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On Fixed Point Theorems for Contraction Mappings in n-Normed Spaces

*Mehmet KIR** and *Hukmi KIZILTUNC**

Department of Mathematics, Faculty of Science, Ataturk University, 25240, Erzurum, Turkey

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Abstract: In this paper, we introduce contraction mappings, $\varphi-$contraction mappings in n-normed spaces and we show that the mappings have a unique fixed point in n-Banach spaces. Also, taking advantage of the authors [15] and [16] we give a new type of contraction mappings in n-normed spaces. Thus, our results allow the work of the fixed point theory in n-normed spaces.

Keywords: $n-$normed spaces, $n-$Banach spaces, fixed point, contraction mappings.
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1 Introduction and Preliminaries

In 1963 S.Gahler introduced the concept of 2-normed space. Since 1963, S. Gähler, Y. J. Cho, R. W. Frees, C. R. Diminnie, R. E. Ehret, K. Iséki, A. White and many others have studied on both 2-normed spaces and 2-metric spaces. Recently, H. Gunavan and M. Mashadi defined n-normed space (for more details [1–7]).

The origins of the fixed point theory based on the use of good approximations to construct the existence and uniqueness of solutions, especially, for differential equations. This method is associated with the names of such celebrated mathematicians as Cauchy, Liouville, Lipschitz, Peano, Fredholm and especially Picard. In fact that the precursors of a fixed point theoretic approach are explicit in the work of Picard. However, it is the Polish mathematician Stefan Banach who is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. In spite of their being a long years old, the study in metric fixed point theory was limited to minor extensions of Banach's contraction mapping principal and its manifold applications. The theory gained new impetus largely as a result of the pioneering work of Felix Browder in the mid-nineteen sixties and the development of nonlinear functional analysis as an active and vital branch of mathematics. Pivotal in this development were the 1965 existence theorems of Browder, Göhde, and Kirk and the early metric results of Edelstein. By the end of the decade, a rich fixed point theory for nonexpansive mappings was clearly emerging and it was equally clear that such mappings play a main role in many aspects of nonlinear functional analysis with links to variational inequalities and the theory of monotone and accretive operators (for more information [8–14]).

Definition 1. *[9] Let E be a nonempty set and $T : E \to E$ a selfmap. We say that $x \in E$ is a fixed point of T if $T(x) = x$ and denote by F_T or $Fix(T)$ the set of all fixed points of T.*

Let E be any set and $T : E \to E$ a selfmap. For any given $x \in E$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we recall $T^n(x)$ the n^{th} iterative of x under T.

For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \ \ n = 1,2,... \tag{1}$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x.

Definition 2. *[3] Let $n \in \mathbb{N}$ and E be a real vector space of dimension $d \geq n$. A real valued function $\|\cdot,\cdots,\cdot\|$ on E^n satisfying the following*

n_1) $\|x_1,\cdots,x_n\| = 0$ if and only if $x_1,...,x_n$ are linearly dependent;

n_2) $\|x_1,\cdots,x_n\|$ is invariant under permutation;

n_3) $\|x_1,\cdots,x_{n-1},cx_n\| = |c|\,\|x_1,\cdots,x_{n-1},x_n\|$ for all $c \in \mathbb{R}$,

* Corresponding author e-mail: mehmetkir04@gmail.com,hukmu@atauni.edu.tr

$n_4)$ $\|x_1,\cdots,x_{n-1},y+z\| \leq \|x_1,\cdots,x_{n-1},y\| + \|x_1,\cdots,x_{n-1},z\|$,

is called a $n-$norm on E and the pair $(E,\|\cdot,\cdots,\cdot\|)$ is called $n-$normed space.

Definition 3. *[3] A sequence $\{x_n\}$ in a n-normed space $(E,\|\cdot,\cdots,\cdot\|)$ is said to be a Cauchy sequence if $\lim\limits_{m,n\to\infty}\|x_n-x_m,x_2,\cdots,x_n\| = 0$ for all $x_2,...,x_n \in E$.*

Definition 4. *[3] A sequence $\{x_n\}$ in a n-normed space $(E,\|\cdot,\cdots,\cdot\|)$ is said to be convegent if there is a point x in E such that $\lim\limits_{n\to\infty}\|x_n-x,x_2,\cdots,x_n\| = 0$ for all $x_2,...,x_n$ in E. If $\{x_n\}$ converges to x we write $x_n \to x$ as $n\to\infty$.*

Definition 5. *[3] A linear n-normed space is said to be complete if every Cauchy sequence is convergent to an element of E. A complete n-normed space E is called n-Banach space.*

Definition 6. *[2] A subset L of E of the form $\{x+ty : t\in\mathbb{R}\}$, where x and y are in E and y is a non-zero element, will be called a line.*

2 Contraction Mappings and Their Fixed Point Theorems in n-Normed Space

In this section, we introduce the new definitions which are $\varphi-$contraction mappings, contraction mappings in n-normed space. Then, we show that these mappings have a unique fixed point in n-Banach spaces.

Definition 7.*Let E be a linear n-normed space then the mapping $T : E \to E$ is said to be a contraction if there exist some $k\in[0,1)$ such that*

$$\|Tx-Ty,x_2,\cdots,x_n\| \leq k\,\|x-y,x_2,\cdots,x_n\|, \quad \text{for all } x,y,x_2,...,x_n\in E.$$

Definition 8.*Let E be a linear n-normed space then the mapping $T : E\to E$ is called contractive if*

$$\|Tx-Ty,x_2,\cdots,x_n\| < \|x-y,x_2,\cdots,x_n\|, \quad \text{for all } x,y,x_2,\cdots,x_n\in E.$$

*Example 1.*Let $(E,\|\cdot,\cdots,\cdot\|)$ be a n-normed space and S be a subset of the line $L=\{x+ty : t\in\mathbb{R}\backslash\{0\}\}$. Define $T : S\to L$ by $T(x+ty) = \frac{t}{1+\|y,x_2,...,x_n\|}y$, such that $y,x_2,...,x_n \in S$ are linearly independent. Then, if $x+t_1y$, $x+t_2y$ are in S and $z\in E$, we have

$$\begin{aligned}
&\|T(x+t_1y)-T(x+t_2y),x_2,\cdots,x_{n-1}\| \\
&= \left\|\frac{t_1}{1+\|y,x_2,\cdots,x_n\|}y - \frac{t_2}{1+\|y,x_2,\cdots,x_n\|}y,x_2,\cdots,x_n\right\| \\
&= \left|\frac{t_1-t_2}{1+\|y,x_2,\cdots,x_n\|}\right|\|y,x_2,\cdots,x_n\| \\
&< |t_1-t_2|\,\|y,x_2,\cdots,x_n\| \\
&= \|(x+yt_1)-(x+yt_2),x_2,\cdots,x_n\|.
\end{aligned}$$

Therefore, T is a contractive mapping in S.

Now, we extend the definition of contraction mapping by using a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined as following.

Definition 9. *[9] Let $\varphi : \mathbb{R}_+\to\mathbb{R}_+$ be a function. In connection with the function φ we consider the following properties:*

(i_φ) φ is monotone increasing, i.e., $t_1 \geqslant t_2$ implies $\varphi(t_1)\geqslant\varphi(t_2)$;
(ii_φ) $\varphi(t) < t$ for all $t>0$;
(iii_φ) $\varphi(0)=0$;
(iv_φ) φ is continuous;
(v_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t\geq 0$;
(vi_φ) $\sum_{n=0}^{\infty}\varphi^n(t)$ converges for all $t>0$;
(vii_φ) $t-\varphi(t)\to 0$ as $t\to\infty$;
$(viii_\varphi)$ φ is subadditive.

We have some important relationships between conditions of Definition 9 as followings;

Lemma 1.*([9])*

1) (i_φ) and (ii_φ) imply (iii_φ);
2) (ii_φ) and (iv_φ) imply (iii_φ);
3) (i_φ) and (v_φ) imply (ii_φ).

Definition 10. *[9] A function φ satisfying (i_φ) and (v_φ) is said to be a comparison function.*

Lemma 2.*([9])*
1) *Any comparison function satisfies (iii_φ);*
2) *Any comparison function satisfying $(viii_\varphi)$ satisfies (iv_φ), too;*
3) *If φ is a comparison function, then, for any $k\in\mathbb{N}^*$, φ^k is a comparison function, too;*
4) *If φ is a comparison function, then the function $s : \mathbb{R}_+\to\mathbb{R}_+$ $s(t) = \sum_{k=0}^{\infty}\varphi^k(t)$ satisfies (i_φ) and (iii_φ).*

We can give some examples for function φ as follows;
1. $\varphi : \mathbb{R}_+\to\mathbb{R}_+$, $\varphi(t) = kt$, $k\in[0,1)$, satisfies all the conditions (i_φ) - $(viii_\varphi)$.
2. $\varphi : \mathbb{R}_+\to\mathbb{R}_+$, $\varphi(t) = \frac{t}{t+1}$, satisfies (i_φ), (v_φ) and (vii_φ).

Now, we extend the definition of contraction mappings by using a comparison function $\varphi : \mathbb{R}_+\to\mathbb{R}_+$.

Definition 11.*Let $(E,\|\cdot,\cdots,\cdot\|)$ be a linear n-normed space. A mapping $T : E\to E$ is said to be a $\varphi-$contraction if there exists a comparison function $\varphi : \mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\|Tx-Ty,x_2,\cdots,x_n\| \leq \varphi(\|x-y,x_2,\cdots,x_n\|), \quad \text{for all } x,y,x_2,...,x_n\in E.$$

*Remark.*In Definition 11 if we take $\varphi(t) = kt$, $k\in[0,1)$ we obtain definition of contraction mappings to n-normed spaces. It is clear that Definition 11 is an extended of Definition 7.

Lemma 3.*Let$(E,\|\cdot,\cdots,\cdot\|)$ be a linear n-normed space then every $\varphi-$contraction $T : E\to E$ is sequentially continuous.*

Proof. Let $\{x_n\}$ be a sequence in E and $\{x_n\} \to x \in E$ that means $\|x_n - x, x_2, ..., x_n\| \to 0$ as $n \to \infty$

$$\begin{aligned}\|Tx_n - Tx, x_2, \cdots, x_n\| &\leq \varphi(\|x_n - x, x_2, \cdots, x_n\|) \\ &< \|x_n - x, x_2, \cdots, x_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus, $Tx_n \to Tx$.

Now, we are in a position to give the definition of closed set and bounded set in n-normed spaces.

Definition 12. *Let $(E, \|\cdot, \cdots, \cdot\|)$ be a linear n-normed space, C be a subset of E then the closure of C is $\overline{C} = \{x \in E;$ there is a sequence x_n of C such that $x_n \to x\}$. We say, C is sequentially closed if $C = \overline{C}$.*

Definition 13. *Let $(E, \|\cdot, \cdots, \cdot\|)$ be a linear n-normed space, B be a nonempty subset of E and $e \in B$ then B is said to be $e-$bounded if there exist some $M > 0$ such that $\|e, x_2, \cdots, x_n\| \leq M$ for all $x_2, \cdots, x_n \in B$. If for all $e \in B$, B is $e-$bounded then B is called a bounded set.*

Theorem 1. *Let $(E, \|., \cdots, .\|)$ be a linear n-Banach space and K be a nonempty closed and bounded subset of E. A selfmap $T : K \to K$ be $\varphi-$contraction then T has a unique fixed point in K.*

Proof. Let $a_0 \in K$ and $\{a_n\}_{n=0}^{\infty}$ be sequence in K such that

$$a_n = Ta_{n-1} = T^n a_0\,, \quad n = 1, 2, ...$$

Because of T is φ contraction and from (1) for all $a_0, a_1 \in K$ we have

$$\begin{aligned}&\left\|T^2(a_0) - T^2(a_1), x_2, \cdots, x_n\right\| \\ &= \|T(Ta_0) - T(Ta_1), x_2, \cdots, x_n\| \\ &\leq \varphi(\|Ta_0 - Ta_1, x_2, \cdots, x_n\|) \\ &\leq \varphi(\varphi(\|a_0 - a_1, x_2, \cdots, x_n\|)) \\ &= \varphi^2(\|a_0 - a_1, x_2, \cdots, x_n\|). \qquad (2)\end{aligned}$$

Similarly, we obtain that

$$\|T^n a_0 - T^n a_1, x_2, \cdots, x_n\| \leq \varphi^n(\|a_0 - a_1, x_2, \cdots, x_n\|), \quad \text{for all } n \in \mathbb{N}.$$

Now, we show that $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. Let $m, n > 0$, with $m > n$, take $m = n + p$

$$\begin{aligned}&\|a_n - a_m, x_2, \cdots, x_n\| \\ &= \left\|a_n - a_{n+p}, x_2, \cdots, x_n\right\| \\ &= \left\|\left[(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + +(a_{n+p-1} - a_{n+p})\right], x_2, \cdots, x_n\right\| \\ &\leq \|a_n - a_{n+1}, x_2, \cdots, x_n\| + \|a_{n+1} - a_{n+2}, x_2, \cdots, x_n\| \\ &\quad + ... + \left\|a_{n+p-1} - a_{n+p}, x_2, \cdots, x_n\right\| \\ &= \|T^n a_0 - T^n a_1, x_2, \cdots, x_n\| + \left\|T^{n+1} a_0 - T^{n+1} a_1, x_2, \cdots, x_n\right\| \\ &\quad + ... + \left\|T^{n+p-1} a_0 - T^{n+p-1} a_1, x_2, \cdots, x_n\right\| \\ &\leq \varphi^n(\|a_0 - a_1, x_2, \cdots, x_n\|) + \varphi^{n+1}(\|a_0 - a_1, x_2, \cdots, x_n\|) \\ &\quad + ... + \varphi^{n+p-1}(\|a_0 - a_1, x_2, \cdots, x_n\|). \qquad (3)\end{aligned}$$

Note that K is bounded so there is a constant $M > 0$ such that $\|a_0 - a_1, x_2, \cdots, x_n\| \leq M$ for all $x_2, ..., x_n \in K$. In (3) we make use of the definition of comparison function φ, that is

$$\|a_n - a_m, x_2, \cdots, x_n\| \leq \varphi^n(M) + \varphi^{n+1}(M) + \cdots + \varphi^{n+p-1}(M).$$

From definition of φ, we obtain

$$\begin{aligned}&\lim_{n\to\infty} \|a_n - a_m, x_2, \cdots, x_n\| \\ &= \lim_{n\to\infty} \left\|a_n - a_{n+p}, x_2, \cdots, x_n\right\| \\ &\leq \lim_{n\to\infty} \varphi^n(M) + \lim_{n\to\infty} \varphi^{n+1}(M) + ... + \lim_{n\to\infty} \varphi^{n+p-1}(M) \\ &= 0.\end{aligned}$$

Hence, $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. The $\{a_n\}_{n=0}^{\infty}$ converges to a in K that K is a closed and bounded subset of E. Also, by continuity of T, we have

$$Ta = \lim Ta_n = \lim a_{n+1} = a \quad \text{, as } n \to \infty.$$

Therefore, T has a fixed point in K. Now, we prove that the fixed point is unique. Let $a' \in K$ and assume that a' is an other fixed point of T. From (1) we have $Ta' = a'$.

Using definition of φ function we have

$$\begin{aligned}\left\|a - a', x_2, \cdots, x_n\right\| &= \left\|Ta - Ta', x_2, \cdots, x_n\right\| \\ &\leq \varphi\left(\left\|a - a', x_2, \cdots, x_n\right\|\right). \qquad (4)\end{aligned}$$

The inequalty (4) contradiction to property $\varphi(t) \leq t$. This implies that

$$\left\|a - a', x_2, \cdots, x_n\right\| = 0$$

Hence, we have $a = a'$ in K so the fixed point is unique. This is completes the proof.

Theorem 2. *Let $(E, \|., ..., .\|)$ be a linear n-normed space and K be a nonempty closed and bounded subset of E. Let $T : K \to K$ be a contraction then T has a unique fixed point on X.*

Proof. *If we take $\varphi(t) = kt$, $k \in [0, 1)$ then, we obtain the proof as a result of Theorem 1.*

Theorem 3. *Let S be a subset of the line $L = \{x + ty : t \in \mathbb{R}_+\}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function. Define $T : S \to L$ by $T(x + ty) = \varphi(t)y$ then T is contractive mapping in S.*

Proof. Let $x + t_1 y$, $x + t_2 y \in S$ under condition $t_1 > t_2$. For all $x_2, ..., x_n \in E$, from Definition 10, we have $\varphi(t_1) > \varphi(t_2)$ and we obtain the following

$$\begin{aligned}&\|T(x + t_1 y) - T(x + t_2 y), x_2, ..., x_n\| \\ &= \|\varphi(t_1)y - \varphi(t_2)y, x_2, ..., x_n\| \\ &= |\varphi(t_1) - \varphi(t_2)|\, \|y, x_2, ..., x_n\| \\ &< |t_1 - t_2|\, \|y, x_2, ..., x_n\| \\ &= \|t_1 y - t_2 y, x_2, ..., x_n\| \\ &= \|(x + t_1 y) - (x + t_2 y), x_2, ..., x_n\|.\end{aligned}$$

Thus, we arrive at the desired result.

In the next section, we will give a new type of contraction mappings in n-normed spaces. We will make the definition taking advantage of the authors [15] and [16].

3 The Concept of n-Contraction Mappings in n-Normed Space

In 2004, Chu et al. [15] defined the concept of n-Lipschitz mapping and n-isometry which are suitable for representing the notion of n-distance preserving mappings in linear n-normed space and studied the Aleksandrov problem in linear n-normed spaces (for more details, [15], [16]).

In this section we introduce the concept of n-contraction mappings and give some new fixed point theorems for n-contraction mappings in n-Banach spaces.

Definition 14. *Let E be a linear n-normed space. We call T an n-contraction mapping if there is a $k \in [0,1)$ such that*

$$\|Tx_1 - Tx_0, Tx_2 - Tx_0, ..., Tx_n - Tx_0\| \le k\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| \tag{5}$$

for all $x_0, x_1, ..., x_n \in E$.

Theorem 4. *Let $(E, \|., \cdots, .\|)$ be a linear n-Banach space and K be a nonempty closed and bounded subset of E. A selfmap $T : K \to K$ be $n-$contraction then the sequences $\{a_n\}$ generated from arbitrary $y_0 \in K$ by*

$$a_n = T^n b_i,\ n = 0, 1, 2, ... \tag{6}$$

$$b_i = a_0 + \frac{i}{c}(a_1 - a_0),\ i = 0, 1, 2, ..., n;\ c \in \mathbb{N} \tag{7}$$

converges to some fixed point of T.

Proof. For $i = 0, 1, 2, ..., n$, $y_i \in E$, $a_n \in E$, for $n = 0, 1, 2, ...$

$$\begin{aligned}&\left\|T^2 b_1 - T^2 b_0, ..., T^2 b_n - T^2 b_0\right\| \\ &\le k\|Tb_1 - Tb_0, ..., Tb_n - Tb_0\| \\ &\le k^2 \|b_1 - b_0, ..., b_n - b_0\|,\end{aligned}$$

continuing this process, we easly arrive at

$$\|T^n b_1 - T^n b_0, ..., T^n b_n - T^n b_0\| \le k^n \|b_1 - b_0, ..., b_n - b_0\|. \tag{8}$$

Now, we show $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. Let $m, n \in \mathbb{N}$, with $m > n$, take $m = n + p$

$$\begin{aligned}&\|a_n - a_m, x_2, ..., x_n\| \\ &\le \|a_n - a_{n+1}, x_2, ..., x_n\| + \|a_{n+1} - a_{n+2}, x_2, ..., x_n\| \\ &\quad ... + \left\|a_{n+p-1} - a_{n+p}, x_2, ..., x_n\right\|.\end{aligned} \tag{9}$$

Also, for all $x_2, ..., x_n \in K$ we have

$$\begin{aligned}\|a_{n+1} - a_n, x_2, ..., x_n\| &= \left\|T^{n+1} b_i - T^n b_i, x_2, ..., x_n\right\| \\ &\le k^n \|Tb_i - b_i, x_2, ..., x_n\|,\end{aligned}$$

countining this process, we arrive at

$$1)\ \|a_{n+1} - a_n, x_2, ..., x_n\| \le k^n \|Tb_i - b_i, x_2, ..., x_n\| \tag{10}$$

$$2)\ \|a_{n+2} - a_{n+1}, x_2, ..., x_n\| \le k^n \|Tb_i - b_i, x_2, ..., x_n\| \tag{11}$$

$$3)\ \|a_{n+3} - a_{n+2}, x_2, ..., x_n\| \le k^n \|Tb_i - b_i, x_2, ..., x_n\| \tag{12}$$

$$\vdots$$

$$p)\ \left\|a_{n+p} - a_{n+p-1}, x_2, ..., x_n\right\| \le k^n \|Tb_i - b_i, x_2, ..., x_n\|. \tag{13}$$

Substituting (10)-(13) into (9) and simplifying, we have

$$\|a_n - a_m, x_2, ..., x_n\| \le k^n.p\,\|Tb_i - b_i, x_2, ..., x_n\|.$$

Note that K is bounded, there is a constant $M > 0$ such that $\|Tb_i - b_i, x_2, ..., x_n\| \le M$ for all l $x_2, ..., x_n \in K$. Thus, leads to the following:

$$\|a_n - a_m, x_2, ..., x_n\| \le k^n pM. \tag{14}$$

When we take $n \to \infty$ in (14), we obtain that

$$\begin{aligned}\lim_{n\to\infty} \|a_n - a_m, x_2, ..., x_n\| &= \lim_{n\to\infty} \left\|a_n - a_{n+p}, x_2, ..., x_n\right\| \\ &\le \lim_{n\to\infty} k^n pM \\ &= 0.\end{aligned}$$

Hence, $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. Obviously that K is a closed and bounded subset of E. Therefore, we consider that $\{a_n\}_{n=0}^{\infty}$ converges to "a" in K such that $a = b_t$, $i < t < n$. Additionally, from continuity of T, we see that

$$Ta = T(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} Ta_n = \lim_{n\to\infty} a_{n+1} = a.$$

This implies that b_t is fixed point of T. Now, we prove that the fixed point is unique. Let $b_{t_2} = a' \in K$ and assume that b_{t_2} is an other fixed point of T. Then $Tb_{t_2} = b_{t_2} = a'$, $i < t_2 < n$.

Note that if T n-contraction, for $x, y, x_2, ..., x_n \in K$ we have

$$\begin{aligned}&\|Tx - Ty, x_2, ..., x_n\| \\ &\le \|Tx - Ty, Tx_2 - Ty, ..., x_n\| + \|Tx - Ty, x_2 + Ty - Tx_2, ..., x_n\| \\ &\le \|Tx - Ty, Tx_2 - Ty, ..., x_n\| \\ &\quad \vdots \\ &\le \|Tx - Ty, Tx_2 - Ty, ..., Tx_n - Ty\| \\ &\le k\|x - y, x_2 - y, ..., x_n - y\| \\ &\le k\|x - y, x_2, ..., x_n\|.\end{aligned}$$

Therefore, if T is n-contraction then T is contraction in n-normed space.Thus, we can use this fact to show uniqueness of fixed point of T.

$$\begin{aligned}\left\|a - a', x_2, \cdots, x_n\right\| &= \left\|Ta - Ta', x_2, \cdots, x_n\right\| \\ &\le k\left\|a - a', x_2, \cdots, x_n\right\|.\end{aligned} \tag{15}$$

This is contradiction to $k \in [0,1)$. Therefore, the fixed point is unique for n-contraction mapping T.

Now we extend the definition of n-contraction mapping by using a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 15. *Let $(E, \|\cdot, \cdots, \cdot\|)$ be a linear n-normed space. A mapping $T : E \to E$ is said to be a $\varphi - n -$ contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|Tx_1 - Tx_0, Tx_2 - Tx_0, \cdots, Tx_n - Tx_0\| \le \varphi\left(\|x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0\|\right),$$

for all $x_1, x_0, x_2, \ldots, x_n \in E$.

Remark. In Definition 15, if we take $\varphi(t) = kt$, $k \in [0,1)$ we obtain definition of n- contraction mappings to n-normed spaces. It is clear that Definition 15 is an extended of Definition 14.

Theorem 5. *Let $(E, \|., \cdots, .\|)$ be a linear n-Banach space and K be a nonempty closed and bounded subset of E. A selfmap $T : K \to K$ be $\varphi - n-$contraction then the sequences $\{a_n\}$ generated from arbitrary $b_0 \in K$ by*

$$a_n = T^n b_i,\ n = 0, 1, 2, \ldots$$

$$b_i = a_0 + \frac{i}{c}(a_1 - a_0),\ i = 0, 1, 2, \ldots, n;\ c \in \mathbb{N}$$

converges to some fixed point of T.

Proof. *For $i = 0, 1, 2, \ldots, n$, $y_i \in E$, $a_n \in E$, for $n = 0, 1, 2, \ldots$*

$$\begin{aligned} &\left\|T^2 b_1 - T^2 b_0, \ldots, T^2 b_n - T^2 b_0\right\| \\ &\le \varphi\left(\|T b_1 - T b_0, \ldots, T b_n - T b_0\|\right) \\ &\le \varphi^2\left(\|b_1 - b_0, \ldots, b_n - b_0\|\right), \end{aligned}$$

continuing this process, we easly arrive at

$$\|T^n b_1 - T^n b_0, \ldots, T^n b_n - T^n b_0\| \le \varphi^n\left(\|b_1 - b_0, \ldots, b_n - b_0\|.\right) \tag{16}$$

Now, we show $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. Let $m, n \in \mathbb{N}$, with $m > n$, take $m = n + p$

$$\begin{aligned} &\|a_n - a_m, x_2, \ldots, x_n\| \\ &\le \|a_n - a_{n+1}, x_2, \ldots, x_n\| + \|a_{n+1} - a_{n+2}, x_2, \ldots, x_n\| \\ &\quad \ldots + \left\|a_{n+p-1} - a_{n+p}, x_2, \ldots, x_n\right\|. \end{aligned} \tag{17}$$

Also, for all $x_2, \ldots, x_n \in K$ we have

$$\begin{aligned} \|a_{n+1} - a_n, x_2, \ldots, x_n\| &= \left\|T^{n+1} b_i - T^n b_i, x_2, \ldots, x_n\right\| \\ &\le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right), \end{aligned} \tag{18}$$

continuing this process, we arrive at

$$1)\ \|a_{n+1} - a_n, x_2, \ldots, x_n\| \le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right) \tag{19}$$

$$2)\ \|a_{n+2} - a_{n+1}, x_2, \ldots, x_n\| \le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right)$$

$$3)\ \|a_{n+3} - a_{n+2}, x_2, \ldots, x_n\| \le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right)$$

$$\vdots$$

$$p)\ \left\|a_{n+p} - a_{n+p-1}, x_2, \ldots, x_n\right\| \le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right). \tag{20}$$

Substituting (19)-(20) into (17) and simplifying, we have

$$\|a_n - a_m, x_2, \ldots, x_n\| \le \varphi^n\left(\|T b_i - b_i, x_2, \ldots, x_n\|\right) p.$$

Note that K is bounded, there is a constant $M > 0$ such that $\|T b_i - b_i, x_2, \ldots, x_n\| \le M$ for all $l\ x_2, \ldots, x_n \in K$. Thus, it leads to the following:

$$\|a_n - a_m, x_2, \ldots, x_n\| \le \varphi^n(M)\, p. \tag{21}$$

When we take $n \to \infty$ in (21), we obtain that

$$\begin{aligned} \lim_{n\to\infty} \|a_n - a_m, x_2, \ldots, x_n\| &= \lim_{n\to\infty} \left\|a_n - a_{n+p}, x_2, \ldots, x_n\right\| \\ &\le \lim_{n\to\infty} \varphi^n(M)\, p \\ &= 0. \end{aligned} \tag{22}$$

Hence, $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence in K. Obviously that K is a closed and bounded subset of E. Therefore, we consider that $\{a_n\}_{n=0}^{\infty}$ converges to "a" in K such that $a = b_t$, $i < t < n$. Additionally, from continuity of T, we see that

$$Ta = T(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} Ta_n = \lim_{n\to\infty} a_{n+1} = a.$$

This implies that b_t is fixed point of T. Now, we prove that the fixed point is unique. Let $b_{t_2} = a' \in K$ and assume that b_{t_2} is an other fixed point of T. Then $T b_{t_2} = b_{t_2} = a'$, $i < t_2 < n$.

Note that when T $\varphi - n$-contraction, for $x, y, x_2, \ldots, x_n \in K$ we have

$$\begin{aligned} &\|Tx - Ty, x_2, \ldots, x_n\| \\ &\le \|Tx - Ty, Tx_2 - Ty, \ldots, x_n\| + \|Tx - Ty, x_2 + Ty - Tx_2, \ldots, x_n\| \\ &\le \|Tx - Ty, Tx_2 - Ty, \ldots, x_n\| \\ &\quad \vdots \\ &\le \|Tx - Ty, Tx_2 - Ty, \ldots, Tx_n - Ty\| \\ &\le \varphi\left(\|x - y, x_2 - y, \ldots, x_n - y\|\right) \\ &\le \|x - y, x_2, \ldots, x_n\|. \end{aligned} \tag{23}$$

Therefore, if T is $\varphi - n$-contraction then T is contraction in n-normed space.Thus, we can use this fact to show uniqueness of fixed point of T.

$$\begin{aligned} \left\|a - a', x_2, \cdots, x_n\right\| &= \left\|Ta - Ta', x_2, \cdots, x_n\right\| \\ &\le \varphi\left(\left\|a - a', x_2, \cdots, x_n\right\|\right). \end{aligned} \tag{24}$$

This is contradiction to property $\varphi(t) \le t$. Therefore, the fixed point is unique. This completes the proof.

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