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Ruin Probability in a Generalized Risk Process under Rates of Interest with Dependent Structures

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Abstract: The aim of this paper is to build recursive and integral equations for ruin probabilities of generalized risk processes under rates of interest with homogenous markov chain claims and homogenous markov chain premiums, while the interest rates follow a first-order autoregressive processe. Generalized Lundberg inequalities for ruin probabilities of this processe are derived by using recursive technique.

Keywords: Integral equation, Recursive equation, Ruin probability, Homogeneous Markov chain.

1 Introduction

In classical risk model, the claim number process was assumed to be a Poisson process and the individual claim amounts were described as independent and identically distributed random variables. In recent years, the classical risk process has been extended to more practical and real situations. For most of the investigations treated in risk theory, it is very significant to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital. Teugels and Sundt [9], [10] studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang [12] established both exponential and non-exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu and Wang [11] given upper bounds for ruin probabilities in a risk model with interest force with independent premiums and claims, Markov chain interest rates. Cai [1], [2] investigated the ruin probabilities in two risk models with independent premiums and claims, the author used a first-order autoregressive process to model the rates of in interest. Cai and Dickson [3] obtained Lundberg inequalities for ruin probabilities in two discrete-time risk process with a Markov chain interest model, independent premiums and claims. Fenglong Guo and Dingcheng Wang [4] built Lundberg inequalities for ruin probabilities in two discrete-time risk process with the premiums, claims and rates of interest have autoregressive oving average (ARMA) dependent structures simultaneously. P. D. Quang [5] used recursive technique to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and claims, Markov chain premiums. P. D. Quang [6] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and premiums, Markov chain claims. P. D. Quang [7] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates, Markov chain claims and Markov chain premiums. P. D. Quang [8] also used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent premiums, Markov chain claims and Markov chain interests.

In this paper, we study the models considered by Cai and Dickson [3] to the case homogenous markov chain claims and homogenous markov chain premiums, while the interest rates follow a first-order autoregressive processe. Recursive and integral equations for the finite-time and ultimate ruin probabilities are established by using recursive technique. Generalized Lundberg inequalities for ruin probabilities are derived.

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We let $X=\{X_n\}_{n\geq 0}$ be premiums, $Y=\{Y_n\}_{n\geq 0}$ be claims and $I=\{I_n\}_{n\geq 0}$ be interests. Suppose that the premiums are collected at the end of each period, then the surplus process $\{U_n\}_{n\geq 0}$ with initial u can be written as

$$U_n=(U_{n-1}+X_n)(1+I_n)-Y_n, \tag{1}$$

which is quivalent to

$$U_n=u.\prod_{k=1}^{n}(1+I_k)+\sum_{k=1}^{n}\left[X_k(1+I_k)-Y_k\right]\prod_{j=k+1}^{n}(1+I_j). \tag{2}$$

where throughout this paper, we denote $\prod_{t=a}^{b}x_t=1$ and $\sum_{t=a}^{b}x_t=0$ if $a>b$.

We assume that:

Assumption 1. $U_0=u>0$.

Assumption 2. $X=\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain such that for any n, X_n takes values in a set of non-negative numbers $E_X=\{x_1,x_2,...,x_m,...\}$ with $X_o=x_i\in E_X$ and

$$p_{ij}=P\left[X_{m+1}=x_j\middle|X_m=x_i\right],(m\in N);x_i,x_j\in E_X \text{ where } 0\leq p_{ij}\leq 1,\ \sum_{j=1}^{+\infty}p_{ij}=1.$$

Assumption 3. $Y=\{Y_n\}_{n\geq 0}$ is a homogeneous Markov chain such that for any n, Y_n takes values in a set of non-negative numbers $E_Y=\{y_1,y_2,...,y_n,...\}$ with $Y_o=y_r\in E_Y$ and

$$q_{rs}=P\left[Y_{m+1}=y_s|Y_m=y_r\right],(m\in N);y_r,y_s\in E_Y \text{ where } 0\leq q_{rs}\leq 1,\ \sum_{j=1}^{+\infty}q_{rs}=1.$$

Assumption 4. $I=\{I_n\}_{n\geq 0}$ is a first-order autoregressive process,

$$I_n=aI_{n-1}+Z_n,\ n=1,2,..., \tag{3}$$

where, $I_o=i_o\geq 0$ and $0\leq a<1$ are two constants and $Z=\{Z_n\}_{n\geq 1}$ is a sequence of independent and identically distributed non-negative random variables with the distribution function $F(z)=P(Z_1\leq z)$.

Assumption 5. X, Y and I are assumed to be independent.

We define the finite time and ultimate ruin probabilities in model (1) with assumption 1 to assumption 5, respectively, by

$$\psi_n(u,x_i,y_r,i_o)=P\left(\bigcup_{k=1}^{n}(U_k<0)\middle|U_o=u,X_o=x_i,Y_o=y_r,I_o=i_o\right), \tag{4}$$

$$\psi(u,x_i,y_r,i_o)=P\left(\bigcup_{k=1}^{\infty}(U_k<0)\middle|U_o=u,X_o=x_i,Y_o=y_r,I_o=i_o\right). \tag{5}$$

It is clear that

$$\lim_{n\to\infty}\psi_n(u,x_i,y_r,i_o)=\psi(u,x_i,y_r,i_o).$$

In this paper, we derive probability inequalities for $\psi_n(u,x_i,y_r,i_o)$ and $\psi(u,x_i,y_r,i_o)$.

2 Recursive and integral equations for ruin probabilities

We first give the recursive equation for $\psi_n(u,x_i,y_r,i_o)$ and the integral equation for $\psi(u,x_i,y_r,i_o)$.

Theorem 2.1. Let model (1) satisfies assumption 1 to assumption 5 then for $n=1,2,3,\ldots$

$$\psi_{n+1}(u,x_i,y_r,i_o)=$$

$$\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{F\left(\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}\right)+\int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty}\psi_n\left((u+x_j)(1+ai_o+z)-y_s,x_j,y_s,i\right)dF(z)\right\}, \tag{6}$$

and

$$\psi(u,x_i,y_r,i_o) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}\left\{F\left(\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}\right)+\int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty}\psi\left((u+x_j)(1+ai_o+z)-y_s,x_j,y_s,i\right)dF(z)\right\}. \quad (7)$$

with $i = ai_0 + z$.

Proof.
Give $X_1 = x_j, Y_1 = y_s$. From (1), we have $U_1 = (u+X_1)(1+I_1)-Y_1 = (u+x_j)(1+ai_o+Z_1)-y_s$.
Let

$$B = \{U_o = u, X_o = x_i; Y_o = y_r, I_o = i_o\},\ A_{js} = \{X_1 = x_j, Y_1 = y_s\},$$
$$A_1 = \left\{Z_1 \geq \frac{Y_1-(u+X_1)(1+ai_o)}{u+x_j}\right\},\ A_2 = \left\{Z_1 < \frac{Y_1-(u+X_1)(1+ai_o)}{u+x_j}\right\}.$$

Thus, we have

$$P(U_1<0|A_1\cap A_{js}\cap B) = 0, \quad (8)$$

and

$$P(U_1<0|A_2\cap A_{js}\cap B) = 1 \Rightarrow P\left(\bigcup_{k=1}^{n+1}(U_k<0)\middle|A_2\cap A_{js}\cap B\right) = 1. \quad (9)$$

Let $\{\tilde{X}_n\}_{n\geq 0}, \{\tilde{Y}_n\}_{n\geq 0}, \{\tilde{Z}_n\}_{n\geq 1}$ be independent copies of $\{X_n\}_{n\geq 0}, \{Y_n\}_{n\geq 0}, \{Z_n\}_{n\geq 1}$ with $\tilde{X}_o = X_1 = x_j, \tilde{Y}_o = Y_1 = y_s$. Given $Z_1 = z$, consider a process $\{\tilde{I}_n\}_{n\geq 0}$ defines as

$$\tilde{I}_n = a\tilde{I}_{n-1} + \tilde{Z}_n,$$

where $\tilde{I}_o = ai_0 + z = i$. Trivially, $\{\tilde{I}_n\}_{n\geq 0}$ has a similar structure to that of $I = \{I_n\}_{n\geq 0}$ but with different initial values. Thus, (8) implies

$$\begin{aligned}
&P\left(\bigcup_{k=1}^{n+1}(U_k<0)\middle|A_1\cap A_{js}\cap B\right) = P\left(\bigcup_{k=2}^{n+1}(U_k<0)\middle|A_1\cap A_{js}\cap B\right)\\
&= P\left(\bigcup_{k=2}^{n+1}\left([(u+x_j)(1+ai_o+Z_1)-y_s]\prod_{m=2}^{k}(1+I_m)+\sum_{m=2}^{k}(X_m(1+I_m)-Y_m)\prod_{p=m+1}^{k}(1+I_p)<0\right)\middle|A_1\cap A_{js}\cap B\right)\\
&= P\left(\bigcup_{k=1}^{n}\left(\tilde{U}_o\prod_{m=1}^{k}(1+\tilde{I}_m)+\sum_{m=1}^{k}\left(\tilde{X}_m(1+\tilde{I}_m)-\tilde{Y}_m\right)\prod_{p=m+1}^{k}(1+\tilde{I}_p)<0\right)\right|\\
&\qquad \left(\tilde{U}_o = (u+x_j)(1+ai_o+Z_1)-y_s, \tilde{X}_o = x_j, \tilde{Y}_o = y_s, \tilde{I}_o = ai+Z_1\right)\cap A_1\cap B\Big) \quad (10)
\end{aligned}$$

That, (1) implies

$$\psi_{n+1}(u,x_i,y_r,i_o) = P\left(\bigcup_{k=1}^{n+1}(U_k<0)\middle|U_o = u, X_o = x_i, Y_o = y_r, I_o = i_o\right) = P\left(\bigcup_{k=1}^{n+1}(U_k<0)|B\right)$$

Thus, we have

$$\begin{aligned}
&\psi_{n+1}(u,x_i,y_r,i_o)\\
&= \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}P\left(\bigcup_{k=1}^{n+1}(U_k<0)\middle|A_{js}\cap B\right)\\
&= \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}\left\{P\left(\bigcup_{k=1}^{n+1}(U_k<0)|A_1\cap A_{js}\cap B\right).P(A_1|A_{js}\cap B)+P\left(\bigcup_{k=1}^{n+1}(U_k<0)|A_2\cap A_{js}\cap B\right).P(A_2|A_{js}\cap B)\right\}.
\end{aligned} \quad (11)$$

Thus, from (9), (10) and (11), we have

$$\psi_{n+1}(u,x_i,y_r,i_o)$$
$$= \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}\left\{ \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} dF(z) + \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} \psi_n((u+x_j)(1+ai_o+z)-y_s,x_j,y_s,i)dF(z) \right\}$$
$$= \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}\left\{ F\left(\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}\right) + \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} \psi_n((u+x_j)(1+ai_o+z)-y_s,x_j,y_s,i)dF(z) \right\} \quad (12)$$

where $i = ai_o + z$.
When $n = 0$, we have

$$\psi_1(u,x_i,y_r,i_o) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}F\left(\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}\right) \quad (13)$$

Thus, from the dominated convergence theorem, the integaral equation for $\psi(u,x_i,y_r,i_o)$ in Theorem 2.1 follows immediately by letting $n\to\infty$ in (12).

Next, we establish probability inequalities for ruin probabilities of model (1).

3 Probability inequality for ruin probability

To establish probability inequalities for ruin probabilities of model (1), we first prove the following Lemma.

Lemma 3.1. Let model (1) satisfies assumption 1 to assumption 5, $M = \sup\{x_i \in E_X\} < +\infty$ and $E(I_1^k) < +\infty\,(k=1,2)$. Any $x_i \in E_X$ and $y_r \in E_Y$, if

$$E(Y_1|Y_o=y_r) < E\left(X_1(1+I_1)|X_o=x_i\right) \text{ and } P\left(Y_1-X_1(1+I_1)>0|X_o=x_i,Y_o=y_r\right)>0, \quad (14)$$

then, there exists a unique positive constant R_{ir} satisfying:

$$E\left(e^{R_{ir}[Y_1-X_1(1+I_1)]}\Big|X_o=x_i,Y_o=y_r\right)=1. \quad (15)$$

Proof.
Define

$$f_{ir}(t) = E\left\{e^{t[Y_1-X_1(1+I_1)]}\Big|X_o=x_i,Y_o=y_r\right\}-1;\, t\in(0,+\infty).$$

We have

$$f_{ir}(t) = E\left\{e^{tY_1}\big|Y_o=y_r\right\}.E\left\{e^{-tX_1(1+I_1)}\Big|X_o=x_i\right\}-1,$$

where $g_r(t) = E\left\{e^{tY_1}\big|Y_o=y_r\right\}$ and $h_i(t) = E\left\{e^{-tX_1(1+I_1)}\Big|X_o=x_i\right\}$.
From Y_1 is a discrete random variable which takes values in $E_Y = \{y_1,y_2,...,y_n,...\}$ then

$$g_r(t) = E\left\{e^{tY_1}\big|Y_o=y_r\right\} = \sum_{s=1}^{+\infty} q_{rs}e^{ty_s}$$

have n-th derivatived function on $(0,+\infty)$ (any $n\in N^* = N\setminus\{0\}$).
From X_1 is a discrete random variable which takes values in $E_X = \{x_1,x_2,...,x_m,...\}$ then

$$h_i(t) = E\left\{e^{-tX_1(1+I_1)}\Big|X_o=x_i\right\} = \sum_{j=1}^{+\infty} p_{\text{ij}}\int_0^{+\infty} e^{-tx_j(1+z)}f(z)dz$$

with $f(z) = F'(z)$.
We have

$$h_i(t) = \sum_{j=1}^{+\infty} p_{\text{ij}} \int_0^{+\infty} e^{-tx_j(1+z)} f(z)dz \le \sum_{j=1}^{+\infty} p_{\text{ij}} \int_0^{+\infty} f(z)dz = 1,$$

$$\sum_{j=1}^{+\infty} p_{\text{ij}} \int_0^{+\infty} x_j(1+z)e^{-tx_j(1+z)} f(z)dz \le \sum_{j=1}^{+\infty} p_{\text{ij}} x_j \int_0^{+\infty} (1+z)f(z)dz \le \sum_{j=1}^{+\infty} p_{\text{ij}} M\left[1+E(I_1)\right] = M\left[1+E(I_1)\right]$$

and

$$\sum_{j=1}^{+\infty} p_{\text{ij}} \int_0^{+\infty} [x_j(1+z)]^2 e^{-tx_j(1+z)} f(z)dz \le \sum_{j=1}^{+\infty} p_{\text{ij}} x_j^2 \int_0^{+\infty} 2(1+z^2)f(z)dz$$
$$\le 2\sum_{j=1}^{+\infty} p_{\text{ij}} M^2\left[1+E(I_1^2)\right] = 2M^2\left[1+E(I_1^2)\right]$$

Thus, $h_i(t)$ has n-th derivative function on $(0,+\infty)$ with $n = 1,2$.
Therefore, $f_{ir}(t)$ has n-th derivative function on $(0,+\infty)$ with $n = 1,2$ and

$$f'_{ir}(t) = E\left\{ [Y_1 - X_1(1+I_1)]\, e^{t[Y_1-X_1(1+I_1)]} \middle| X_o = x_i, Y_o = y_r \right\}$$
$$f''_{ir}(t) = E\left\{ [Y_1 - X_1(1+I_1)]^2 e^{t[Y_1-X_1(1+I_1)]} \middle| X_o = x_i, Y_o = y_r \right\} \ge 0.$$

Which implies that

$$f_{ir}(t) \text{ is a convex function with } f_{ir}(0) = 0. \tag{16}$$

and

$$f'_{ir}(0) = E\left\{[Y_1 - X_1(1+I_1)]|X_o = x_i, Y_o = y_r\right\} = E\left(Y_1|Y_o = y_r\right) - E\left(X_1(1+I_1)|X_o = x_i\right) < 0. \tag{17}$$

By $P\left(Y_1 - X_1(1+I_1) > 0|X_o = x_i, Y_o = y_r\right) > 0$, we can find some constant $\delta > 0$ such that

$$P\left(Y_1 - X_1(1+I_1) > \delta > 0|X_o = x_i, Y_o = y_r\right) > 0$$

Then, we can get that

$$f_{ir}(t) = E\left\{ e^{t[Y_1-X_1(1+I_1)]} \middle| X_o = x_i, Y_o = y_r \right\} - 1$$
$$\ge E\left(\left\{ e^{t[Y_1-X_1(1+I_1)]} \middle| X_o = x_i, Y_o = y_r \right\}.1_{\{Y_1-X_1(1+I_1)>\delta|X_o=x_i,Y_o=y_r\}}\right) - 1$$
$$\ge e^{t\delta}.P(\{Y_1 - X_1(1+I_1) > \delta|X_o = x_i, Y_o = y_r\} - 1.$$

Imply

$$\lim_{t\to+\infty} f_{ir}(t) = +\infty \tag{18}$$

From (16), (17) and (18) there exists a unique positive constant R_{ir} satisfying (14).

Let $R_o = \inf\left\{R_{ir} > 0 : E\left(e^{R_{ir}[Y_1-X_1(1+I_1)]}\middle| X_o = x_i, Y_o = y_r\right) = 1\left(x_i \in E_X, y_r \in E_Y\right)\right\}$.

Remark 3.1. $E\left(e^{R_o[Y_1-X_1(1+I_1)]}|X_0 = x_i, Y_0 = y_r\right) \le 1, \forall x_i \in E_X, y_r \in E_Y$.

Use Lemma 3.1 and Theorem 2.1, we now obtain a probability inequality for $\psi(u,x_i,y_r,i_o)$ by an inductive approach.

Theorem 3.1. Let model (1) satisfies assumption 1 to assumption 5. Under the conditions of Lemma 3.1 and $R_o > 0$ then, for any $x_i \in E_X$ and $y_r \in E_Y$

$$\psi(u,x_i,y_r,i_o) \le \beta E\left[e^{R_oY_1}\middle|Y_o = y_r\right] E\left[e^{-R_o(u+X_1)(1+I_1)}\middle| X_o = x_i\right], \tag{19}$$

where

$$\beta^{-1} = \inf_{z>0} \frac{e^{R_ouz}\int_0^z e^{-R_out}dF(t)}{F(z)}, \beta \le 1. \tag{20}$$

Proof.
Firstly, we have

$$\beta^{-1} = \inf_{z>0} \frac{\int_0^z e^{R_o u(z-t)} dF(t)}{F(z)} \geq \inf_{z>0} \frac{\int_0^z dF(t)}{F(z)} = 1 \Leftrightarrow \frac{1}{\beta} \geq 1 \Leftrightarrow \beta \leq 1.$$

For any $z > 0$, we have

$$F(z) = \left[\frac{e^{R_o uz}. \int_0^z e^{-R_o ut} dF(t)}{F(z)} \right]^{-1} .e^{R_o uz}. \int_0^z e^{-R_o ut} dF(t) \leq \beta .e^{R_o uz}. \int_0^z e^{-R_o ut} dF(t). \tag{21}$$

Then, any $u > 0, i_o \geq 0, x_i \in E_X$ and $y_r \in E_Y$

$$\begin{aligned}
\psi_1(u, x_i, y_r, i_o) &= \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} F\left(\frac{y_s - (u + x_j)(1 + ai_o)}{u + x_j} \right) \\
&\leq \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o u \left[\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j} - z \right]} dF(z) \\
&= \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o u \left[\frac{y_s-(u+x_j)(1+ai_o+z)}{u+x_j} \right]} dF(z) \\
&\leq \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o \left[y_s-(u+x_j)(1+ai_o+z) \right]} dF(z) \\
&\leq \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \int_0^{+\infty} e^{R_o \left[y_s-(u+x_j)(1+ai_o+z) \right]} dF(z) = \beta E\left[e^{R_o Y_1} \middle| Y_o = y_r \right] .E\left[e^{-R_o(u+X_1)(1+I_1)} \middle| X_o = x_i \right].
\end{aligned}$$

Hence

$$\psi_1(u, x_i, y_r, i_o) \leq \beta E\left[e^{R_o Y_1} \middle| Y_o = y_r \right] .E\left[e^{-R_o(u+X_1)(1+I_1)} \middle| X_o = x_i \right]. \tag{22}$$

Under an inductive hypothesis, we assume for

$$\psi_n(u, x_i, y_r, i_o) \leq \beta E\left[e^{R_o Y_1} \middle| Y_o = y_r \right] .E\left[e^{-R_o(u+X_1)(1+I_1)} \middle| X_o = x_i \right]. \tag{23}$$

Then (22) implies (23) hold with $n = 1$.
For $x_j \in E_X$, $y_s \in E_Y$, $z \geq \dfrac{y_s - (u + x_j)(1 + ai_o)}{u + x_j}$ and $I_1 \geq 0$, we have

$$\begin{aligned}
\psi_{n+1}\left((u + x_j)(1 + ai_o + z) - y_s, x_j, y_s, i \right) &\leq \beta^* E\left[e^{R_o^* Y_1} \middle| Y_o = y_s \right] .E\left[e^{-R_o^* \left[(u+x_j)(1+ai_o+z) - y_s + X_1 \right](1+I_1)} \middle| X_o = x_j \right] \\
&= \beta^* E\left[e^{R_o^* Y_1} \middle| Y_o = y_s \right] .E\left[e^{-R_o^* \left[(u+x_j)(1+ai_o+z) - y_s \right](1+I_1) - R_o^* X_1 (1+I_1)} \middle| X_o = x_j \right] \\
&\leq \beta^* E\left[e^{R_o^* Y_1} \middle| Y_o = y_s \right] .E\left[e^{-R_o^* \left[(u+x_j)(1+ai_o+z) - y_s \right] - R_o^* X_1 (1+I_1)} \middle| X_o = x_j \right] \\
&= \beta^* E\left[e^{R_o^* Y_1} \middle| Y_o = y_s \right] .E\left[e^{-R_o^* X_1 (1+I_1)} \middle| X_o = x_j \right] .e^{-R_o^* \left[(u+x_j)(1+ai_o+z) - y_s \right]} \\
&\leq \beta^* .e^{-R_o^* \left[(u+x_j)(1+ai_o+z) - y_s \right]}.
\end{aligned} \tag{24}$$

where

$$\beta^{*-1} = \inf_{z>0} \frac{e^{R_o^* uz} \int_0^z e^{-R_o^* ut} dF(t)}{F(z)}, E\left(e^{R_o^*(Y_1 - X_1(1+I_1))} \middle| X_o = x_j, Y_o = y_s \right) = 1 \text{ and } R_o^* \geq R_o > 0.$$

For any $z > 0$

$$\frac{e^{R_o uz} \int_0^z e^{-R_o ut} dF(t)}{F(z)} = \frac{\int_0^z e^{R_o u(z-t)} dF(t)}{F(z)} \leq \frac{\int_0^z e^{R_o^* u(z-t)} dF(t)}{F(z)} = \frac{e^{R_o^* uz} \int_0^z e^{-R_o^* ut} dF(t)}{F(z)}$$

then

$$\beta^{-1} = \inf_{z>0} \frac{e^{R_o uz} \int_0^z e^{-R_o ut} dF(t)}{F(z)} \leq \beta^{*-1} = \inf_{z>0} \frac{e^{R_o^* uz} \int_0^z e^{-R_o^* ut} dF(t)}{F(z)} \Leftrightarrow \frac{1}{\beta} \leq \frac{1}{\beta^*} \Leftrightarrow \beta^* \leq \beta.$$

We get $R_o^* [(u+x_j)(1+ai_o+z) - y_s] \geq R_o [(u+x_j)(1+ai_o+z) - y_s] > 0$ then (24) becomes

$$\psi_n((u+x_j)(1+ai_o+z) - y_s, x_j, y_s, i) \leq \beta . e^{-R_o\left[(u+x_j)(1+ai_o+z)-y_s\right]} \tag{25}$$

Therefore, by Lemma 3.1, (6) and (25), we get

$$\begin{aligned}
&\psi_{n+1}(u, x_i, y_r, i_o) \\
&= \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ F\left(\frac{y_s - (u+x_j)(1+ai_o)}{u+x_j} \right) + \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} \psi_n((u+x_j)(1+ai_o+z) - y_s, x_j, y_s, i)\, dF(z) \right\} \\
&\leq \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o u\left[\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j} - z\right]} dF(z) + \beta \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} e^{-R_o\left[(u+x_j)(1+ai_o+z)-y_s\right]} dF(z) \right\} \\
&= \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o u\left[\frac{y_s-(u+x_j)(1+ai_o+z)}{u+x_j}\right]} dF(z) + \beta \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} e^{-R_o\left[(u+x_j)(1+ai_o+z)-y_s\right]} dF(z) \right\} \\
&\leq \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \beta \int_0^{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}} e^{R_o\left[y_s-(u+x_j)(1+ai_o+z)\right]} dF(z) + \beta \int_{\frac{y_s-(u+x_j)(1+ai_o)}{u+x_j}}^{+\infty} e^{-R_o\left[(u+x_j)(1+ai_o+z)-y_s\right]} dF(z) \right\} \\
&= \beta \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \int_0^{+\infty} e^{R_o\left[y_s-(u+x_j)(1+ai_o+z)\right]} dF(z) \\
&= \beta E\left[e^{R_o Y_1} \middle| Y_o = y_r \right] . E\left[e^{-R_o(u+X_1)(1+I_1)} \middle| X_o = x_i \right].
\end{aligned}$$

Hence

$$\psi_{n+1}(u, x_i, y_r, i_o) \leq \beta E\left[e^{R_o Y_1} \middle| Y_o = y_r \right] . E\left[e^{-R_o(u+X_1)(1+I_1)} \middle| X_o = x_i \right]$$

Hence, for any $n = 1, 2, \ldots$, (23). Therefore, (19) follows by letting $n \to \infty$ in (23).

Remark 3.2.
Let $A(u,x_i,y_r,i_o) = \beta E\left[e^{R_oY_1}\middle| Y_o = y_r\right].E\left[e^{-R_o(u+X_1)(1+I_1)}\middle| X_o = x_i\right].$
From $I_1 \geq 0, X_1 \geq 0$ and $\beta \leq 1$, we have

$$\begin{aligned} A(u,x_i,y_r,i_o) &\leq \beta E\left[e^{R_oY_1}\middle| Y_o = y_r\right] E\left[e^{-R_ou(1+I_1)-R_oX_1(1+I_1)}\middle| X_o = x_i\right] \\ &\leq \beta E\left[e^{R_oY_1}\middle| Y_o = y_r\right] E\left[e^{-R_ou-R_oX_1(1+I_1)}\middle| X_o = x_i\right] \\ &= \beta E\left[e^{R_oY_1}\middle| Y_o = y_r\right] E\left[e^{-R_oX_1(1+I_1)}\middle| X_o = x_i\right].e^{-R_ou} \leq \beta e^{-R_ou} \leq e^{-R_ou}. \end{aligned}$$

Hence, upper bound for ruin probability in (19) is better than e^{-R_ou}.

4 Conclusion

Our main results in this paper, Theorem 2.1 give the recursive equation for $\psi_n(u,x_i,y_r,i_o)$ and the integral equation for $\psi(u,x_i,y_r,i_o)$, Theorem 3.1 built the probability inequality for $\psi(u,x_i,y_r,i_o)$ by an inductive approach.

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