

# Numerical solution for systems of two dimensional integral equations by using Jacobi operational collocation method

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**Abstract:** In this paper, the numerical solution of two dimensional Fredholm and Volterra integral equations will be investigated. For this order, two dimensional collocation method is applied to solve system of two dimensional linear and nonlinear Fredholm and Volterra integral equations. Using the Jacobi polynomials, two dimensional integral equations reduce to a system of algebraic equations. The main aim is the developing the Jacobi operational matrices of integration and product for the solving system of two dimensional Fredholm and Volterra integral equations. These matrices together with the collocation method are applied to reduce the solution of these problems to the solution of a system of algebraic equations. The numerical examples illustrate the efficiency and accuracy of this method.

**Keywords:** collocation method, shifted Jacobi polynomials, two dimensional Fredholm and Volterra integral equations, operational matrices of integration and product, linear and nonlinear systems, convergence

## 1 Introduction

Two dimensional integral equations provide an important tool for modeling a numerous problems in engineering and mechanics [1, 2]. There are many different numerical methods for solving one dimensional integral equations, such as [3, 4, 5, 6, 7, 8, 9]. Some of these methods can be used for solving two dimensional integral equations. Computational complexity of mathematical operations is the most important obstacle for solving integral equations in higher dimensions.

Maleknejad and et al in [10] have applied the Adomian decomposition method to solve the nonlinear mixed Volterra-Fredholm integral equations. Guoqiang, [11], has used the Nyström method for a nonlinear Volterra-Fredholm integral equations. Babolian and et al have used the Homotopy perturbation method and differential transform method for two dimensional linear and nonlinear Volterra integral equations [12]. Hatamzadeh and et al, [13], applied the block-pulse functions to solve two dimensional linear integral equations.

In this study, first two dimensional Jacobi operational matrices of integration and product are obtained. Next, the collocation method is developed for solving the systems of two dimensional integral equations.

The remainder of this paper is organized as follows: The Jacobi polynomials and some their properties and one dimensional matrices of integration and product are introduced in Section 2. Afterwards, these matrices will be extended to two dimensional case. In Section 3, the convergence of the method is studied. Section 4 is devoted to applying two dimensional Jacobi operational matrices for solving systems of two dimensional integral equations. In Section 5, the proposed method is applied to solve several examples. A conclusion is presented in Section 6.

## 2 Jacobi polynomials and Jacobi operational matrices

The Jacobi polynomials, associated with the real parameters  $(\alpha, \beta > -1)$  are a sequence of polynomials  $P_i^{(\alpha, \beta)}(t) (i = 0, 1, 2, \dots)$ , each of degree  $i$ , are orthogonal

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with Jacobi weighted function,  $w(t) = (1 - t)^\alpha(1 + t)^\beta$  over  $I = [-1, 1]$ , and

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) w(t) dt = h_n \delta_{mn},$$

where  $\delta_{mn}$  is Kroneker function and

$$h_n = \frac{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1)}.$$

These polynomials can be generated with the following recurrence formula:

$$P_i^{(\alpha,\beta)}(t) = \frac{(\alpha + \beta + 2i - 1)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} \times \{ \alpha^2 - \beta^2 + t(\alpha + \beta + 2i)(\alpha + \beta + 2i - 2) \} P_{i-1}^{(\alpha,\beta)}(t) - \frac{(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-2}^{(\alpha,\beta)}(t),$$

$i = 2, 3, \dots,$

where  $P_0^{(\alpha,\beta)}(t) = 1$  and  $P_1^{(\alpha,\beta)}(t) = (\alpha + \beta + 2)t/2 + (\alpha - \beta)/2$ .

In order to use these polynomials on the interval  $[0, 1]$ , shifted Jacobi polynomials are defined by introducing the change of variable  $t = 2x - 1$ . In what following, the shifted Jacobi polynomials  $P_i^{(\alpha,\beta)}(2x - 1)$  are denoted by  $P_i^{(\alpha,\beta)}(x)$ , for convenience. Then the shifted Jacobi polynomials  $P_i^{(\alpha,\beta)}(x)$  can be generated from following formula:

$$P_i^{(\alpha,\beta)}(x) = \frac{(\alpha + \beta + 2i - 1)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} \times \{ \alpha^2 - \beta^2 + (2x - 1)(\alpha + \beta + 2i)(\alpha + \beta + 2i - 2) \} P_{i-1}^{(\alpha,\beta)}(x) - \frac{(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-2}^{(\alpha,\beta)}(x),$$

$x \in D = [0, 1], i = 2, 3, \dots,$

where  $P_0^{(\alpha,\beta)}(x) = 1$ , and

$$P_1^{(\alpha,\beta)}(x) = (\alpha + \beta + 2)(2x - 1)/2 + (\alpha - \beta)/2.$$

**Remark.** Of this polynomials, the most commonly used are the shifted Gegenbauer polynomials,  $C_{S,i}^\alpha(x)$ , the shifted Chebyshev polynomials of the first kind,  $T_{S,i}(x)$ , the shifted Legendre polynomials,  $P_{S,i}(x)$ , the shifted Chebyshev polynomials of the second kind,  $U_{S,i}(x)$ . These orthogonal polynomials are related to the shifted Jacobi polynomials by the following relations.

$$C_{S,i}^\alpha(x) = \frac{i! \Gamma(\alpha + \frac{1}{2})}{\Gamma(i + \alpha + \frac{1}{2})} P_i^{(\alpha - \frac{1}{2}, \beta - \frac{1}{2})}(x),$$

$$T_{S,i}(x) = \frac{i! \Gamma(\frac{1}{2})}{\Gamma(i + \frac{1}{2})} P_i^{(-\frac{1}{2}, -\frac{1}{2})}(x),$$

$$P_{S,i}(x) = P_i^{(0,0)}(x), \quad U_{S,i}(x) = \frac{(i + 1)! \Gamma(\frac{1}{2})}{\Gamma(i + \frac{3}{2})} P_i^{(\frac{1}{2}, \frac{1}{2})}(x).$$

The analytic form of the shifted Jacobi polynomials,  $P_i^{(\alpha,\beta)}(x)$ , is given by

$$P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^i \frac{(-1)^{(i-k)} \Gamma(i + \beta + 1) \Gamma(i + k + \alpha + \beta + 1) x^k}{\Gamma(k + \beta + 1) \Gamma(i + \alpha + \beta + 1) (i - k)! k!},$$

Some properties of the shifted Jacobi polynomials are as follows:

- (1)  $P_i^{(\alpha,\beta)}(0) = (-1)^i \binom{i + \alpha}{i}$ ,
- (2)  $P_i^{(\alpha,\beta)}(1) = (-1)^i \binom{i + \beta}{i}$ ,
- (3)  $\frac{d^i}{dx^i} P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n + \alpha + \beta + i + 1)}{\Gamma(n + \alpha + \beta + 1)} P_{n-i}^{(\alpha + i, \beta + i)}(x)$ .

The orthogonality condition of shifted Jacobi polynomials is:

$$\int_0^1 P_j^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = \theta_k \delta_{jk},$$

where  $w^{(\alpha,\beta)}(x)$ , shifted weighted function, is as follows:

$$w^{(\alpha,\beta)}(x) = x^\beta (1 - x)^\alpha,$$

and,  $\theta_k = h_k / 2^{\alpha + \beta + 1}$ .

**Lemma 2.1.** The shifted Jacobi polynomial  $P_n^{(\alpha,\beta)}(x)$  can be obtained in the form of:

$$P_n^{(\alpha,\beta)}(x) = \sum_{i=0}^n p_i^{(n)} x^i,$$

where  $p_i^{(n)}$  are

$$p_i^{(n)} = (-1)^{n-i} \binom{n + \alpha + \beta + i}{i} \binom{n + \alpha}{n - i}.$$

**Proof.** The  $p_i^{(n)}$  can be obtained as,

$$p_i^{(n)} = \frac{1}{i!} \frac{d^i}{dx^i} P_n^{(\alpha,\beta)}(x) |_{x=0}.$$

Now, using properties (1) and (3) in above relation, the lemma can be proved.  $\square$

**Lemma 2.2.** For  $m > 0$ , one has

$$\int_0^1 x^m P_j^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = \sum_{l=0}^j p_l^{(j)} B(m + l + \beta + 1, \alpha + 1),$$

where  $B(s, t)$  is the Beta function and is defined as

$$B(s, t) = \int_0^1 v^{s-1}(1-v)^{t-1} dv = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

**Proof.** Using Lemma 2.1 and  $w^{(\alpha, \beta)} = (1-x)^\alpha x^\beta$  one has

$$\begin{aligned} \int_0^1 x^m P_j^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx &= \sum_{l=0}^j p_l^{(j)} \int_0^1 x^m x^l (1-x)^\alpha x^\beta dx \\ &= \sum_{l=0}^j p_l^{(j)} \int_0^1 (1-x)^\alpha x^{m+l+\beta} dx \\ &= \sum_{l=0}^j p_l^{(j)} B(m+l+\beta+1, \alpha+1). \square \end{aligned}$$

A function  $u(x) \in L^2(D)$  can be expanded as the below formula:

$$u(x) = \sum_{j=0}^{\infty} c_j P_j^{(\alpha, \beta)}(x),$$

where the coefficients  $c_j$  are given by

$$c_j = \frac{1}{\theta_j} \int_0^1 P_j^{(\alpha, \beta)}(x) u(x) w^{(\alpha, \beta)}(x) dx, \quad j = 0, 1, 2, \dots$$

By noting in practice only the first  $(N+1)$ -terms shifted polynomials are considered, then one has

$$u(x) \simeq u_N(x) = \sum_{j=0}^N c_j P_j^{(\alpha, \beta)}(x) = \Phi^T(x)C,$$

where  $C = [c_0, c_1, \dots, c_N]^T$ , and

$$\Phi(x) = [P_0^{(\alpha, \beta)}(x), P_1^{(\alpha, \beta)}(x), \dots, P_N^{(\alpha, \beta)}(x)]^T.$$

Now, two variables Jacobi polynomials can be defined by means of one variable Jacobi polynomials as follows:

**Definition 2.3.** Let  $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$  be the sequence of one variable shifted Jacobi polynomials on  $D = [0, 1]$ . Two variables Jacobi polynomials,  $\{R_{m,n}^{(\alpha, \beta)}(x, y)\}_{m,n=0}^{\infty}$ , are defined on  $D^2 = [0, 1] \times [0, 1]$  as:

$$R_{m,n}^{(\alpha, \beta)}(x, y) = P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y), \quad (x, y) \in D^2.$$

The family  $\{R_{m,n}^{(\alpha, \beta)}(x, y)\}_{m,n=0}^{\infty}$  is orthogonal with weighted function  $W^{(\alpha, \beta)}(x, y) = w^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(y)$  on  $D^2$  and forms a basis for  $L^2(D^2)$ .

**Theorem 2.4.** The basis  $\{R_{m,n}^{(\alpha, \beta)}(x, y)\}$  is orthogonal on  $D^2$ .

**Proof.** One has

$$\begin{aligned} \int_0^1 \int_0^1 R_{m,n}^{(\alpha, \beta)}(x, y) R_{k,l}^{(\alpha, \beta)}(x, y) W^{(\alpha, \beta)}(x, y) dx dy \\ &= \int_0^1 P_m^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx \\ &\quad \times \int_0^1 P_n^{(\alpha, \beta)}(y) P_l^{(\alpha, \beta)}(y) w^{(\alpha, \beta)}(y) dy \\ &= \begin{cases} \theta_m \theta_n, & (m, n) = (k, l), \\ 0, & (m, n) \neq (k, l) \text{ or } m \neq k \text{ or } n \neq l. \square \end{cases} \end{aligned}$$

A function  $u(x, y)$  defined over  $D^2$  may be expanded by the two variables Jacobi polynomials as:

$$u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} R_{m,n}^{(\alpha, \beta)}(x, y), \quad (x, y) \in D^2 \quad (1)$$

where the Jacobi coefficients,  $c_{mn}$ , are obtained as:

$$c_{mn} = \frac{1}{\theta_m \theta_n} \int_0^1 \int_0^1 R_{m,n}^{(\alpha, \beta)}(x, y) u(x, y) W^{(\alpha, \beta)}(x, y) dx dy.$$

If the infinite series in equation (1) is truncated up to their  $(N+1)$ -terms then it can be written as:

$$u(x, y) \simeq u_N(x, y) = \sum_{m=0}^N \sum_{n=0}^N c_{mn} R_{m,n}^{(\alpha, \beta)}(x, y) = \Phi^T(x, y)C,$$

where  $C$  and  $\Phi(x, y)$  are Jacobi coefficients and Jacobi polynomials vectors, respectively:

$$C = [c_{00}, c_{01}, \dots, c_{0N}, \dots, c_{N1}, \dots, c_{NN}]^T,$$

$$\Phi(x, y) = [\Phi_{00}(x, y), \dots, \Phi_{0N}(x, y), \dots, \Phi_{N0}(x, y), \dots, \Phi_{NN}(x, y)]^T$$

$$= [R_{0,0}^{(\alpha, \beta)}(x, y), \dots, R_{0,N}^{(\alpha, \beta)}(x, y), \dots, R_{N,0}^{(\alpha, \beta)}(x, y), \dots, R_{N,N}^{(\alpha, \beta)}(x, y)]^T \quad (2)$$

Similarly, a function of four variables,  $k(x, y, t, s)$ , on  $D^4$  may be approximated with respect to Jacobi polynomials such as:

$$k(x, y, t, s) \simeq \Phi^T(x, y) K \Phi(t, s),$$

where  $\Phi(x, y)$  is two variables Jacobi vector and  $K$  is a  $(N+1)^2 \times (N+1)^2$  known matrix.

### 2.1 One dimensional Jacobi operational matrices

In performing arithmetic and other operations on the Jacobi basis, we frequently encounter the integration of the vector  $\Phi(x)$  and it is necessary to evaluate the product of  $\Phi(x)$  and  $\Phi^T(x)$ , which is called the product matrix for the Jacobi polynomials basis. In this subsection, these operational matrices are derived.

### 2.1.1 One dimensional Jacobi operational matrix of integration

In this subsection, Jacobi operational matrix of the integration is derived. Let

$$\int_0^x \Phi(t) dt \simeq P\Phi(x), \quad (3)$$

where matrix  $P$  is called the Jacobi operational matrix of integration. The entries of this matrix are obtained as follows:

**Theorem 2.5.** Let  $P$  be  $(N+1) \times (N+1)$  operational matrix of integration. Then the elements of this matrix are obtained as:

$$P_{ij} = \frac{1}{\theta_j} \sum_{m=0}^i \sum_{n=0}^j \frac{1}{m+1} p_m^{(i)} p_n^{(j)} B(m+n+\beta+2, \alpha+1),$$

$$i, j = 0, 1, 2, \dots, N.$$

**Proof.** Using equation (3) and orthogonality property of Jacobi polynomials one has:

$$P = \left( \int_0^x \Phi(t) dt, \Phi^T(x) \right) \Delta^{-1},$$

where  $\left( \int_0^x \Phi(t) dt, \Phi^T(x) \right)$  and  $\Delta^{-1}$  are two  $(N+1) \times (N+1)$  matrices defined as follows:

$$\left( \int_0^x \Phi(t) dt, \Phi^T(x) \right) = \left\{ \left( \int_0^x P_i^{(\alpha, \beta)}(t) dt, P_j^{(\alpha, \beta)}(x) \right) \right\}_{i, j=0}^N,$$

$$\Delta^{-1} = \text{diag} \left\{ \frac{1}{\theta_j} \right\}_{j=0}^N.$$

Set

$$\rho_{ij} = \left( \int_0^x P_i^{(\alpha, \beta)}(t) dt, P_j^{(\alpha, \beta)}(x) \right)$$

$$= \int_0^1 \left\{ \int_0^x P_i^{(\alpha, \beta)}(t) dt \right\} P_j^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx.$$

$\int_0^x P_i^{(\alpha, \beta)}(t) dt$  and  $P_j^{(\alpha, \beta)}(x)$  by using Lemma 2.1 can be obtained as:

$$\int_0^x P_i^{(\alpha, \beta)}(t) dt = \sum_{m=0}^i p_m^{(i)} \frac{x^{m+1}}{m+1},$$

$$P_j^{(\alpha, \beta)}(x) = \sum_{n=0}^j p_n^{(j)} x^n, \quad i, j = 0, 1, \dots, N.$$

Therefore,  $\rho_{ij}$  by using Lemma 2.2 can be obtained as follows,

$$\rho_{ij} = \sum_{m=0}^i \sum_{n=0}^j \frac{1}{m+1} p_m^{(i)} p_n^{(j)} \int_0^1 x^{m+1} x^n (1-x)^{\alpha} x^{\beta} dx.$$

$$= \sum_{m=0}^i \sum_{n=0}^j \frac{1}{m+1} p_m^{(i)} p_n^{(j)} B(m+n+\beta+2, \alpha+1).$$

So, the entries of matrix  $P$  is obtained as:

$$P_{ij} = \frac{1}{\theta_j} \sum_{m=0}^i \sum_{n=0}^j \frac{1}{m+1} p_m^{(i)} p_n^{(j)} B(m+n+\beta+2, \alpha+1),$$

$$i, j = 0, 1, 2, \dots, N. \quad \square$$

### 2.1.2 One dimensional Jacobi operational matrix of product

The following property of the product of two Jacobi function vector will be also be applied to solve the Volterra and Volterra - Fredholm integral equations.

$$\Phi(x) \Phi^T(x) Y \simeq \tilde{Y} \Phi(x), \quad (4)$$

where  $\tilde{Y}$  is a  $(N+1) \times (N+1)$  product operational matrix and its entries are determined in terms of the components of the vector  $Y$ . Using equation (4) and by the orthogonality property of Jacobi polynomials the entries  $\tilde{Y}_{i,j}$  can be calculated as follows:

$$\tilde{Y}_{ij} = \frac{1}{\theta_j} \sum_{k=0}^N Y_k \int_0^1 \Phi_i(x) \Phi_j(x) \Phi_k(x) w^{(\alpha, \beta)}(x) dx$$

$$= \frac{1}{\theta_j} \sum_{k=0}^N Y_k \int_0^1 P_i^{(\alpha, \beta)}(x) P_j^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx$$

$$= \frac{1}{\theta_j} \sum_{k=0}^N Y_k h_{ijk},$$

where

$$h_{ijk} = \int_0^1 P_i^{(\alpha, \beta)}(x) P_j^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx.$$

## 2.2 Two dimensional Jacobi operational matrices

In this subsection, two dimensional operational matrices are presented.

### 2.2.1 Two dimensional Jacobi operational matrix of integration for $x$ variable

The operational matrix of integration in  $x$ -direction is defined as follows:

**Theorem 2.6.** The operational matrix of integration in  $x$ -direction is defined as follows.

$$\int_0^x \Phi(t, y) dt \simeq P_x \Phi(x, y) = (P \otimes I) \Phi(x, y),$$

where  $P_x$  is a  $(N+1)^2 \times (N+1)^2$  operational matrix of integration,  $P$  is operational matrix of integration

introduced in subsection 2.1.1 and  $I$  is  $(N + 1) \times (N + 1)$  identity matrix.

**Proof.** Suppose  $R_j$  be  $j$ th row of matrix  $P$ . One has

$$\int_0^x P_j^{(\alpha,\beta)}(t) dt = R_j^T \Phi(x).$$

Also, noting the definition of the vector  $\Phi(x, y)$  one has

$$\Phi(x, y) = [P_0^{(\alpha,\beta)}(x)P_0^{(\alpha,\beta)}(y), \dots, P_0^{(\alpha,\beta)}(x)P_N^{(\alpha,\beta)}(y), \dots, P_N^{(\alpha,\beta)}(x)P_0^{(\alpha,\beta)}(y), \dots, P_N^{(\alpha,\beta)}(x)P_N^{(\alpha,\beta)}(y)]^T. \quad (5)$$

Integrating of equation (5) from 0 to  $x$  yields

$$\begin{aligned} \int_0^x \Phi(t, y) dt &= [P_0^{(\alpha,\beta)}(y) \int_0^x P_0^{(\alpha,\beta)}(t) dt, \dots, P_N^{(\alpha,\beta)}(y) \int_0^x P_N^{(\alpha,\beta)}(t) dt]^T \\ &= [R_0 \cdot \Phi(x) P_0^{(\alpha,\beta)}(y), \dots, R_0 \cdot \Phi(x) P_N^{(\alpha,\beta)}(y), \dots, R_N \cdot \Phi(x) P_0^{(\alpha,\beta)}(y), \dots, R_N \cdot \Phi(x) P_N^{(\alpha,\beta)}(y)]^T \\ &= [R_0 [P_0^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y), \dots, P_N^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y)], \dots, R_N [P_0^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y), \dots, P_N^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y)]]^T, \end{aligned}$$

$$= \begin{bmatrix} P_{00} & 0 & \dots & 0 & P_{01} & 0 & \dots & 0 & \dots & P_{0N} & 0 & \dots & 0 \\ 0 & P_{00} & \dots & 0 & 0 & P_{01} & \dots & 0 & \dots & 0 & P_{0N} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{00} & 0 & 0 & \dots & P_{01} & \dots & 0 & 0 & \dots & P_{0N} \\ P_{10} & 0 & \dots & 0 & P_{11} & 0 & \dots & 0 & \dots & P_{1N} & 0 & \dots & 0 \\ 0 & P_{10} & \dots & 0 & 0 & P_{11} & \dots & 0 & \dots & 0 & P_{1N} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{10} & 0 & 0 & \dots & P_{11} & \dots & 0 & 0 & \dots & P_{1N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ P_{N0} & 0 & \dots & 0 & P_{N1} & 0 & \dots & 0 & \dots & P_{NN} & 0 & \dots & 0 \\ 0 & P_{N0} & \dots & 0 & 0 & P_{N1} & \dots & 0 & \dots & 0 & P_{NN} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{N0} & 0 & 0 & \dots & P_{N1} & \dots & 0 & 0 & \dots & P_{NN} \end{bmatrix} \times \begin{bmatrix} P_0^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_0^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \\ P_1^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_1^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \end{bmatrix}$$

$$= \begin{bmatrix} P_{00}I & P_{01}I & \dots & P_{0N}I \\ P_{10}I & P_{11}I & \dots & P_{1N}I \\ \vdots & \vdots & \ddots & \vdots \\ P_{N0}I & P_{N1}I & \dots & P_{NN}I \end{bmatrix} \begin{bmatrix} P_0^{(\alpha,\beta)}(x) \Phi(y) \\ P_1^{(\alpha,\beta)}(x) \Phi(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) \Phi(y) \end{bmatrix} = (P \otimes I) \Phi(x, y).$$

Where  $\otimes$  denotes the Kronecker product and is defined for two arbitrary matrices  $A$  and  $B$  as  $A \otimes B = (a_{ij}B)$  and  $P_{ij}$  denotes  $(i, j)$ th entry of the matrix  $P$ .  $\square$

### 2.2.2 Two dimensional Jacobi operational matrix of integration for $y$ variable

**Theorem 2.7.** The operational matrix of integration in  $y$ -direction is defined as:

$$\int_0^y \Phi(x, s) ds \simeq P_y \Phi(x, y) = (I \otimes P) \Phi(x, y),$$

where  $P_y$  is a  $(N + 1)^2 \times (N + 1)^2$  operational matrix of integration. **Proof.** Again, integrating of equation (5) from 0 to  $y$  one has

$$\begin{aligned} \int_0^y \Phi(x, s) ds &= [P_0^{(\alpha,\beta)}(x) \int_0^y P_0^{(\alpha,\beta)}(s) ds, \dots, P_N^{(\alpha,\beta)}(x) \int_0^y P_N^{(\alpha,\beta)}(s) ds]^T \\ &= [P_0^{(\alpha,\beta)}(x) R_0 \cdot \Phi(y), \dots, P_0^{(\alpha,\beta)}(x) R_N \cdot \Phi(y), \dots, P_N^{(\alpha,\beta)}(x) R_N \cdot \Phi(y), \dots, P_N^{(\alpha,\beta)}(x) R_N \cdot \Phi(y)]^T \end{aligned}$$

$$= \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0N} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ P_{10} & P_{11} & \dots & P_{1N} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ P_{N0} & P_{N1} & \dots & P_{NN} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & P_{00} & P_{01} & \dots & P_{0N} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & P_{10} & P_{11} & \dots & P_{1N} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & P_{N0} & P_{N1} & \dots & P_{NN} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & P_{00} & P_{01} & \dots & P_{0N} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & P_{10} & P_{11} & \dots & P_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & P_{N0} & P_{N1} & \dots & P_{NN} \end{bmatrix} \times \begin{bmatrix} P_0^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_0^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \\ P_1^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_1^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) P_0^{(\alpha,\beta)}(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) P_N^{(\alpha,\beta)}(y) \end{bmatrix}$$

$$= \begin{bmatrix} P & O & \dots & O \\ O & P & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & P \end{bmatrix} \begin{bmatrix} P_0^{(\alpha,\beta)}(x) \Phi(y) \\ P_1^{(\alpha,\beta)}(x) \Phi(y) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) \Phi(y) \end{bmatrix} = (I \otimes P) \Phi(x, y).$$

Where  $O$  is a  $(N+1) \times (N+1)$  zero matrix.  $\square$

### 2.3 Two dimensional Jacobi operational matrix of product

The following property of the product of two vectors  $\Phi(x, y)$  and  $\Phi^T(x, y)$  will also be used.

$$\Phi(x, y) \Phi^T(x, y) U \simeq \tilde{U} \Phi(x, y), \quad (6)$$

where  $U$  and  $\tilde{U}$  are a  $(N+1)^2 \times 1$  vector and a  $(N+1)^2 \times (N+1)^2$  product operational matrix, respectively. One has

$$] \Phi(x, y) \Phi^T(x, y) U = \begin{bmatrix} \sum_{i=0}^N \sum_{j=0}^N u_{ij} R_{i,j}^{(\alpha,\beta)}(x, y) R_{0,0}^{(\alpha,\beta)}(x, y) \\ \sum_{i=0}^N \sum_{j=0}^N u_{ij} R_{i,j}^{(\alpha,\beta)}(x, y) R_{0,1}^{(\alpha,\beta)}(x, y) \\ \vdots \\ \sum_{i=0}^N \sum_{j=0}^N u_{ij} R_{i,j}^{(\alpha,\beta)}(x, y) R_{N,N}^{(\alpha,\beta)}(x, y) \end{bmatrix}.$$

One puts

$$R_{i,j}^{(\alpha,\beta)}(x, y) R_{k,l}^{(\alpha,\beta)}(x, y) = \sum_{r=0}^N \sum_{s=0}^N a_{rs} R_{r,s}^{(\alpha,\beta)}(x, y). \quad (7)$$

The coefficients  $a_{rs}$  are obtained by the following manner. Multiplying both equation (7) by  $R_{m,n}^{(\alpha,\beta)}(x, y)$ ,  $m, n = 0, 1, 2, \dots, N$ , and integrating of the result from 0 to 1 yields:

$$\begin{aligned} & \int_0^1 \int_0^1 R_{i,j}^{(\alpha,\beta)}(x, y) R_{k,l}^{(\alpha,\beta)}(x, y) R_{m,n}^{(\alpha,\beta)}(x, y) W^{(\alpha,\beta)}(x, y) dx dy \\ &= \sum_{r=0}^N \sum_{s=0}^N a_{rs} \int_0^1 \int_0^1 R_{r,s}^{(\alpha,\beta)}(x, y) R_{m,n}^{(\alpha,\beta)}(x, y) W^{(\alpha,\beta)}(x, y) dx dy \\ &= a_{mn} \theta_m \theta_n. \end{aligned}$$

Therefore

$$\begin{aligned} a_{mn} &= \frac{1}{\theta_m \theta_n} \int_0^1 \int_0^1 R_{i,j}^{(\alpha,\beta)}(x, y) R_{k,l}^{(\alpha,\beta)}(x, y) R_{m,n}^{(\alpha,\beta)}(x, y) \times \\ & \quad W^{(\alpha,\beta)}(x, y) dx dy \\ &= \frac{1}{\theta_m \theta_n} \int_0^1 P_i^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx \times \\ & \quad \int_0^1 P_j^{(\alpha,\beta)}(y) P_l^{(\alpha,\beta)}(y) P_n^{(\alpha,\beta)}(y) w^{(\alpha,\beta)}(y) dy. \end{aligned}$$

Now suppose

$$\omega_{ikm} = \int_0^1 P_i^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx,$$

one gets  $i, k, m = 0, 1, \dots, N$  So,

$$a_{mn} = \frac{\omega_{ikm} \omega_{jln}}{\theta_m \theta_n}.$$

Substituting  $a_{mn}$  into equation (7) one has:

$$R_{i,j}^{(\alpha,\beta)}(x, y) R_{k,l}^{(\alpha,\beta)}(x, y) = \sum_{m=0}^N \sum_{n=0}^N \frac{\omega_{ikm} \omega_{jln}}{\theta_m \theta_n} R_{m,n}^{(\alpha,\beta)}(x, y).$$

If only the components of  $\Phi(x, y)$  are retained, then the matrix  $\tilde{U}$  in the equation (6) is obtained as

$$\tilde{U} = [\tilde{u}_{ij}], \quad i, j = 0, 1, \dots, N. \quad (8)$$

In the equation (8),  $\tilde{u}_{ij}$ ,  $i, j = 0, 1, \dots, N$ , are  $(N+1) \times (N+1)$  matrices given by

$$\tilde{u}_{ij} = \frac{1}{\theta_j} \sum_{n=0}^N B_n \omega_{ijn}, \quad i, j = 0, 1, \dots, N,$$

and  $B_n$  are  $(N+1) \times (N+1)$  matrices as

$$[B_n]_{kl} = \frac{1}{\theta_l} \sum_{m=0}^N u_{nm} \omega_{klm}, \quad k, l = 0, 1, \dots, N.$$

### 3 Convergence analysis

In this section, the theorems on convergence analysis and error estimation of the proposed method are provided.

**Theorem 3.1.** Suppose  $u(x, y) \in C^N[0, 1] \times C^N[0, 1]$  and  $\sum_{i=0}^N \sum_{j=0}^N c_{ij} P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y)$  be an approximation for  $u(x, y)$ . Then for the coefficients  $c_{ij}$  one has:

$$|c_{ij}| \leq \frac{1}{2^{2(i+j)}} A_{i,j}^{(\alpha,\beta)} \max_{(x,y) \in D} \left| \frac{\partial^{i+j} u(x, y)}{\partial x^i \partial y^j} \right|, \quad D = [0, 1] \times [0, 1],$$

where  $A_{i,j}^{(\alpha,\beta)}$  are independent of the function  $u(x, y)$ .

**Proof.** According to the assumption,

$$u(x, y) \simeq \sum_{i=0}^N \sum_{j=0}^N c_{ij} P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y),$$

where the coefficients  $c_{ij}$ ,  $i, j = 0, 1, \dots, N$  are obtained as follows:

$$c_{ij} = \frac{1}{\theta_i \theta_j} \int_0^1 \int_0^1 u(x, y) P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y) W^{(\alpha,\beta)}(x, y) dx dy, \quad (9)$$

Consider Taylor expansion about points  $x = 0$  and  $y = 0$ . For each  $i, j = 0, 1, \dots, N$  one has:

$$u(x, y) = \sum_{m=0}^{i+j-1} \sum_{n=0}^m \frac{x^{m-n} y^n}{(m-n)! n!} \frac{\partial^m u(0, 0)}{\partial x^{m-n} \partial y^n}$$



$$+ \sum_{n=0}^{i+j} \frac{x^{i+j-n}y^n}{(i+j-n)!n!} \frac{\partial^{i+j}u(\xi_x, \xi_y)}{\partial x^{i+j-n}\partial y^n}, (\xi_x, \xi_y) \in [0, x] \times [0, y]. \tag{10}$$

Substituting equation (10) in equation (9) leads to:

$$c_{ij} = \frac{1}{\theta_i\theta_j} \left\{ \sum_{m=0}^{i+j-1} \sum_{n=0}^m \frac{\partial^m u(0,0)}{\partial x^{m-n}\partial y^n} \frac{1}{(m-n)!n!} \times \int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) x^{m-n} y^n P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y) dx dy \right. \\ + \sum_{n=0}^{i+j} \frac{1}{(i+j-n)!n!} \int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) x^{i+j-n} y^n P_i^{(\alpha,\beta)}(x) \times \\ \left. P_j^{(\alpha,\beta)}(y) \frac{\partial^{i+j}u(\xi_x, \xi_y)}{\partial x^{i+j-n}\partial y^n} dx dy \right\} \\ = \frac{1}{\theta_i\theta_j} \left\{ \sum_{m=0}^{i+j-1} \sum_{n=0}^m \frac{\partial^m u(0,0)}{\partial x^{m-n}\partial y^n} \frac{1}{(m-n)!n!} \times \int_0^1 w^{(\alpha,\beta)}(x) x^{m-n} P_i^{(\alpha,\beta)}(x) dx \int_0^1 w^{(\alpha,\beta)}(y) y^n P_j^{(\alpha,\beta)}(y) dy \right. \\ + \sum_{n=0}^{i+j} \frac{1}{(i+j-n)!n!} \int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) x^{i+j-n} y^n P_i^{(\alpha,\beta)}(x) \times \\ \left. P_j^{(\alpha,\beta)}(y) \frac{\partial^{i+j}u(\xi_x, \xi_y)}{\partial x^{i+j-n}\partial y^n} dx dy \right\} \\ = \frac{1}{\theta_i\theta_j} \sum_{n=0}^{i+j} \frac{1}{(i+j-n)!n!} \int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) x^{i+j-n} y^n \times \\ P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y) \frac{\partial^{i+j}u(\xi_x, \xi_y)}{\partial x^{i+j-n}\partial y^n} dx dy,$$

then

$$c_{ij} \leq \frac{1}{\theta_i\theta_j} \sum_{n=0}^{i+j} \frac{1}{(i+j-n)!n!} \max_{(x,y) \in D} \left| \frac{\partial^{i+j}u(x,y)}{\partial x^{i+j-n}\partial y^n} \right| \times \int_0^1 w^{(\alpha,\beta)}(x) x^{i+j-n} P_i^{(\alpha,\beta)}(x) dx \int_0^1 w^{(\alpha,\beta)}(y) y^n P_j^{(\alpha,\beta)}(y) dy.$$

The last summation will be nonzero only for  $n = j$ . Therefore

$$c_{ij} \leq \frac{1}{\theta_i\theta_j} \frac{1}{i!j!} \max_{(x,y) \in D} \left| \frac{\partial^{i+j}u(x,y)}{\partial x^i \partial y^j} \right| \int_0^1 w^{(\alpha,\beta)}(x) x^i P_i^{(\alpha,\beta)}(x) dx \times \int_0^1 w^{(\alpha,\beta)}(y) y^j P_j^{(\alpha,\beta)}(y) dy.$$

Thereupon

$$|c_{ij}| \leq \frac{1}{2^{2(i+j)} 2^{2(\alpha+\beta+1)}} \times \frac{\Gamma(\alpha+\beta+2i+2)\Gamma(\alpha+\beta+2j+2)}{\Gamma(\alpha+i+1)\Gamma(\alpha+j+1)\Gamma(\beta+i+1)\Gamma(\beta+j+1) i!j!}$$

$$\times \max_{(x,y) \in D} \left| \frac{\partial^{i+j}u(x,y)}{\partial x^i \partial y^j} \right| \int_0^1 |w^{(\alpha,\beta)}(x) x^i P_i^{(\alpha,\beta)}(x)| dx \times \int_0^1 |w^{(\alpha,\beta)}(y) y^j P_j^{(\alpha,\beta)}(y)| dy \\ = \frac{1}{2^{2(i+j)}} A_{i,j}^{(\alpha,\beta)} \max_{(x,y) \in D} \left| \frac{\partial^{i+j}u(x,y)}{\partial x^i \partial y^j} \right|,$$

where

$$A_{i,j}^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+2i+2)}{2^{2(\alpha+\beta+1)}\Gamma(\alpha+i+1)\Gamma(\alpha+j+1)} \times \frac{\Gamma(\alpha+\beta+2j+2)}{\Gamma(\beta+i+1)\Gamma(\beta+j+1) i!j!} \times \int_0^1 |w^{(\alpha,\beta)}(x) x^i P_i^{(\alpha,\beta)}(x)| dx \int_0^1 |w^{(\alpha,\beta)}(y) y^j P_j^{(\alpha,\beta)}(y)| dy. \quad \square$$

The last inequality shows that the coefficients decrease when  $i, j$  (in fact  $N$ ) increase. Therefore, function  $u(x, y)$  can be approximated using the finite numbers of the Jacobi polynomials.

**Theorem 3.2.** Suppose  $u(x, y) \in C^N[0, 1] \times C^N[0, 1]$ . Then the bound of the error for the approximate solution resulted is as follows:

$$\|u(x, y) - \Phi^T(x, y)C\| \leq \frac{2^{2N}}{(2N)!} M,$$

where  $M = \max\{M_0, M_1, \dots, M_{2N}\}$  and

$$M_i = \max_{(x,y) \in D} \left\| \frac{\partial^{2N}u(x,y)}{\partial x^{2N-i}\partial y^i} \right\|, \quad i = 0, 1, \dots, 2N.$$

**Proof.** Noting the least square property, consider polynomial  $S_N(x, y)$ , of degree at most  $N$  with respect to both variables  $x$  and  $y$ , which interpolates  $u(x, y)$  in the domain  $D$ . Therefore

$$\int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) (u(x,y) - \Phi^T(x,y)C)^2 dx dy \leq \int_0^1 \int_0^1 W^{(\alpha,\beta)}(x,y) (u(x,y) - S_N(x,y))^2 dx dy$$

Now, consider the Taylor expansion of function  $u(x, y)$  about point  $(0, 0)$  in  $D$ . The bound of the error is obtained as follows:

$$u(x, y) - S_N(x, y) = \sum_{i=0}^{2N} \frac{x^{2N-i}y^i}{(2N-i)!i!} \frac{\partial^{2N}u(\xi_x, \xi_y)}{\partial x^{2N-i}\partial y^i},$$

where  $(\xi_x, \xi_y) \in [0, x] \times [0, y]$ . Therefore

$$\|u(x, y) - S_N(x, y)\| \leq \sum_{i=0}^{2N} \frac{1}{(2N-i)!i!} \max_{(\xi_x, \xi_y) \in [0,x] \times [0,y]} \left\| \frac{\partial^{2N}u(\xi_x, \xi_y)}{\partial x^{2N-i}\partial y^i} \right\|.$$

But

$$\max_{(\xi_x, \xi_y) \in [0, x] \times [0, y]} \left\| \frac{\partial^{2N} u(\xi_x, \xi_y)}{\partial x^{2N-i} \partial y^i} \right\| \leq \max_{(x, y) \in D} \left\| \frac{\partial^{2N} u(x, y)}{\partial x^{2N-i} \partial y^i} \right\| = M_i.$$

Therefore

$$\|u(x, y) - S_N(x, y)\| \leq \sum_{i=0}^{2N} \frac{M_i}{(2N-i)!i!}.$$

Let set  $M = \max\{M_0, M_1, \dots, M_{2N}\}$ . Hence

$$\begin{aligned} \|u(x, y) - S_N(x, y)\| &\leq M \sum_{i=0}^{2N} \frac{1}{(2N-i)!i!} = \frac{M}{(2N)!} \sum_{i=0}^{2N} \binom{2N}{i} \\ &= \frac{M 2^{2N}}{(2N)!}, \end{aligned}$$

Therefore

$$\|u(x, y) - \Phi^T(x, y)C\| \leq \frac{2^{2N}}{(2N)!} M. \quad \square$$

#### 4 Solution of the systems of two dimensional integral equations

In this section, the presented operational matrices are applied to solve the systems of linear and nonlinear Fredholm, Volterra and Volterra-Fredholm integral equations.

##### 4.1 System of two dimensional Fredholm integral equations

In this paper, a system of Fredholm integral equations is considered as follows:

$$\begin{aligned} u_i(x, y) + \sum_{j=1}^m \int_0^1 \int_0^1 k_{ij}(x, y, t, s) g_{ij}(u_1(t, s), \dots, u_n(t, s)) \\ \times ds dt = f_i(x, y), \quad (x, y) \in D, \quad i = 1, \dots, n. \end{aligned} \quad (11)$$

where  $k_{ij}(x, y, t, s) \in L^2(D^2)$ ,  $f_i(x, y)$  are known functions, and  $g_{ij}(x, y, t, s)$  are linear or nonlinear functions in terms of unknown functions  $u_1(x, y), \dots, u_n(x, y)$ . To solve the system (11), the functions  $u_i(x, y)$ ,  $k_{ij}(x, y, t, s)$  and  $g_{ij}(x, y, t, s)$  can be approximated as follows:

$$u_i(x, y) \simeq \Phi^T(x, y)C_i,$$

$$k_{ij}(x, y, t, s) \simeq \Phi^T(x, y)K_{ij}\Phi(t, s),$$

$$g_{ij}(x, y) \simeq \Phi^T(x, y)G_{ij}, \quad i = 1, \dots, n, \quad j = 1, \dots, m.$$

where  $G_{ij}$  and  $K_{ij}$  are known vectors and matrices, respectively. Also

$$C_i = [c_{i0}^0, c_{i0}^1, \dots, c_{i0}^N, \dots, c_{iN}^0, \dots, c_{iN}^N]^T, \quad i = 1, \dots, n.$$

Substituting above approximations into system (11) leads to the following algebraic system:

$$\Phi^T(x, y)C_i + \Phi^T(x, y) \sum_{j=1}^m \{K_{ij}AG_{ij}\} - f_i(x, y) = 0, \quad (12)$$

where  $A$  is a  $(N+1)^2 \times (N+1)^2$  matrix as follows:

$$A = \int_0^1 \int_0^1 \Phi(t, s)\Phi^T(t, s)ds dt.$$

The algebraic system (12) has  $n(N+1)^2$  unknown coefficients  $c_{ijk}^i$ . So,  $n(N+1)^2$  collocating points are needed for collocating the algebraic system resulted. For this purpose, the  $(N+1)$  roots of Jacobi polynomials  $P_{N+1}^{(\alpha, \beta)}(x)$  and  $P_{N+1}^{(\alpha, \beta)}(y)$  are considered in the  $x$ - and  $y$ -directions. The domain of two dimensional is represented by a tensor product points  $\{x_i\}_{i=0}^N$  and  $\{y_j\}_{j=0}^N$  which are roots of  $P_{N+1}^{(\alpha, \beta)}(x)$  and  $P_{N+1}^{(\alpha, \beta)}(y)$ . Each the equations of the system (12) is collocating in the resulted tensor points  $\{(x_i, y_j)\}_{i, j=0}^N$ . Finally, collocating the equations (12) gives  $n(N+1)^2$  linear or nonlinear equations which nonlinear equations can be solved using the well-known Newton's iterative method.

##### 4.2 System of two dimensional Volterra integral equations

A system of two dimensional Volterra integral equations can be presented as follows:

$$\begin{aligned} u_i(x, y) + \sum_{j=1}^{m_1} \int_0^x k_{ij}(x, y, t, s) g_{ij}(u_1(t, y), \dots, u_n(t, y)) dt \\ + \sum_{k=1}^{m_2} \int_0^y h_{ik}(x, y, t, s) l_{ik}(u_1(x, s), \dots, u_n(x, s)) ds = f_i(x, y), \\ (x, y) \in D, \quad i = 1, \dots, n, \end{aligned} \quad (13)$$

where  $k_{ij}(x, y, t, s)$  and  $h_{ik}(x, y, t, s) \in L^2(D^2)$ ,  $f_i(x, y)$  are known functions, and  $g_{i,j}(x, y)$  and  $l_{i,k}(x, y)$  are linear or nonlinear functions in terms of unknown functions  $u_1(x, y), \dots, u_n(x, y)$ . The functions  $u_i(x, y)$ ,  $g_{ij}(x, y)$ ,  $l_{ik}(x, y)$ ,  $k_{ij}(x, y, t, s)$  and  $h_{ik}(x, y, t, s)$  can be approximated as follows:

$$u_i(x, y) \simeq \Phi^T(x, y)C_i, \quad g_{ij}(x, y) \simeq \Phi^T(x, y)G_{ij},$$

$$l_{ik}(x, y) \simeq \Phi^T(x, y)L_{ik} \quad k_{ij}(x, y, t, s) \simeq \Phi^T(x, y)K_{ij}\Phi(t, y),$$

$$h_{ik}(x, y, t, s) \simeq \Phi^T(x, y)H_{ik}\Phi(x, s),$$

$$i = 1, \dots, n, \quad j = 1, \dots, m_1, \quad k = 1, \dots, m_2,$$

where  $C_i$ ,  $G_{ij}$  and  $L_{ik}$  are  $(N+1)^2 \times 1$  vectors and  $K_{ij}$  and  $H_{ik}$  are  $(N+1)^2 \times (N+1)^2$  known matrices. Substituting above approximations into system (13) leads to:



$$\Phi^T(x,y)C_i + \Phi^T(x,y)\{\sum_{j=1}^{m_1} K_{ij}\tilde{G}_{ij}\}P_x\Phi(x,y) + \Phi^T(x,y)\{\sum_{k=1}^{m_2} H_{ik}\tilde{L}_{ik}\}P_y\Phi(x,y) - f_i(x,y) = 0, \quad (14)$$

$$i = 1, \dots, n, \quad j = 1, \dots, m_1, \quad k = 1, \dots, m_2,$$

where  $\tilde{G}_{ij}$  and  $\tilde{L}_{ik}$  are operational matrices of product and their entries are in terms of the components of vectors  $G_{ij}$  and  $L_{ij}$ .  $P_x$  and  $P_y$  are operational matrices of integration in  $x$ - and  $y$ -directions, respectively. Each the equations of the system (14) is collocated in tensor points  $\{(x_i, y_j)\}_{i,j=0}^N$  (stated in subsection 4.1). Finally, equations (14) give  $n(N + 1)^2$  linear or nonlinear equations which nonlinear equations can be solved using the Newton's iterative method.

### 4.3 System of two dimensional Volterra-Fredholm integral equations

In this paper, a system of Volterra-Fredholm integral equations is considered as follows:

$$u_i(x,y) + \sum_{j=1}^m \int_0^y \int_0^1 k_{ij}(x,y,t,s)g_{ij}(u_1(t,s), \dots, u_n(t,s))dsdt = f_i(x,y), \quad (x,y) \in D, \quad i = 1, \dots, n. \quad (15)$$

To solve system (15) the functions  $u_i(x,y)$ ,  $g_{ij}(x,y)$  and  $k_{ij}(x,y,t,s)$  can be approximated as follows:

$$u_i(x,y) \simeq \Phi^T(x,y)C_i, \quad g_{ij}(x,y) \simeq \Phi^T(x,y)G_{ij},$$

$$k_{ij}(x,y,t,s) \simeq \Phi^T(x,y)K_{ij}\Phi(t,s),$$

$$i = 1, \dots, n, \quad k = 1, \dots, m,$$

where  $C_i$  and  $G_{ij}$  are  $(N + 1)^2 \times 1$  vectors and  $K_{ij}$  are  $(N + 1)^2 \times (N + 1)^2$  known matrices. Substituting above approximations into system (15) leads to,

$$\Phi^T(x,y)C_i + \Phi^T(x,y)\sum_{j=1}^m \{K_{ij}\tilde{G}_{ij}\}P_yB - f_i(x,y) = 0, \quad (16)$$

$$i = 1, \dots, n, \quad j = 1, \dots, m,$$

where  $B$  is  $(N + 1)^2$  matrix as follows:

$$B = \int_0^1 \Phi(t,y) dt,$$

and  $K_{ij}$ ,  $\tilde{G}_{ij}$  and  $P_y$  are  $(N + 1)^2 \times (N + 1)^2$  known matrices, operational matrices of product and operational matrix of integration, respectively. Collocating each the equations of the system (16) in tensor points  $\{(x_i, y_j)\}_{i,j=0}^N$  leads to  $n(N + 1)^2$  linear or nonlinear. Unknown coefficients  $c_{jk}^i$  are determined by solving the system resulted.

**Table 1:** Different values of exact and approximate solutions in points (0.1i, 0.1i) for  $N = 4$  and  $\alpha = \beta = 0$  of Example 1.

$(x_i, y_j)$	$u_{exact}$	$u_{approximate}$
(0.1, 0.1)	-0.0004995835	-0.0004990535
(0.2, 0.2)	-0.0039866844	-0.0039862814
(0.3, 0.3)	-0.0133990533	-0.0134002600
(0.4, 0.4)	-0.0315756024	-0.0315774590
(0.5, 0.5)	-0.0612087190	-0.0612086874
(0.6, 0.6)	-0.1047986311	-0.1047956186
(0.7, 0.7)	-0.1646104689	-0.1646073175
(0.8, 0.8)	-0.2426346326	-0.2426367636
(0.9, 0.9)	-0.3405510285	-0.3405573796
(1, 1)	-0.4596976941	-0.4596795534

## 5 Illustrative examples

In this section, the proposed method is applied to solve some systems of two dimensional integral equations.

**Example 1.** Consider the following two dimensional linear Fredholm integral equation.

$$u(x,y) - \int_0^1 \int_0^1 (ssin(x) + ty)u(t,s)dt ds = f(x,y), \quad (17)$$

where

$$f(x,y) = x \cos(y) - \frac{3}{4}y + \frac{5}{6}\sin(x) - \frac{y}{3}\sin(1) - \frac{1}{2}\sin(x) \cos(1) - \frac{1}{2}\sin(x) \sin(1),$$

and exact solution is  $u(x,y) = x \cos(y) - y$ . Function  $u(x,y)$  and kernel are approximated as:

$$u(x,y) \simeq \Phi^T(x,y)C, \quad ssin(x) + ty \simeq \Phi^T(x,y)K\Phi(t,s).$$

Equation (17) is written by using above relations as:

$$\Phi^T(x,y)C - \Phi^T(x,y)KBC - f(x,y) = 0, \quad (18)$$

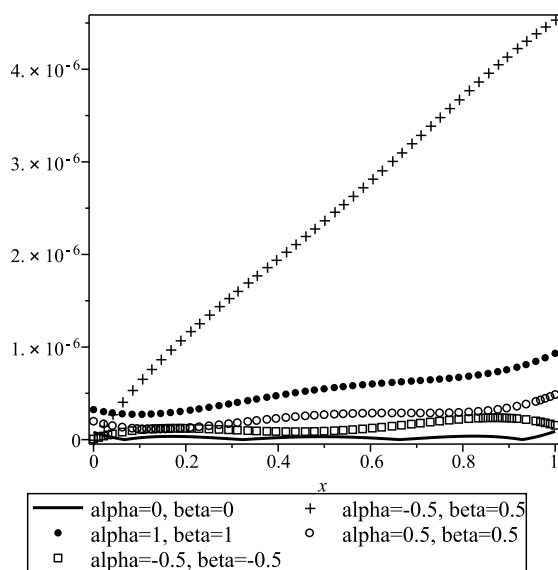
where

$$B = \int_0^1 \int_0^1 \Phi(t,s)\Phi^T(t,s)dt ds.$$

Setting  $N = 4$  and using the roots of  $P_5^{(\alpha,\beta)}(x)$  and  $P_5^{(\alpha,\beta)}(y)$  in the  $x$ - and  $y$ -directions, equation (18) is collocated in 25 inner tensor points for different values of parameters  $\alpha$  and  $\beta$ . Hereby, the equation (17) reduces the problem to solve a system of linear algebraic equations and unknown coefficients are obtained for some values of parameters  $\alpha$  and  $\beta$ . Table 1 displays different values of the exact and approximate solutions in points  $(x,y) = (0.1i, 0.1i)$ , ( $i = 1, 2, \dots, 10$ ) for  $\alpha = \beta = 0$ . Table 2 shows errors of the approximate solutions in  $L^2(D)$  for different values of  $\alpha$  and  $\beta$ . Figure 1 indicates the maximum absolute errors for some values  $\alpha$  and  $\beta$  and  $y = 0.5$ .

**Table 2:** Comparison of the errors in  $L^2(D)$  for different values  $\alpha$  and  $\beta$  of Example 1.

$(\alpha, \beta)$	$Error(L^2)$
(0,0)	$2.7496 \times 10^{-6}$
(-0.5, -0.5)	$4.1629 \times 10^{-6}$
(1, 1)	$3.9061 \times 10^{-6}$
(0.5, 0.5)	$3.1072 \times 10^{-6}$
(-0.5, -0.5)	$3.1597 \times 10^{-6}$
(-0.1, -0.1)	$7.9793 \times 10^{-4}$



**Fig. 1:** Plot of the maximum of the absolute errors in Example 1.

**Example 2.** Consider the following linear system of Volterra integral equations.

$$\begin{cases} u_1(x, y) = f_1(x, y) - \int_0^x (2u_1(t, y) + u_2(t, y)) dt, \\ u_2(x, y) = f_2(x, y) + \int_0^y (u_2(x, s) - u_1(x, s)) ds, \end{cases} (x, y) \in D, \quad (19)$$

where

$$f_1(x, y) = 3x^2y + 2x^3y + \frac{1}{2}x^2y^3 - x,$$

$$f_2(x, y) = xy^3 - \frac{1}{4}xy^4 + \frac{3}{2}x^2y^2 + y - 1$$

and exact solutions are  $u_1(x, y) = 3x^2y$  and  $u_2(x, y) = xy^3 - 1$ . With  $N = 3$ , functions and kernels are approximated as:

$$u_1(x, y) \simeq \Phi^T(x, y)C_1, \quad u_2(x, y) \simeq \Phi^T(x, y)C_2,$$

$$1 \simeq \Phi^T(x, y)K\Phi(t, y).$$

Substituting above approximations in system (19) leads to the following algebraic system:

$$\begin{cases} \Phi^T(x, y)C_1 - f_1(x, y) + \Phi^T(x, y)K\{2\tilde{C}_1 + \tilde{C}_2\}P_x\Phi(x, y) = 0, \\ \Phi^T(x, y)C_2 - f_2(x, y) - \Phi^T(x, y)K\{\tilde{C}_2 - \tilde{C}_1\}P_y\Phi(x, y) = 0, \end{cases} \quad (20)$$

where  $\tilde{C}_1$  and  $\tilde{C}_2$  are operational matrices of product,  $P_x$  and  $P_y$  are operational matrices of integration in the  $x$ - and  $y$ -directions, respectively. Now using the roots of  $P_4^{(\alpha, \beta)}(x)$  and  $P_4^{(\alpha, \beta)}(y)$  in the  $x$ - and  $y$ -directions, each equations of system (20) is collocated in 16 inner tensor points for different values of parameters  $\alpha$  and  $\beta$ , the coefficients are obtained as follows. Thereupon, the exact solutions are acquired.

$$\underline{\underline{\alpha = \beta = 0}}:$$

$$C_1 = \left[ \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{3}{4}, \frac{3}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{7}{8}, \frac{9}{40}, \frac{1}{8}, \frac{1}{40}, \frac{1}{8}, \frac{9}{40}, \frac{1}{8}, \frac{1}{40}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\underline{\underline{\alpha = \beta = 0.5}}:$$

$$C_1 = \left[ \frac{15}{32}, \frac{5}{16}, 0, 0, \frac{1}{2}, \frac{1}{3}, 0, 0, \frac{3}{20}, \frac{1}{10}, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{57}{64}, \frac{7}{48}, \frac{3}{40}, \frac{1}{70}, \frac{7}{96}, \frac{7}{72}, \frac{1}{20}, \frac{1}{105}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\underline{\underline{\alpha = -\beta = 0.5}}:$$

$$C_1 = \left[ \frac{3}{32}, \frac{3}{16}, 0, 0, \frac{9}{32}, \frac{9}{16}, 0, 0, \frac{1}{8}, \frac{1}{4}, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{251}{256}, \frac{9}{128}, \frac{5}{96}, \frac{1}{80}, \frac{5}{128}, \frac{9}{64}, \frac{5}{48}, \frac{1}{40}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\underline{\underline{\alpha = -\beta = -0.5}}:$$

$$C_1 = \left[ \frac{45}{32}, \frac{15}{16}, 0, 0, \frac{45}{32}, \frac{15}{16}, 0, 0, \frac{3}{8}, \frac{1}{4}, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{151}{256}, \frac{63}{128}, \frac{7}{32}, \frac{3}{80}, \frac{35}{128}, \frac{21}{64}, \frac{7}{48}, \frac{1}{40}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\underline{\underline{\alpha = \beta = -0.5}}:$$

$$C_1 = \left[ \frac{9}{16}, \frac{9}{8}, 0, 0, \frac{3}{2}, 3, 0, 0, \frac{1}{2}, 1, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{27}{32}, \frac{15}{32}, \frac{1}{4}, \frac{1}{20}, \frac{5}{16}, \frac{15}{16}, \frac{1}{2}, \frac{1}{10}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\underline{\underline{\alpha = \beta = 1}}:$$

$$C_1 = \left[ \frac{9}{20}, \frac{9}{40}, 0, 0, \frac{3}{8}, \frac{3}{16}, 0, 0, \frac{1}{10}, \frac{1}{20}, 0, 0, 0, 0, 0, 0 \right],$$

$$C_2 = \left[ -\frac{9}{10}, \frac{3}{28}, \frac{1}{20}, \frac{1}{112}, \frac{1}{20}, \frac{3}{56}, \frac{1}{40}, \frac{1}{224}, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

**Example 3.** Third example covers the system of nonlinear Volterra - Fredholm integral equation.

$$\begin{cases} u_1(x,y) = f_1(x,y) + \int_0^y \int_0^1 u_1^2(t,s) dt ds, \\ u_2(x,y) = f_2(x,y) + (x-y^2) \int_0^y \int_0^1 u_2^2(t,s) dt ds, \end{cases} \quad (x,y) \in D, \quad (21)$$

where

$$f_1(x,y) = -y^2 + 2xy - \frac{1}{90}y^3(4.45y + 18y^2),$$

$$f_2(x,y) = 1 + y^2 \sin(x) - 0.01667y(-60 - 18.3879y^2 - 3.27211y^4)(y^2 - x)$$

and exact solutions are  $u_1(x,y) = -y^2 + 2xy$  and  $u_2(x,y) = 1 + y^2 \sin(x)$ . With  $N = 4$ , solutions and kernels are approximated as:

$$u_1(x,y) \simeq \Phi^T(x,y)C_1, \quad u_2(x,y) \simeq \Phi^T(x,y)C_2,$$

$$1 \simeq \Phi^T(x,y)K\Phi(t,s), \quad u_1^2(x,y) \simeq \Phi^T(x,y)U_1,$$

$$u_2^2(x,y) \simeq \Phi^T(x,y)U_2.$$

Substituting above approximations in system (21) leads to the following algebraic system.

$$\begin{cases} \Phi(x,y)C_1 - f_1(x,y) - \Phi^T(x,y)K\tilde{U}_1P_yB = 0, \\ \Phi(x,y)C_2 - f_2(x,y) - (x-y^2)\Phi^T(x,y)K\tilde{U}_2P_yB = 0, \end{cases} \quad (22)$$

where  $\tilde{U}_1$  and  $\tilde{U}_2$  are operational matrices of product,  $P_y$  is operational matrix of integration and  $B$  is a  $(N+1)^2$  vector as:

$$B = \int_0^1 \Phi(t,y) dt.$$

Using roots of  $P_5^{(\alpha,\beta)}(x)$  and  $P_5^{(\alpha,\beta)}(y)$  in the  $x$ - and  $y$ -directions, each equations of system (22) is collocated in 25 inner tensor points. The problem reduces to solve a system of nonlinear algebraic equations which will be solved by means of Newton iterative method and 50 unknown coefficients are determined for some values of parameters  $\alpha$  and  $\beta$ . For  $u_1(x,y)$ , the exact solution is obtained. The nonzero components of vector  $C_1$  for the various values of parameters  $\alpha$  and  $\beta$  are as follows:

$$\underline{\underline{\alpha = \beta = 0}} : c_{00}^1 = \frac{1}{6}, c_{02}^1 = -\frac{1}{6}, c_{05}^1 = \frac{1}{2}, c_{06}^1 = \frac{1}{2},$$

$$\underline{\underline{\alpha = \beta = \frac{1}{2}}} : c_{00}^1 = \frac{3}{16}, c_{02}^1 = -\frac{1}{10}, c_{05}^1 = \frac{1}{3}, c_{06}^1 = \frac{2}{9},$$

$$\underline{\underline{\alpha = \beta = -\frac{1}{2}}} : c_{00}^1 = \frac{1}{8}, c_{02}^1 = -\frac{1}{3}, c_{05}^1 = 1, c_{06}^1 = 2,$$

$$\underline{\underline{\alpha = -\beta = \frac{1}{2}}} : c_{01}^1 = -\frac{1}{8}, c_{02}^1 = -\frac{1}{6}, c_{05}^1 = \frac{1}{4}, c_{06}^1 = \frac{1}{2},$$

$$\underline{\underline{\alpha = -\beta = -\frac{1}{2}}} : c_{00}^1 = \frac{1}{2}, c_{02}^1 = \frac{1}{8}, c_{05}^1 = \frac{3}{4}, c_{06}^1 = \frac{1}{2},$$

**Table 3:** Different values of exact and approximate solutions in points  $(0.1i, 0.1i)$  for  $N = 4$  and  $\alpha = \beta = -0.5$  of Example 3.

$(x_i, y_j)$	$u_{2_{exact}}$	$u_{2_{approximate}}$
(0.1, 0.1)	1.000998334	1.000997067
(0.2, 0.2)	1.007946773	1.007939665
(0.3, 0.3)	1.026596819	1.026582292
(0.4, 0.4)	1.062306935	1.062283999
(0.5, 0.5)	1.119856385	1.119823318
(0.6, 0.6)	1.203271290	1.203228160
(0.7, 0.7)	1.315666667	1.315619935
(0.8, 0.8)	1.459107898	1.459071923
(0.9, 0.9)	1.634494799	1.634481860
(1, 1)	1.841470985	1.841458774

**Table 4:** comparison of the errors in  $L^2(D)$  for different values  $\alpha$  and  $\beta$  of Example 3.

$(\alpha, \beta)$	$Error(L^2)$
(0, 0)	$8.5374 \times 10^{-5}$
(-0.5, -0.5)	$6.2993 \times 10^{-5}$
(-0.5, 0.5)	$6.4008 \times 10^{-5}$
(0.5, 0.5)	$6.4328 \times 10^{-5}$
(0.5, -0.5)	$6.4887 \times 10^{-5}$
(1, 1)	$6.5354 \times 10^{-5}$

$$\underline{\underline{\alpha = \beta = 1}} : c_{00}^1 = \frac{1}{5}, c_{02}^1 = -\frac{1}{15}, c_{05}^1 = \frac{1}{4}, c_{06}^1 = \frac{1}{8},$$

$$\underline{\underline{\alpha = \beta = 2}} : c_{00}^1 = \frac{3}{14}, c_{02}^1 = -\frac{1}{28}, c_{05}^1 = \frac{1}{6}, c_{06}^1 = \frac{1}{8},$$

Table 3 shows different values of the exact and approximate solutions for  $u_2(x,y)$  in points  $(x_i, y_i) = (0.1i, 0.1i)$ ,  $(i = 1, 2, \dots, 10)$  for  $\alpha = \beta = -0.5$ . Table 4 displays the maximum absolute errors for values of  $\alpha$  and  $\beta$ .

## 6 Conclusion

Analytic solution of the two dimensional integral equations are usually difficult. In many cases, it is required to approximate solutions. In this paper, the system of two dimensional linear and nonlinear integral equations was solved by using collocation method. For this purpose, the shifted Jacobi collocation method was employed to solve a class of systems of Fredholm and Volterra integral equations. First, a general formulation for two dimensional Jacobi operational matrix of integration has been derived. This matrix is used to approximate numerical solution of system of linear and nonlinear Volterra integral equations. Proposed approach is based on the shifted Jacobi collocation method. The solutions obtained using the proposed method shows that this method is a powerful mathematical tool for solving the integral equations. Proving the convergence of the method, consistency and stability are ensured automatically. Moreover, only a small number of shifted

Jacobi polynomials is needed to obtain a satisfactory result.

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