Applied Mathematics & Information Sciences **4**(2) (2010), 227–235
– An International Journal

Iterative Methods for Nonlinear Equations Using Homotopy Perturbation Technique

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Received February 21, 2008; Revised November 16, 2008

In this paper, we suggest and analyze some iterative methods for solving nonlinear equations using the homotopy perturbation technique coupled with system of equations. In addition, we show that the homotopy polynomial is exactly the the Adomian polynomial and this establishes the equivalence between the homotopy perturbation and decomposition techniques. Our method of establishing the equivalence between these methods is very simple as compared with others.

Keywords: Homotopy method, iterative method, convergence, Newton method, coupled system of equations.

2010 AMS Subject Classifications: 65N30.

1 Introduction

It is well known that a wide class of problems, which arise in various branches of pure and applied sciences, can be studied via the nonlinear equations of the type $f(x) = 0$ using the novel and innovative techniques. In recent years, considerable attention has been focused to solve these nonlinear equations both analytically and numerically. Several iterative type methods have been developed using quite different techniques such as Taylor's series, quadrature formulas, homotopy, interpolation, decomposition and its various modification. For recent developments, see [1-7, 9-25] and the references therein.

In this paper, we consider the modified homotopy perturbation technique to suggest a number of iterative methods for solving the nonlinear equations. First of all, we rewrite the given nonlinear equations as an equivalent coupled system of equations using the Taylor's series and technique of He [6]. This approach enables us to express the given nonlinear equation as a sum of linear and nonlinear equations. This way of writing the given equation is known as the decomposition and has played the central role in solving the problems. Several kind of decomposition of the nonlinear equations have been suggested and analyzed recently, see [1-3, 5, 9-24] and the references therein.

In this paper, we use the system of coupled equations to express the given nonlinear equations as a sum of linear and nonlinear operators. We then use the modified homotopy perturbation technique involving the auxiliary parameter h. This auxiliary parameter is usually known as the controlling parameter and can be selected arbitrarily to deduce several new iterative methods for solving the nonlinear equations. In brief, we show that the iterative methods obtained by using the Adomian decomposition method by Chun [5] can be obtained as special cases of these new methods. This fact motivated to establish the equivalence between the Adomian decomposition and the homotopy perturbation method. This also explains the fact all the solution obtained by the Adomian decomposition and homotopy perturbation technique are the same. These methods of deriving these solution are quite different from each other.

2 Iterative Methods

Consider the nonlinear equation of the type

$$f(x) = 0. \tag{2.1}$$

We assume that α is a simple root of (2.1) and γ is an initial guess sufficiently close to α. We can rewrite the nonlinear equation (2.1) as a coupled system using the Taylor's series and technique of He [6] as

$$\begin{aligned} f(\gamma) \quad &+ \quad f'(\gamma)(x-\gamma) + g(x) = 0, && (2.2)\\ g(x) \quad &= \quad f(x) - f(\gamma) + f'(\gamma)(x-\gamma), && (2.3) \end{aligned}$$

where γ is the initial approximation for a zero of (2.1). We can rewrite (2.3) in the form

$$x = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{g(x)}{f'(\gamma}. \tag{2.4}$$

From the relation (2.3), it is clear that

$$g(x_0) = f(x_0). \tag{2.5}$$

We remark that the equation (2.5) plays a very important role in the derivation of the iterative method, see Chun [5]. It is worth mentioning that He [6] and Luo [8] have considered a very special case for which $f(x_0) = 0$. In this case, from (2.5), we note that $g(x_0) = 0$. We also rectify this aspect in this paper. Also see Noor [9, 10] for a different approach.

We rewrite equation (2.4) in the following equivalent useful form

$$x = c + N(x), \tag{2.6}$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \tag{2.7}$$

$$N(x) = -\frac{g(x)}{f'(\gamma)}. \tag{2.8}$$

Here $N(x)$ is a nonlinear operator. We use the homotopy perturbation technique to develop a class of new iterative methods for solving nonlinear equations of the type (2.1).

Following the homotopy perturbation technique [1,2], one usually defines $H(\nu, p)$: $R \times [0,1] \times R \longrightarrow R$ as

$$H(\nu, p) = (1 - ph)[L(\nu) - L(x_))] + ph[L(\nu) + N(\nu) - c] = 0, \tag{2.9}$$

or

$$H(\nu, p) = L(\nu) - L(x_0) + hpL(x_0) + ph[N(\nu) - c] = 0, \tag{2.10}$$

where $p \in [0,1]$ is an embedding parameter, $h \neq 0$ is an auxiliary parameter, and x_0 is an initial approximation. We would like to emphasize that one has great freedom to select the initial guess x_0 and auxiliary parameter h.

Clearly from (2.9) and (2.10), we have

$$H(\nu, 0, T) = L(\nu) - L(x_0) = x - c = 0, \tag{2.11}$$

$$H(\nu, 1, T) = h\{L(\nu) + N(\nu) - c\} = h\{\nu - c - N(\nu)\} = 0. \tag{2.12}$$

The embedding parameter p increases monotonically from zero to unity as trivial problem $H(\nu, 0, T) = x - c = 0$ is continuously deformed to original problem $H(\nu, 1, T) = h\{\nu - c - N(\nu)\} = 0$. The changing process of p from zero to unity is called deformation. $L(\nu) - L(x_0)$ and $L(\nu) + N(\nu) - c$ are homotopic. The basis assumption is that the solution of equations (2.9) and (2.10) can be expressed as a power series in p

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots . \tag{2.13}$$

The approximate solution of (2.6) can be readily obtained as

$$x = \lim_{p \longrightarrow 1} \nu = x_0 + x_1 + x_2 + \cdots . \tag{2.14}$$

For the application of the homotopy perturbation method to (2.1), we can rewrite (2.6) as follows by expanding $N(x)$ into a Taylor series around x_0 as

$$\nu - c - ph\left\{N(x_0) + (x - x_0)N'(x_0) + \frac{(x - x_0)^2}{2!}N''(x_0) + \cdots\right\} = 0. \tag{2.15}$$

Substituting (2.13) into (2.15), we have

$$x_0 + px_1 + p^2x_2 + \cdots - c - ph\{N(x_0) + p(x_1 + px_2 + \cdots)N'(x_0) + \cdots\} = 0. \tag{2.16}$$

Equating the coefficients of the identical powers of p, we obtain

$$p^0: \ x_0 - c = 0, \tag{2.17}$$
$$p^1: \ x_1 - hN(x_0) = 0, \tag{2.18}$$
$$p^2: \ x_2 - hx_1N'(x_0) = 0, \tag{2.19}$$
$$p^3: \ x_3 - hx_2N'(x_0) - \frac{1}{2}hx_1^2N''(x_0) = 0, \tag{2.20}$$
$$\vdots$$

From (2.7) and (2.17), we have

$$x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \tag{2.21}$$
$$x_1 = hN(x_0), \tag{2.22}$$
$$x_2 = hx_1N'(x_0), \tag{2.23}$$
$$x_3 = h\{x_2N'(x_0) + \frac{1}{2}x_1^2n''(x_0)\}, \tag{2.24}$$

and so on.

From (2.3), (2.5) and (2.8), we have

$$N(x_0) = -\frac{g(x_0)}{f'(\gamma)} = -\frac{f(x_0)}{f'(\gamma)}, \tag{2.25}$$
$$N'(x_0) = 1 - \frac{f'(x_0)}{f'(\gamma)}, \tag{2.26}$$
$$N''(x_0) = -\frac{f''(x_0)}{f'(\gamma)}. \tag{2.27}$$

Note that x is approximated by

$$x = \lim_{m \longrightarrow \infty} X_m = x_0 + x_1 + x_2 + \cdots + x_m. \tag{2.28}$$

For $m = 0$,

$$x \approx x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)},$$

which gives us the following iterative method for solving the nonlinear equations (2.1).

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is the well known Newton method and has quadratic convergence. Chun [5] obtained Algorithm 2.1 using the Adomian decomposition technique.

For $m = 1$, we have

$$\begin{aligned} x &\approx X_1 = x_0 + x_1 = x_0 + hN(x_0) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - h\frac{g(x_0)}{f'(\gamma)} \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - h\frac{f(x_0)}{f'(\gamma)}. \end{aligned}$$

This formulation gives us the following iterative method.

Algorithm 2.2. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - h\frac{f(y_n)}{f'(x_n)} = x_n - \frac{f(x_n) + hf(y_n)}{f'(x_n)}. \end{aligned}$$

For $h = 1$, Algorithm 2.2 is exactly the well known [5, 9-23] two-step Newton method which was obtained by using the Adomian decomposition method. For different values of h, one can obtain a number of iterative methods for solving the nonlinear equations (2.1).

In a similar way, for $m = 2$, we have

$$\begin{aligned} x &\approx X_2 = x_0 + x_1 + x_3 \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - h\frac{g(x_0)}{f'(\gamma)} - h\frac{g'(x_0)}{f'(\gamma)} \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - h\frac{f(x_0)}{f'(\gamma)} - h\left\{1 - \frac{f'(x_0)}{f'(\gamma)}\right\}\frac{f(x_0)}{f'(\gamma)}. \end{aligned}$$

This formulation enables to suggest the following iterative method for solving nonlinear equations (2.1).

Algorithm 2.3. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - 2h\frac{f(y_n)}{f'(x_n)} + h\frac{f(y_n)f'(y_n)}{[f(x_n)]^2} \\ &= x_n - \frac{f(x_n) + 2hf(y_n)}{f'(x_n)} + h\frac{f(y_n)f'(y_n)}{[f(x_n)]^2}. \end{aligned}$$

We would like to mention that $h = 1$, Algorithm 2.3 was obtained in [5, 9-24] by using the Adomian decomposition method. Here we have shown that the homotopy perturbation technique can be used to obtain the same iterative methods. Using the standard technique and ideas, one can show that Algorithm 2.3 has fourth-order convergence. For different values of the auxiliary parameter h, one can obtain several new and previously known methods for solving nonlinear equations. This shows that Algorithm 2.3 is quite general and flexible. Essentially using the technique and ideas of [5, 9-24], one can consider the convergence criteria of Algorithms.

3 Equivalence

Now we establish the equivalence between the Adomian decomposition method and the homotopy perturbation technique and this is one of the main motivation of this paper.

Combining (2.21)-(2.28), we have

$$\begin{aligned} x &= \nu_0 + h\left\{N(\nu_0) + \nu_1 N'(\nu_0) + \frac{1}{2!}v_1^2 N''(\nu_0) + \cdots\right\} \\ &= \nu_0 + h\sum_{m=1}^{\infty} \nu_m, \end{aligned}$$

where ν_m are homotopy polynomials and are given by

$$\nu_m = \frac{1}{m!}\frac{d^m N(\nu)}{dv^m}|_{m=0}. \tag{3.1}$$

Recall that the Adomian decomposition method consists in finding the series solution of the nonlinear equation (2.1) of the form

$$x = \sum_{n=0}^{\infty} x_n \tag{3.2}$$

and the nonlinear term $N(x)$ is decomposed as

$$N(x) = \sum_{n=0}^{\infty} A_n, \tag{3.3}$$

where A_n are polynomials which are called the Adomian's polynomials depending on $x_o, x_1, x_2, \ldots$, which are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{3.4}$$

From (2.6), (2.28) and (3.1), we have

$$\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n. \tag{3.5}$$

Combining (2.14), (2.17)-(2.24) and (3.1)-(3.5), we conclude that

$$\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n = c + \sum_{n=1}^{\infty} \nu_n = \nu_0 + \sum_{n=1}^{\infty} \nu_n,$$

which shows that both the Adomian decomposition method and homotopy perturbation technique are equivalent.

Remark 3.1. We would like to emphasize that the equivalence between Adomian decomposition and homotopy perturbation technique has played a central part in all the previous works. This is the main reason that all the results obtained by both methods are the same. This fact has been pointed out by several others authors in recent years. However, we would like to remark that our method of establishing the equivalence between these methods is very simple as compared with other methods. It is an open problem to extend the Adomian decomposition method and homotopy perturbation technique for solving the variational inequalities, which have are being used to study a wide class of problems in pure and applied sciences. For the formulation, applications and numerical methods for solving variational inequalities, see [11,12] and the references therein.

Acknowledgement

The author would like to express his gratitude to Dr. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

References

[1] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified decomposition method, *Appl. Math. Computation* **145** (2003), 887–893.

[2] S. Abbasbandy, Y. Tan, and S. J. Liao, Newton-homotopy analysis method for nonlinear equations, *Appl. Math. Computation* **188** (2007), 1794–1800.

[3] F. M. Allan, Derivation of the Adomian decomposition method using the homotopy analysis method, *Appl. Math. Comput.* **190** (2007), 6–14.

[4] R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS publishing company, Boston, USA, 2001.

[5] C. Chun, Iterative methods improving Newton's method by the decomposition method, *Computers Math. Appl.* **50** (2005), 1559–1568.

[6] J. H. He, A new iterative method for solving algebraic equations, *Appl. Math. Computation* **135** (2005), 81–84.

[7] A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equations*, McGraw-Hill, New York, 1970.

[8] X. Luo, A note on the new iteration for solving algebraic equations, *Appl. Math. Computation* **171**(2005), 1177–1183.

[9] K. Inayat Noor, M. Aslam Noor, and S. Momani, Modified Householder iterative method for nonlinear equations, *Appl. Math. Computation* **190** (2007), 1534–1539.

[10] K. Inayat Noor and M. Aslam Noor, Predictor-corrector Halley method for nonlinear equations, *Appl. Math. Computation* **188** (2007), 1587–1591.

[11] M. Aslam Noor, General variational inequalities, *Appl. Math. Letters* **1** (1988), 119–121.

[12] M. Aslam Noor, Some developments in general variational inequalities, *Appl. Math. Computation* **152** (2004), 199–277.

[13] M. Aslam Noor, New family of iterative methods for nonlinear equations, *Appl. Math. Computation* **197** (2007), 553–558.

[14] M. Aslam Noor, Fifth-order convergent iterative method for solving nonlinear equations using quadrature formula, *J. Math. Control Sci. Appl.* **1** (2007), 241–249.

[15] M. Aslam Noor, New classes of iterative methods for nonlinear equations, *Appl. Math. Comput.* **191** (2007), 128–131.

[16] M. Aslam Noor, Homotopy perturbation method for solving nonlinear equations, *J. Math. Anal. Approx. Theory* **2** (2007), 111–117.

[17] M. Aslam Noor, Some iterative methods for solving nonlinear equations using the homotopy perturbation technique, *Inter. J. Comput. Math.*, in press.

[18] M. Aslam Noor, *Numerical Analysis and Optimization*, Lecture Notes, Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan, 2006–2009.

[19] M. Aslam Noor and K. Inayat Noor, Some iterative schemes for nonlinear equations, *Appl. Math. Computation* **183** (2006), 774–779.

[20] M. Aslam Noor and K. Inayat Noor, Three-step iterative methods for nonlinear equations, *Appl. Math. Computation* **183** (2006), 322–327.

[21] M. Aslam Noor, K. Inayat Noor, and Th. M. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.* **47** (1993), 285–312.

[22] M. Aslam Noor and F. A. Shah, Variational iteration technique for solving nonlinear equations, *J. Appl. Math. Computing*, in press.

[23] M. Aslam Noor and M. Waseem, Some iterative methods for solving a system of nonlinear equations, *Comput. Math. Appl.* **57** (2009), 101-106.

[24] M. Aslam Noor, S. Tauseef Mohyud-Din, and A. Waheed, Variations of parameters method for solving fifth-order boundary value problems, *Appl. Math. Infor. Sci.* **2** (2008), 135–141.

[25] J. F. Traub, *Methods for Solution of Equations*, Prentice-Hall, Englewood, Cliffs, NJ, 1964.

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