; j_3 = i_3, \\ & i_1 = 1, \quad i_2 = 0, \; i_3 \in V_0^M, \\
i_1\gamma, & j_1 = i_1 - 1, \; j_2 = 0, \; j_3 = i_3, \\ & i_1 \in V_2^S, \quad i_2 = 0, \; i_3 \in V_0^M, \\
(i_1-1)\gamma, & j_1 = i_1 - 1, \; j_2 = 1, \; j_3 = i_3, \\ & i_1 \in V_2^S, \quad i_2 = 1, \; i_3 \in V_1^M, \\
i_3\alpha, & j_1 = i_1, \quad j_2 = i_2, \; j_3 = i_3 - 1, \\ & i_1 \in V_1^S, \quad i_2 = 0, \; i_3 \in V_1^M, \\
(i_3-1)\alpha, & j_1 = i_1, \quad j_2 = i_2, \; j_3 = i_3 - 1, \\ & i_1 \in V_1^S, \quad i_2 = 1, \; i_3 \in V_2^M, \\
-(\lambda\overline{\delta}_{i_3M} + (i_3\alpha+\beta)\overline{\delta}_{i_30} + & j_1 = i_1, \quad j_2 = i_2, \; j_3 = i_3, \\ \quad i_1\gamma), & i_1 \in V_1^S, \quad i_2 = 0, \; i_3 \in V_0^M, \\
-(\lambda\overline{\delta}_{i_3M} + (i_3-1)\alpha\overline{\delta}_{i_31} + & j_1 = i_1, \quad j_2 = i_2, \; j_3 = i_3, \\ \quad (i_1-1)\gamma + \sum_{j=1}^{N} p_j\mu), & i_1 \in V_1^S, \quad i_2 = 0, \; i_3 \in V_0^M, \\
0, & \text{otherwise.}
\end{cases}
$$

It may be noted that the matrices A_{i_1}, and B_{i_1}, $i_1 = 1,2,\ldots,S$, are square matrices of size $(2M+1)\times(2M+1)$. C_{i_1}, and H_{i_1}, $i_1 = 1,2,3,\ldots,S$, are square matrices of size $(M+1)\times(M+1)$. D_{i_1}, and K_{i_1}, $i_1 = 1,2,3,\ldots,S$, are square matrices of size $M\times M$. F and G are matrices of size $M\times(M+1)$ and $(M+1)\times M$ respectively.

4 Computation of steady state probability vector

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t),Y(t),X(t)), t \geq 0\}$ on the finite space E is irreducible. Hence the limiting distribution

$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[L(t) = i_1, Y(t) = i_2, X(t) = i_3 | L(0), Y(0), X(0)]$,

exists. Let $\Phi = (\phi^{(\mathbf{1})}, \ldots, \phi^{(\mathbf{S})})$,
partitioning the vector, $\phi^{(\mathbf{i_1})}$ as follows, for $i_1 \geq 1$

$\phi^{(\mathbf{i_1})} = (\phi^{(\mathbf{i_1,0})}, \phi^{(\mathbf{i_1,1})}), \qquad i_1 = 1,2,\ldots,S;$

which is partitioned as follows,

$$
\begin{aligned}
\phi^{(\mathbf{i_1,0})} &= (\phi^{(i_1,0,0)}, \phi^{(i_1,0,1)}, \ldots, \phi^{(i_1,0,M)}), & i_1 = 1,2,\ldots,S; \\
\phi^{(\mathbf{i_1,1})} &= (\phi^{(i_1,1,1)}, \ldots, \phi^{(i_1,1,M)}), & i_1 = 1,2,\ldots,S;
\end{aligned}
$$

where $\phi^{(i_1,i_2,i_3)}$ denotes the steady state probability for the state (i_1,i_2,i_3) of the process, exists and is given by

$$\Phi\Theta = 0 \quad \text{and} \quad \sum_{(i_1,i_2,i_3)}\sum\sum \phi^{(i_1,i_2,i_3)} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

$$\phi^{(i_1)}A_{i_1} + \phi^{(i_1+1)}B_{i_1+1} = 0, \qquad i_1 = 1,2,\ldots,S-1, \tag{2}$$
$$\phi^{(S)}A_S + \phi^{(1)}B_1 = 0. \tag{3}$$

The limiting probability distribution $\phi^{(i_1)}, i_1 = 1,\ldots,S$, can be obtained using the following algorithm.

Algorithm:

1. Solve the following system of equations to find the value of $\phi^{(1)}$

$$\phi^{(1)}\left[B_1+\left\{(-1)^{S-1}\overset{S-1}{\underset{r=1}{\Omega}}A_rB_{r+1}^{-1}\right\}\right]=0,$$

and

$$\phi^{(1)}\left[\sum_{i_1=2}^{S}\left((-1)^{i_1-1}\overset{i_1-1}{\underset{r=1}{\Omega}}A_rB_{r+1}^{-1}\right)+I\right]\mathbf{e} = 1.$$

2. Compute the values of

$$\Pi_{i_1}=(-1)^{i_1-1}\overset{r=i_1-1}{\underset{k=1}{\Omega}}A_kB_{k+1}^{-1} \qquad i_1=1,2,\ldots,S$$

3. Using $\phi^{(1)}$ and $\Pi_{i_1}, i_1 = 1,\ldots,S$ calculate the value of $\phi^{(i_1)}, i_1 = 1,\ldots,S$. That is,

$$\phi^{(i_1)} = \phi^{(1)}\Pi_{i_1}, \quad i_1=1,\ldots,S.$$

5 Performance measures of the system

In this section some performance measures of the system under consideration in the steady state are derived.

5.1 Mean inventory level

Let η_{II} denote the mean inventory level in the steady state. Since $\phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of customers in the system and the status of server, $\phi^{(i_1)}\mathbf{e}$ gives the probability of i_1 item in the inventory in the steady state. Hence η_{II} is given by

$$\eta_{II}=\sum_{i_1=1}^{S} i_1\phi^{(i_1)}\mathbf{e}$$

5.2 Expected reorder rate

Let η_{RR} denote the mean reorder rate in the steady state. We note that a reorder is triggered when the inventory level drops from 1 to 0. This will occur when

1. a service completion of the customer if the server is busy and only one customer in the system or
2. a service completion of the customer if the server is busy and the system has more than one customer or
3. a failure of one item if the server is on vacation and the number of customers in the waiting area lies between zero to M.

This leads to

$$\eta_{RR}=\sum_{j=1}^{N}p_j\mu\phi^{(1,1,1)}+\sum_{i_3=2}^{M}\sum_{j=1}^{N}p_j\mu\phi^{(1,1,i_3)}+\sum_{i_3=0}^{M}\gamma\phi^{(1,0,i_3)} \tag{4}$$

5.3 Mean failure rate

Since $\phi^{(i_1,i_2,i_3)}$ is a vector of probabilities with the inventory level is i_1, status of the server is i_2 and the number of customer in the waiting area is i_3, the mean failure rate η_{FR} in the steady state is given by

$$\eta_{FR}=\sum_{i_3=0}^{M}\gamma\phi^{(1,0,i_3)}+\sum_{i_1=2}^{S}\sum_{i_3=0}^{M}i_1\gamma\phi^{(i_1,0,i_3)}+\sum_{i_1=2}^{S}\sum_{i_3=1}^{M}(i_1-1)\gamma\phi^{(i_1,1,i_3)} \tag{5}$$

5.4 Mean loss rate of the customers

Let η_{LR} denote the mean loss rate of the customer in the steady state. Any arriving customer finds the waiting area is full and leave the system without getting service. These customers are considered to be lost. Thus we obtain

$$\eta_{LR} = \sum_{i_1=1}^{S} \sum_{i_2=0}^{1} \lambda \phi^{(i_1,i_2,i_3)} \tag{6}$$

5.5 Mean waiting time

Let η_{WT} denote the mean waiting time in the steady state and is given by

$$\eta_{WT} = \frac{\eta_{WW}}{\eta_{ARA}}, \tag{7}$$

where

$$\eta_{WW} = \sum_{i_1=1}^{S} \sum_{i_2=0}^{1} \sum_{i_3=1}^{M} i_3 \phi^{(i_1,i_2,i_3)}$$

and the mean arrival rate (Ross [12]), η_{ARA} is given by

$$\eta_{ARA} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{M-1} \lambda \phi^{(i_1,1,i_3)} + \sum_{i_1=1}^{S} \sum_{i_3=0}^{M-1} \lambda \phi^{(i_1,0,i_3)}$$

5.6 Mean Reneging rate

Let η_{IP} denote the mean reneging rate of the customers in the steady state which is given by

$$\eta_{IP} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{M} i_3 \alpha \phi^{(i_1,0,i_3)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{M} (i_3-1) \alpha \phi^{(i_1,1,i_3)} \tag{8}$$

5.7 Fraction of time the server is on vacation

Let η_{FV} denote the fraction of time the server is on vacation in the steady state and is given by

$$\eta_{FV} = \sum_{i_1=1}^{S} \sum_{i_3=0}^{M} \phi^{(i_1,0,i_3)} \tag{9}$$

5.8 Mean number of customers waiting while server is on vacation

Let η_{WV} denote the mean number of customers waiting while server is on vacation in the steady state and is given by

$$\eta_{WV} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{M} i_3 \phi^{(i_1,0,i_3)} \tag{10}$$

5.9 Mean number of customers in the system when the server is providing service

Let η_{WP} denote the mean number of customers in the waiting area when the server is busy in the steady state and is given by

$$\eta_{WB} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{M} i_3 \phi^{(i_1,1,i_3)} \tag{11}$$

5.10 Probability that server is busy

Let P_{SB} denote probability that server is busy in the steady state which is given by

$$P_{SB} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{M} \phi^{(i_1,1,i_3)} \tag{12}$$

6 Optimal cost analysis

In this section, we discuss the problem of minimizing the steady-state expected cost rate under the following cost structure.

c_h : The inventory carrying cost per unit item per unit time
c_s : Set up cost (ordering cost) per order
c_p : Failure cost per unit item per unit time
c_w : Waiting time cost of a customer per unit time
c_l : balking cost per customer per unit time.,
c_r : reneging cost per customer per unit time,

Then the long run total expected cost rate is given by

$$TC(S,N,M) = c_h\eta_{II} + c_s\eta_{RR} + c_p\eta_{FR} + c_w\eta_{WT} + c_l\eta_{LR} + c_r\eta_{IP} \tag{13}$$

Substituting the values of η's we get TC(S, N, M)=

$$c_h\sum_{i_1=1}^{S} i_1\phi^{(i_1)}\mathbf{e} + c_s\left[\sum_{j=1}^{N} p_j\mu\phi^{(1,1,1)} + \sum_{i_3=2}^{M}\sum_{j=1}^{N} p_j\mu\phi^{(1,1,i_3)} + \sum_{i_3=0}^{M}\gamma\phi^{(1,0,i_3)}\right] +$$
$$c_p\left[\sum_{i_3=0}^{M}\gamma\phi^{(1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_3=0}^{M} i_1\gamma\phi^{(i_1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_3=1}^{M}(i_1-1)\gamma\phi^{(i_1,1,i_3)}\right] +$$
$$c_w\frac{\eta_{WW}}{\eta_{ARA}}, + c_l\sum_{i_1=1}^{S}\sum_{i_2=0}^{1}\lambda\phi^{(i_1,i_2,i_3)} + c_r\left[\sum_{i_1=1}^{S}\sum_{i_3=1}^{M} i_3\alpha\phi^{(i_1,0,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{M}(i_3-1)\alpha\phi^{(i_1,1,i_3)}\right]$$

7 Conclusion

The stochastic model discussed here is useful in studying a perishable inventory system at a service facility with multiple server vacations. The joint probability distribution of the number of customers in the waiting area and the inventory level is derived in the steady state. Various system performance measures are derived and the long-run total expected cost rate is calculated. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH-distributions.

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