Applied Mathematics & Information Sciences **2**(1) (2008), 31–41
– An International Journal

New Abundant Exact Solutions for the System of $(2+1)$-Dimensional Burgers Equations

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Received May 15, 2007; Accepted August 11, 2007

A Bäcklund transformation and a recurrence formula are given for the system of (2+1)-dimensional Burgers equations. Also, new various sequences of exact solutions for the system are obtained by using combinations of the Bäcklund transformations and the generalized tanh function expansion method.

Keywords: Tanh function method, Bäcklund transformations, Burgers equations, exact solutions.

2000 Mathematics Subject Classification: 94A17, 94A15.

1 Introduction

The investigation of exact solutions of nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena, for example, in fluid mechanics, plasma physics, atmospheric science, optical fiber communications, etc. In the past decades, there has been significant progress in the development of methods such as the inverse scattering method [1, 2, 3], Hirota's bilinear method [4], Bäcklund transformations method [5, 6, 7], Darboux transformations method [8], similarity transformation method [9, 10], homogeneous balance method [11,12], the sine-cosine method [13,14], tanh function method [15, 16], Jacobi elliptic function method [17, 18, 19], Painlevé expansion method [20], and so on.

Bäcklund transformation is a useful tool for generating solutions to certain NLPDEs, especially, soliton equations. Using Bäcklund transformations for NLPDEs, one obtains a new solutions to the equation from a known one [21–24]. Up to now, much research

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has been devoted to the search for Bäcklund transformation, e.g., from the Painlevé Property [20], from the Ablowitz-Kaup-Newell-Seger (AKNS) scheme [1], from nonclassical symmetries [21-24], from the homogenous balance method [11, 12] and singular manifold method [20], etc.

In this paper, using combinations of the Bäcklund transformation method and the generalized tanh function expansion method, new various sequences of exact solutions are obtained for the system of (2+1)-dimensional Burgers equations [25].

In section 2, Bäcklund transformations and recurrence formula are introduced by singular manifold method. In section 3, we apply the generalized tanh function expansion method to the system of (2+1)-dimensional Burgers equations. The new abundant exact solutions for the system of (2+1)-dimensional Burgers equations are illustrated in section 4. The conclusion is then given in section 5.

2 Bäcklund Transformation

The integrable (2+1)-dimensional Burgers equation is given by [25]

$$u_t = u_{xx} + 2u_x \partial_y^{-1} u_x. \tag{2.1}$$

Starting from the general solution formula, some interesting nonlinear phenomena for Eq. (2.1) is reported. Under the transformations $u_x = v_y$, Eq. (2.1) is converted into a set of couple NLPDEs

$$u_t = u_{xx} + 2v\, u_x, \qquad u_x = v_y. \tag{2.2}$$

According to the singular manifold method [20], Peng and Yomba [26] truncated the Painlevé expansion of Eq. (2.2) at the constant level term

$$u = \frac{u_0}{\varphi} + u_1, \qquad v = \frac{v_0}{\varphi} + v_1, \tag{2.3}$$

where φ is the singular manifold and $\{u_1,\, v_1\}$ is an arbitrary seed solution of Eq. (2.2). Substituting Eq. (2.3) into Eq. (2.2) and equating the coefficients of like powers of φ yields

$$u_0 = \varphi_y, \qquad v_0 = \varphi_x, \tag{2.4}$$

where φ satisfies the equation

$$\varphi_t = \varphi_{xx} + 2v_1\, \varphi_x, \tag{2.5}$$

which is called the singular manifold equation. Eqs. (2.3)-(2.5) constitute an auto-Bäcklund transformation for Eq. (2.2). If we take $u_1 = \varphi$, $v_1 = \partial_y^{-1}\varphi_x$, then

$$u = \frac{\varphi_y}{\varphi} + \varphi, \tag{2.6}$$

where φ satisfies

$$\varphi_t = \varphi_{xx} + 2\varphi_x \partial_y^{-1} \varphi_x. \tag{2.7}$$

Eqs. (2.6) and (2.7) are another auto-Bäcklund transformation for Eq. (2.1). If we take $u_1 = 0,\ v_1 = 0$, the Cole-Hopf type transformation or hetro-Bäcklund transformation

$$u = \frac{\varphi_x}{\varphi}, \tag{2.8}$$

where φ satisfies

$$\varphi_t = \varphi_{xx}, \tag{2.9}$$

is obtained for the (2+1)-dimensional Burger equation (2.1). Therefore the Bäcklund transformations for the system of Burgers Eq. (2.2) take the form

$$u_{N+1} = \frac{u_{Ny}}{u_N} + u_N, \quad v_{N+1} = \frac{u_{Nx}}{u_N} + v_N + h_{N+1}, \tag{2.10}$$

where h_{N+1} is a constant that can be determined. We turn to the application of the Bäcklund transformation for the integrable equations. Their power lies in that they may be used to generate additional solutions of the integrable equation. Here u_{N+1} and v_{N+1} quantities refer to new solution and u_N and v_N quantities refer to old solution. This means that, on the basis of a known solution to the system of Burgers equations, we are able to find new solutions of the system.

3 Tanh Function Method

The tanh method is a powerful solution method for the computation of exact travelling wave solution. Now we introduce the travelling wave transformations

$$u(x, y, t) = u(\xi), \quad v(x, y, t) = v(\xi), \quad \xi = k_0(k_1\, x + k_2\, y + t), \tag{3.1}$$

where k_0, k_1 and k_2 are arbitrary constants that can be determined. Under the transformations (3.1), Eqs. (2.2) become ordinary differential equations (ODEs) with constant coefficients

$$k_0 k_1^2 u''(\xi) + \Big(2k_1 v(\xi) - 1\Big) u'(\xi) = 0, \;\; k_1 u'(\xi) - k_2 v'(\xi) = 0, \;\; ' = \frac{d}{d\xi}. \tag{3.2}$$

We assume that the solutions of Eqs. (3.2) are in the form

$$u(\xi) = \sum_{i=0}^{r} a_i F^i(\xi), \quad v(\xi) = \sum_{i=0}^{s} b_i F^i(\xi), \tag{3.3}$$

where $F(\xi)$ is the solution of the Riccati equation

$$F'(\xi) = A + B\,F(\xi) + C\,F^2(\xi), \tag{3.4}$$

where A, B and C are arbitrary constant. Balancing the highest derivative term with the nonlinear term in ODE (3.2) we have $r = s = 1$. The solution (3.3) of the system (2.2) becomes

$$u(\xi) = a_0 + a_1\,F(\xi), \quad v(\xi) = b_0 + b_1\,F(\xi). \tag{3.5}$$

Substituting (3.5) into (2.2) gives rise to

$$k_0 k_1^2 F'' + 2b_1 k_1 F + 2b_0 k_1 - 1 = 0, \;\; (a_1 k_1 - b_1 k_2)F' = 0. \tag{3.6}$$

From the second equation of Eq. (3.6) we obtain

$$k_2 = \frac{a_1 k_1}{b_1}. \tag{3.7}$$

Integrating the first equation of Eq. (3.6) we obtain the following equation

$$F' = \alpha + \frac{1 - 2b_0 k_1}{k_0 k_1^2} F - \frac{b_0}{k_0 k_1} F^2. \tag{3.8}$$

The Eq. (3.8) is the same as Eq. (3.4) when

$$A = \alpha, \quad B = \frac{1 - 2b_0 k_1}{k_0 k_1^2}, \quad C = -\frac{b_0}{k_0 k_1}. \tag{3.9}$$

4 Exact Solutions for the System of Burgers Equations

We obtain the following cases of soliton like-solutions and triangular periodic solutions of the system of Burgers equation (2.2):

Case 1. When $A = C = 1$ and $B = 0$, then (3.4) or (3.8) has the solution $\tan[\xi]$ where $\xi = k_0\,t + k_0\,x/(2b_0) - a_1\,y$. The solution of the system of Burgers equations (2.2) takes the form

$$u_1 = a_0 + a_1 \tan[\xi], \quad v_1 = b_0 - \frac{k_0}{2b_0}\tan[\xi]. \tag{4.1}$$

Considering (4.1) the seed solution and substituting it in the Bäcklund transformation (2.10), we find new solution as the following:

$$u_2 = \frac{a_0^2 - a_1^2 + 2a_0 a_1 \tan[\xi]}{a_0 + a_1 \tan[\xi]}, \quad v_2 = b_0 + \frac{k_0}{2b_0}\Big[\frac{a_1 - a_0 \tan[\xi]}{a_0 + a_1 \tan[\xi]}\Big]. \tag{4.2}$$

Similarly, we can find new solution from (4.2) by substituting it in the Bäcklund transformation (2.10) as the following:

$$
\begin{aligned}
u_3 &= \frac{a_0(a_0^2-3a_1^2)+a_1(3a_0^2-a_1^2)\tan[\xi]}{a_0^2-a_1^2+2a_0a_1\tan[\xi]},\\
v_3 &= b_0+\frac{k_0}{2b_0}\Big[\frac{2a_0a_1^2+(a_1^2-a_0^2)\tan[\xi]}{a_0^2-a_1^2+2a_0a_1\tan[\xi]}\Big].
\end{aligned}
\tag{4.3}
$$

and so on.

Case 2. When $A=C=-1$ and $B=0$, then (3.4) or (3.8) has the solution $\cot[\xi]$ where $\xi=k_0\,t+k_0\,x/(2b_0)+a_1\,y$. The solutions of the system of Burgers equations (2.2) are in the forms

$$u_1=a_0+a_1\cot[\xi],\qquad v_1=b_0+\frac{k_0}{2b_0}\cot[\xi], \tag{4.4}$$

$$u_2=\frac{a_0^2-a_1^2+2a_0a_1\cot[\xi]}{a_0+a_1\cot[\xi]},\qquad v_2=b_0-\frac{k_0}{2b_0}\Big[\frac{a_1-a_0\cot[\xi]}{a_0+a_1\cot[\xi]}\Big], \tag{4.5}$$

$$
\begin{aligned}
u_3 &= \frac{a_0(a_0^2-3a_1^2)+a_1(3a_0^2-a_1^2)\cot[\xi]}{a_0^2-a_1^2+2a_0a_1\cot[\xi]},\\
v_3 &= b_0+\frac{k_0}{2b_0a_1}\Big[a_0+\frac{(a_0^2+a_1^2)(a_0+a_1\cot[\xi])}{a_1^2-a_0^2-2a_0a_1\cot[\xi]}\Big],
\end{aligned}
\tag{4.6}
$$

and so on.

Case 3. When $A=1$, $C=-1$ and $B=0$, then (3.4) or (3.8) has the solutions $\tanh[\xi]$ or $\coth[\xi]$ where $\xi=k_0\,t+k_0\,x/(2b_0)+a_1\,y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1=a_0+a_1\tanh[\xi],\qquad v_1=b_0+\frac{k_0}{2b_0}\tanh[\xi], \tag{4.7}$$

$$u_2=\frac{a_0^2+a_1^2+2a_0a_1\tanh[\xi]}{a_0+a_1\tanh[\xi]},\qquad v_2=b_0+\frac{k_0}{2b_0}\Big[\frac{a_1+a_0\tanh[\xi]}{a_0+a_1\tanh[\xi]}\Big], \tag{4.8}$$

$$
\begin{aligned}
u_3 &= \frac{a_0(a_0^2+3a_1^2)+a_1(3a_0^2+a_1^2)\tanh[\xi]}{a_0^2+a_1^2+2a_0a_1\tanh[\xi]},\\
v_3 &= b_0+\frac{k_0}{2b_0a_1}\Big[a_1+\frac{(a_0^2-a_1^2)(a_1+a_0\tanh[\xi])}{a_0^2+a_1^2+2a_0a_1\tanh[\xi]}\Big],
\end{aligned}
\tag{4.9}
$$

and so on.

Case 4. When $A = 1/2$, $C = -1/2$ and $B = 0$, then (3.4) or (3.8) has the solutions $\tanh[\xi] \pm \imath\, \sec\mathrm{h}[\xi]$, $\coth[\xi] \pm\, \csc\mathrm{h}[\xi]$, $\tanh[\xi]/(1 \pm\, \sec\mathrm{h}[\xi])$, $\coth[\xi]/(1 \pm \imath\, \csc\mathrm{h}[\xi])$, $\tanh[\xi/2]$ or $\coth[\xi/2]$ where $\xi = k_0\, t + k_0\, x/(2b_0) + 2a_1\, y$ and $\imath = \sqrt{-1}$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + a_1\Big(\tanh[\xi] \pm \imath\, \sec\mathrm{h}[\xi]\Big), \quad v_1 = b_0 + \frac{k_0}{4b_0}\Big(\tanh[\xi] \pm \imath\, \sec\mathrm{h}[\xi]\Big), \quad (4.10)$$

$$\begin{aligned} u_2 &= \frac{a_0^2 + a_1^2 + 2a_0a_1(\tanh[\xi] \pm \imath\, \sec\mathrm{h}[\xi])}{a_0 + a_1(\tanh[\xi] \pm \imath\, \sec\mathrm{h}[\xi])}, \\ v_2 &= b_0 + \frac{1}{2b_0}\Big[\frac{\imath\, k_0}{2} - \frac{a_1 + \imath\, a_0 \tanh[\xi/2]}{a_1 - a_0\, \imath + (a_0 - a_1\, \imath)\tanh[\xi/2]}\Big], \end{aligned} \quad (4.11)$$

$$\begin{aligned} u_3 &= \frac{(a_1^4 - a_0^4)\imath + (a_0^4 + 6a_0^2a_1^2 + a_1^2)\sinh[\xi] + 4a_0a_1(a_0^2 + a_1^2)\cosh[\xi]}{a_0(a_1^2 - a_0^2)\imath + a_0(a_0^2 + 3a_1^2)\sinh[\xi] + a_1(3a_0^2 + a_1^2)\cosh[\xi]}, \\ v_3 &= h_3 + \Big[\frac{k_0(\imath + 1)(a_0^2 - a_1^2)}{4b_0(a_0 + a_1\, \imath)}\Big]\Big[\frac{a_0 - a_1 - (a_0 + a_1)\exp[\xi]}{(a_0 - a_1)^2 + (a_0 + a_1)^2\, \imath\, \exp[\xi]}\Big], \end{aligned} \quad (4.12)$$

where $h_3 = a_0(4b_0^2 + k_0\, \imath) + a_1(k_0 + 4b_0^2\, \imath)/\big[4b_0(a_0 + a_1\, \imath)\big]$ and so on.

Case 5. When $A = C = 1/2$ and $B = 0$, then (3.4) or (3.8) has the solutions $\tan[\xi] \pm\, \sec[\xi]$, $\pm\, \cot[\xi] - \csc[\xi]$, $\tan[\xi]/(1 \pm\, \sec[\xi])$, $-\cot[\xi]/(1 \pm\, \csc[\xi])$, $\tan[\xi/2]$ or $-\cot[\xi/2]$ where $\xi = k_0\, t + k_0\, x/(2b_0) - 2a_1\, y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + a_1(\tan[\xi] + \sec[\xi]), \quad v_1 = b_0 - \frac{k_0}{4b_0}(\tan[\xi] - \sec[\xi]), \quad (4.13)$$

$$\begin{aligned} u_2 &= \frac{a_0^2 - a_1^2 + 2a_0a_1(\tan[\xi] + \sec[\xi])}{a_0 + a_1(\tan[\xi] + \sec[\xi])}, \\ v_2 &= b_0 - \frac{k_0}{4b_0}\Big[1 + \frac{2(a_1 - a_0\tan[\xi/2])}{a_0 + a_1 + (a_1 - a_0)\tan[\xi/2]}\Big], \end{aligned} \quad (4.14)$$

$$\begin{aligned} u_3 &= \frac{a_0^4 - a_1^4 - (a_0^4 - 6a_0^2a_1^2 - a_1^2)\sin[\xi] + 4a_0a_1(a_0^2 - a_1^2)\cos[\xi]}{a_0(a_0^2 + a_1^2) + a_0(3a_1^2 - a_0^2)\sin[\xi] + a_1(3a_0^2 - a_1^2)\cos[\xi]}, \\ v_3 &= h_3 + \frac{k_0(a_0^2 + a_1^2)}{2b_0(a_0 + a_1)}\Big[\frac{a_1 - a_0\tan[\xi/2]}{a_0^2 + 2a_0a_1 - a_1^2 + (a_1^2 + 2a_0a_1 - a_0^2)\tan[\xi/2]}\Big], \end{aligned} \quad (4.15)$$

where $h_3 = a_0(4b_0^2 + k_0) + a_1(k_0 + 4b_0^2)/\big[4b_0(a_0 + a_1)\big]$ and so on.

Case 6. When $A = C = -1/2$ and $B = 0$, then (3.4) or (3.8) has the solutions $\cot[\xi]/(1 \pm\, \csc[\xi])$, $-\tan[\xi]/(1 \pm\, \sec[\xi])$, $\pm\, \sec[\xi] - \tan[\xi]$, $\cot[\xi] \pm\, \csc[\xi]$, $-\tan[\xi/2]$

or $\cot[\xi/2]$ where $\xi = k_0\, t + k_0\, x/(2b_0) + 2a_1\, y$. The solutions of the system of Burgers equations (2.2) are

$$u_1 = a_0 + \frac{a_1 \cot[\xi]}{1 + \csc[\xi]}, \quad v_1 = b_0 + \frac{k_0}{4b_0} \frac{\cot[\xi]}{1 + \csc[\xi]}, \tag{4.16}$$

$$\begin{aligned} u_2 &= a_0 + a_1 \Big[\frac{\cot[\xi]}{1 + \csc[\xi]} + \frac{2a_1 \csc[\xi]}{a_0(\csc[\xi]) - 1) + a_1 \cot[\xi]} \Big], \\ v_2 &= b_0 + \frac{k_0}{4b_0} \Big[1 - \frac{2(a_0 + a_1 \cot[\xi/2])}{a_0 - a_1 + (a_0 + a_1) \cot[\xi/2]} \Big]. \end{aligned} \tag{4.17}$$

Case 7. When $A = 1$, $C = -4$ and $B = 0$, then (3.4) or (3.8) has the solutions $\coth[\xi]/(1 + \coth^2[\xi])$, $\tanh[\xi]/(1 + \tanh^2[\xi])$, $(1/2)\tanh[2\xi]$ or $(1/2)\coth[2\xi]$ where $\xi = k_0\, t + k_0\, x/(2b_0) + (a_1/4)\, y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + \frac{a_1 \coth[\xi]}{1 + \coth^2[\xi]}, \quad v_1 = b_0 + \frac{2k_0}{b_0} \Big[\frac{\coth[\xi]}{1 + \coth^2[\xi]} \Big], \tag{4.18}$$

$$u_2 = \frac{4a_0 a_1 + (4a_0^2 + a_1^2)\coth[2\xi]}{2a_1 + 4a_0 \coth[2\xi]}, \quad v_2 = b_0 + \frac{k_0}{b_0} \Big[\frac{2a_0 + a_1 \coth[2\xi]}{a_1 + 2a_0 \coth[\xi]} \Big], \tag{4.19}$$

and so on.

Case 8. When $A = 1$, $C = 4$ and $B = 0$, then (3.4) or (3.8) has the solutions $\tan[\xi]/(1 - \tan^2[\xi])$, $\cot[\xi]/(\cot^2[\xi] - 1)$, $(1/2)\tan[2\xi]$ or $-(1/2)\cot[2\xi]$ where $\xi = k_0\, t + k_0\, x/(2b_0) - (a_1/4)\, y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + \frac{a_1 \tan[\xi]}{1 - \tan^2[\xi]}, \quad v_1 = b_0 - \frac{2k_0}{b_0} \Big[\frac{\tan[\xi]}{1 - \tan^2[\xi]} \Big], \tag{4.20}$$

$$u_2 = \frac{a_1^2 - 4a_0^2 - 4a_0 a_1 \tan[2\xi]}{4a_1 + 2a_0 \tan[2\xi]}, \quad v_2 = b_0 + \frac{k_0}{b_0} \Big[\frac{a_1 - 2a_0 \tan[2\xi]}{2a_0 + a_1 \tan[2\xi]} \Big], \tag{4.21}$$

and so on.

Case 9. When $A = -1$, $C = -4$ and $B = 0$, then (3.4) or (3.8) has the solutions $\cot[\xi]/(1 - \cot^2[\xi])$, $\tan[\xi]/(\tan^2[\xi] - 1)$, $-(1/2)\tan[2\xi]$ or $(1/2)\cot[2\xi]$ where $\xi = k_0\, t + k_0\, x/(2b_0) + (a_1/4)\, y$. The solutions of the system of Burgers equations (2.2) are

$$u_1 = a_0 + \frac{a_1}{2} \cot[2\xi], \quad v_1 = b_0 - \frac{k_0}{b_0} \cot[2\xi], \tag{4.22}$$

$$u_2 = \frac{4a_0^2 - a_1^2 + 4a_0 a_1 \cot[2\xi]}{4a_0 + 2a_1 \cot[2\xi]}, \quad v_2 = b_0 + \frac{k_0}{b_0} \Big[\frac{2a_0 \cot[2\xi] - a_1}{2a_0 + a_1 \cot[2\xi]} \Big], \tag{4.23}$$

and so on.

Case 10. When $A = 1$, $C = 2$ and $B = -2$, then (3.4) or (3.8) has the solutions $\tan[\xi]/(1+\tan[\xi])$, $\cot[\xi]/(\cot[\xi]-1)$, $(1/2)(1+\tan[\xi])$ or $1/2)(1-\cot[\xi])$ where $\xi = (2b_0k_1-1)/(2k_1^2)(t+k_1\,x)-(a_1/2)\,y$. The solutions of the system of Burgers equations (2.2) take the form

$$u_1 = a_0 + \frac{a_1\tan[\xi]}{1+\tan[\xi]}, \quad v_1 = b_0 - \frac{1}{k_0}\Big[\frac{b_0k_1+(1-b_0k_1)\tan[\xi]}{1+\tan[\xi]}\Big], \tag{4.24}$$

$$\begin{aligned} u_2 &= \frac{2a_0(1+\tan[\xi])\Big[a_0+(a_0+2a_1)\tan[\xi]\Big]-a_1^2}{2(1+\tan[\xi])\Big[a_0+(a_0-a_1)\tan[\xi]\Big]}, \\ v_2 &= b_0 + \frac{2b_0k_0-1}{2k_1}\Big[\frac{a_1-(2a_0+a_1)\tan[\xi]}{a_0+(a_0+a_1)\tan[\xi]}\Big], \end{aligned} \tag{4.25}$$

and so on.

Case 11. When $A = 1$, $C = 2$ and $B = 2$, then (3.4) or (3.8) has the solutions $\tan[\xi]/(1-\tan[\xi])$, $-\cot[\xi]/(\cot[\xi]+1)$, $(\tan[\xi]-1)/2$ or $-(1+\cot[\xi])/2$ where $\xi = (1-2b_0k_1)/(2k_1^2)\big[t+k_1\,x\big]-(a_1/2)\,y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 - \frac{a_1}{2}(1+\cot[\xi]), \quad v_1 = \frac{1+(1-2b_0k_1)\cot[\xi]}{2k_1}, \tag{4.26}$$

$$\begin{aligned} u_2 &= a_0 + a_1\Big[\frac{a_0+(a_0+a_1)\cot[\xi]}{2a_0-a_1(1+\cot[\xi])}\Big], \\ v_2 &= \frac{1}{2k_1}\Big[1+(1-2b_0k_1)\Big(\cot[\xi]+\frac{a_1\csc^2[\xi]}{2a_0+a_1(1+\cot[\xi])}\Big)\Big], \end{aligned} \tag{4.27}$$

and so on.

Case 12. When $A = -1$, $C = -2$ and $B = 2$, then (3.4) or (3.8) has the solutions $\tan[\xi]/(\tan[\xi]-1)$, $\cot[\xi]/(\cot[\xi]+1)$, $(1-\tan[\xi])/2$ or $(1+\cot[\xi])/2$ where $\xi = (1-2b_0k_1)/(2k_1^2)(t+k_1\,x)+(a_1/2)\,y$. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + \frac{a_1}{2}(1-\tan[\xi]), \quad v_1 = \frac{1+(2b_0k_1-1)\tan[\xi]}{2k_1}, \tag{4.28}$$

$$\begin{aligned} u_2 &= a_0 + a_1\Big[\frac{+a_0-(a_0-a_1)\tan[\xi]}{a_1+2a_0-a_1\tan[\xi]}\Big], \\ v_2 &= \frac{1}{2k_1}\Big[1+(2b_0k_1-1)\Big(\tan[\xi]+\frac{a_1\sec^2[\xi]}{2a_0+a_1(1+\tan[\xi])}\Big)\Big], \end{aligned} \tag{4.29}$$

and so on.

Case 13. When $A = -1$, $C = -2$ and $B = -2$, then (3.4) or (3.8) has the solutions $-\tan[\xi]/(\tan[\xi]+1)$, $\cot[\xi]/(1-\cot[\xi])$, $-(1+\tan[\xi])/2$ or $(\cot[\xi]-1)/2$ where $\xi = (2b_0k_1-1)/(2k_1^2)(t+k_1\,x)+(a_1/2)\,y$. The solutions of the system of Burgers equations (2.2) are

$$u_1 = a_0 + \frac{a_1\cot[\xi]}{1-\cot[\xi]}, \quad v_1 = \frac{1+(2b_0k_1-1)\cot[\xi]}{k_1(1-\cot[\xi])}, \tag{4.30}$$

$$\begin{aligned} u_2 &= \Big[\frac{2a_0^2+2a_0a_1-a_1^2+2a_0(a_0+a_1)\sin[\xi]+a_1(2a_0-a_1)\cos[\xi]}{2(1+\cot[\xi])\Big(a_0+(a_0-a_1)\cot[\xi]\Big)}\Big]\csc^2[\xi], \\ v_2 &= \frac{1-b_0k_1}{k_1}+\frac{2b_0k_1-1}{2k_1}\Big[\frac{2a_0+2(a_0-a_1)\cot[\xi]-a_1\csc^2[\xi]}{(1+\cot[\xi])\Big(a_0+(a_0-a_1)\cot[\xi]\Big)}\Big], \end{aligned} \tag{4.31}$$

and so on.

Case 14. When $A = 0$ and $B = 0$, then (3.4) or (3.8) has the solution $k_0k_1/(b_0\xi - k_0k_1\beta)$ where $\xi = k_0(t+k_1\,x+(a_1k_1/b_1)\,y)$ and β is an arbitrary constant. The solutions of the system of Burgers equations (2.2) take the forms

$$u_1 = a_0 + \frac{a_1k_0k_1}{k_0k_1\beta+b_1\,\xi}, \quad v_1 = \frac{1}{2k_1}+\frac{k_0k_1b_1}{k_0k_1\beta+b_1\xi}, \tag{4.32}$$

$$\begin{aligned} u_2 &= a_0\Big[\frac{2a_1k_0k_1+a_0(k_0k_1\beta+b_1\xi)}{a_1k_0k_1+a_0(k_0k_1\beta+b_1\xi)}\Big], \\ v_2 &= \frac{a_1k_0k_1+a_0\Big(k_0k_1\beta+b_1(\xi+2k_0k_1^2)\Big)}{2k_1\Big(a_1k_0k_1+a_0(k_0k_1\beta+b_1\xi)\Big)}, \end{aligned} \tag{4.33}$$

and so on.

5 Conclusion

We have obtained some traveling wave solutions expressed in terms of hyperbolic and trigonometric functions for the system of (2+1)-dimensional Burgers equations by using the generalized tanh function expansion method. The main advantage of this study is to obtain various sequences of exact solutions of the system of (2+1)-dimensional Burgers equations using Bäcklund transformations. Moreover, this method can be applied to obtain new solutions for other nonlinear evolution equations.

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