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Generalized Anti Fuzzy Interior Ideals in LA-Semigoups

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Abstract: Using the notion of an anti fuzzy point and its *besideness to* and *non-quasicoincidence* with a fuzzy subset, some new concept of an anti fuzzy interior ideals in LA-semigroups are introduced and their interrelations and related properties are investigated. We also introduce the notion of a *strongly besideness* and *strongly non-quasicoincidence* of an anti fuzzy point with a fuzzy subset and characterize anti fuzzy interior ideals of LA-semigroups in terms of these notions.

Keywords: LA-semigroup, beside to, non-quasiconsidence with, $([\alpha],[\beta])$-fuzzy interior ideal.
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1 Introduction

The concept of a fuzzy set was first initiated by Zadeh [1]. Since then it has become a key area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. In [2], Jun et al. introduced the concept of an anti fuzzy bi-ideals of ordered semigroup by using the notion of anti fuzzy points and besideness to and non-quasi-coincidence with a fuzzy set, and investigate their inter-relations and related properties. [3] Shabir et al. studied semigroups by the properties of their anti fuzzy ideals. In [4] Madad et al. studied anti fuzzy ideals in LA-semigroups. An LA-semigroup is a groupoid S whose elements satisfy the left invertive law: $(ab)c = (cb)a$ for all $a,b,c \in S$. An LA-semigroup is the midway structure between a commutative semigroup and a groupoid. It is a useful non-associative structure with wide applications in theory of flocks. In an LA-semigroup the medial law $(ab)(cd) = (ac)(bd)$ holds for all $a,b,c,d \in S$ [5]. If there exists an element e in an LA-semigroup S such that $ex = x$ for all $x \in S$, then S is called an LA-semigroup with left identity e. If an LA-semigroup S has the right identity, then S is a commutative monoid. If an LA-semigroup S contains left identity, then $(ab)(cd) = (dc)(ba)$ holds for all $a,b,c,d \in S$.

In this paper we introduce the concept of generalized anti fuzzy interior ideals in LA-semigroup and investigate their relative properties. We give some interesting characterizations of an LA-semigroup in terms of $([\alpha],[\beta])$-fuzzy interior ideal and introduce the notion of $\left(\underline{[\alpha]},\underline{[\beta]}\right)$-fuzzy interior ideals of an LA-semigroup.

2 Preliminaries

Here onward S will denotes an LA-semigroup.

For subsets A,B of S, we denote $AB = \{ab \in S | a \in A, b \in B\}$. A nonempty subset A of S is called sub LA-semigroup of S if $A^2 \subseteq A$. A is called an interior ideal of S if $A^2 \subseteq A$ and $(SA)S \subseteq A$. By a fuzzy subset ψ of S we mean a mapping $\psi : S \longrightarrow [0,1]$. For any fuzzy subsets ψ_1 and ψ_2 of S define $\psi_1 \circ \psi_2 := S \longrightarrow [0,1]\,, a \longrightarrow (\psi_1 \circ \psi_2)(a)$

$$= \begin{cases} \wedge_{a=pq}\{f(p) \vee g(q)\} & \text{if there exist } p,q \in S \text{ such that } a = pq \\ 1 & \text{otherwise} \end{cases}.$$

For a nonempty family of fuzzy subsets $\{\psi_i\}_{i\in I}$ of S the fuzzy subsets $\cup_{i\in I}\psi i$ and $\cap_{i\in I}\psi i$ of S are defined as follows:

$$\cup_{i\in I}\psi i : S \longrightarrow [0,1]\,, a \longrightarrow (\cup_{i\in I}\psi_i)(a) =: \sup_{i\in I}\{\psi_i(a)\},$$
$$\cap_{i\in I}\psi i : S \longrightarrow [0,1]\,, a \longrightarrow (\cap_{i\in I}\psi_i)(a) =: \inf_{i\in I}\{\psi_i(a)\}.$$

If I is a finite set, say $I = \{1,2,...,n\}$, then clearly

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$\cup_{i\in I}\psi_i(a) = \max\{\psi_1(a), \psi_2(a), ..., \psi_n(a)\}$,
$\cap_{i\in I}\psi_i(a) = \inf\{\psi_1(a), \psi_2(a), ..., \psi_n(a)\}$.

Definition 1.[10] *A fuzzy subset ψ of S called fuzzy interior ideal of S, if it satisfy the following conditions.*

$(B1)$ $(\forall x,y\in S)(\psi(xy)\geq \min\{\psi(x),\psi(y)\})$,
$(B2)$ $(\forall x,y,a\in S)(\psi((xa)y)\geq \psi(a))$.

Lemma 1.[10] *A fuzzy subset ψ of S is fuzzy interior ideal of S if and only if χ_A is interior ideal of S.*

Definition 2.*Let ψ be a fuzzy subset of S. Then for every $t\in[0,1)$ the set $L(\psi;t)=\{x|x\in S, \psi(x)\geq t\}$ is called a level set of ψ.*

Lemma 2.[10] *Let ψ be a fuzzy subset of S. Then ψ is fuzzy interior ideal of S if and only if $L(\psi;t)\neq \Phi$ is an interior ideal of S for every $t\in[0,1)$.*

3 $([\alpha],[\beta])$-fuzzy interior ideals

Definition 3.*A fuzzy subset ψ of S of the form $\psi(y)=\left\{\begin{array}{ll} t\in[0,1) & if\ y=x \\ 1 & otherwise\end{array}\right\}$ is called an anti fuzzy point with support x and value t and is denoted by $\frac{t}{x}$. A fuzzy subset ψ of S is said to be non unit if there exist $x\in S$ such that $\psi(x)<1$. For an anti fuzzy point $\frac{t}{x}$ and a fuzzy subset ψ in S, Jun and Sung [7] gave meaning to the symbol $\frac{t}{x}\alpha\psi$, where $\alpha\in\{[\in],[q],[\in]\vee[q],[\in]\wedge[q]\}$. To say that $\frac{t}{x}[\in]\psi \Longrightarrow \psi(x)\leq t$ and $\frac{t}{x}[q]\psi\Longrightarrow\psi(x)+t<1$.Also $\frac{t}{x}[\in]$ is called besides to and $\frac{t}{x}[q]$ is called non-quasi-coincident with a fuzzy subset ψ of S. Also $\frac{t}{x}[\in]\vee[q]\Longrightarrow\frac{t}{x}[\in]$ or $\frac{t}{x}[q]$ and $\frac{t}{x}[\in]\wedge[q]\Longrightarrow\frac{t}{x}[\in]$ and $\frac{t}{x}[q]$. The symbol $\frac{t}{x}\overline{[\alpha]}\psi$ means that $\frac{t}{x}[\alpha]\psi$ does not hold. A fuzzy point $\frac{t}{x}$ is said to be strongly besides to (respectively strongly non-quasi-coincident) with a fuzzy set ψ written as $\frac{t}{x}\underline{[\in]}\psi\left(resp,.\ \frac{t}{x}\underline{[q]}\psi\right)$. If $\frac{t}{x}\underline{[\alpha]}\psi$ or $\frac{t}{x}\underline{[\beta]}\psi$, then $\frac{t}{x}\left(\underline{[\alpha]\vee[\beta]}\right)$.*

Definition 4.*Let X be a non empty set and A be a subset of X. Then the anti characteristic functions χ_{A^c} is defined by*

$$\chi_{A^c}=\left\{\begin{array}{ll} 0 & if\ x\in A \\ 1 & if\ x\notin A\end{array}\right. .$$

Definition 5.*A fuzzy subset ψ of S called an anti fuzzy interior ideal of S, if it satisfy the following conditions.*

$(B1)$ $(\forall x,y\in S)(\psi(xy)\leq \max\{\psi(x),\psi(y)\})$,
$(B2)$ $(\forall x,y,a\in S)(\psi((xa)y)\leq \psi(a))$.

Lemma 3.*A fuzzy subset ψ of S is an anti fuzzy interior ideal of S if and only if χ_A is interior ideal of S.*

Definition 6.*Let ψ be a fuzzy subset of S. Then for every $t\in[0,1)$ the set $L(\psi;t)=\{x|x\in S,\psi(x)\leq t\}$ is called a lower level set of ψ.*

Lemma 4.*Let ψ be a fuzzy subset of S. Then ψ is an anti fuzzy interior ideal of S if and only if $L(\psi;t)\neq\Phi$ is an interior ideal of S for every $t\in[0,1)$.*

Theorem 1.*For any fuzzy subset ψ of S the conditions (B_1) and (B_2) are equivalent to the following.*

(B_3) $(\forall x,y\in S)(\forall t_1,t_2\in[0,1))\left(\frac{t_1}{x}[\in]\psi,\frac{t_2}{y}[\in]\psi\Longrightarrow\frac{\max\{t_1,t_2\}}{xy}[\in]\psi\right)$.
(B_4)
$(\forall x,y,a\in S)(\forall t\in[0,1))\left(\frac{t}{a}[\in]\psi\Longrightarrow\frac{t}{((xa)y)}[\in]\psi\right)$.

Proof: $(B_1)\Longrightarrow(B_3)$. Let $x,y\in S$ and $t_1,t_2\in[0,1)$ be such that $\frac{t_1}{x}[\in]\psi$,and $\frac{t_2}{y}[\in]\psi$. Then $\psi(x)\leq t_1$ and $\psi(y)\leq t_2$. By (B_1) we have $\psi(xy)\leq\max\{\psi(x),\psi(y)\}\leq\max\{t_1,t_2\}$ and so $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$.

$(B_3)\Longrightarrow(B_1)$. Let $x,y\in S$. Since $\frac{\psi(x)}{x}[\in]\psi$ and $\frac{\psi(y)}{y}[\in]\psi$. Then by (B_3), we have $\frac{\max\{\psi(x),\psi(y)\}}{xy}[\in]\psi$ and so $\psi(xy)\leq\max\{\psi(x),\psi(y)\}$.

$(B_2)\Longrightarrow(B_4)$. Let $x,y,a\in S$ and $t\in[0,1)$ be such that $\frac{t}{a}[\in]\psi$. Then $\psi(a)\leq t$. By (B_2) we have $\psi((xa)y)\leq\psi(a)\leq t$ and so $\frac{t}{((xa)y)}[\in]\psi$.

$(B_4)\Longrightarrow(B_2)$. Let $x,y\in S$. Since $\frac{\psi(a)}{a}[\in]\psi$, by (B_4). We have $\frac{\psi(a)}{((xa)y)}[\in]\psi$ and so $\psi((xa)y)\leq\psi(a)$. □

4 $([\in],[\in]\vee[q])$-Fuzzy Interior Ideals

In[9] Jun et al. introduced the concept of a generalized fuzzy interior ideal of a semigroup. In [8], Jun et al. introduced the concept an (α,β)-fuzzy bi-ideal of an ordered semigroup and characterized ordered semigroups in terms of (α,β) - fuzzy bi-ideals. In [10] Khan et al. introduced the notion of $(\in,\in\vee q)$- fuzzy interior ideals of an Abel Grassmann's groupoid and investigate some of their properties in terms of $(\in,\in\vee q)$- fuzzy interior ideals.

Let ψ be a fuzzy subset of S and $\psi(x)\geq 0.5$ for all $x\in S$. Let $x\in S$ and $t\in[0,1)$ be such that $\frac{t}{x}[\in]\wedge[q]\psi$. Then $\frac{t}{x}[\in]\psi$ and $\frac{t}{x}[q]\psi$ and so $\psi(x)\leq t$ and $\psi(x)+1<t$. It follows that $1>\psi(x)+t\geq\psi(x)+\psi(x)=2\psi(x)$ and so $\psi(x)<0.5$ which is a contradiction. This means that $\{x\in S|\frac{t}{x}[\in]\wedge[q]\psi\}=\emptyset$.

Definition 7.*A fuzzy subset ψ of S is called an $([\alpha],[\beta])$ -fuzzy interior ideal of S, where $\alpha\neq[\in]\wedge[q]$, if it satisfy the following conditions:*

$(B5)$ $(\forall x,y\in S)(\forall t_1,t_2\in[0,1))\left(\frac{t_1}{x}[\alpha]\psi,\frac{t_2}{y}[\alpha]\psi\rightarrow\frac{\max\{t_1,t_2\}}{xy}[\beta]\psi\right)$
$(B6)$
$(\forall x,y,a\in S)(\forall t\in[0,1))\left(\frac{t}{a}[\alpha]\psi\rightarrow\frac{t}{((xa)y)}[\beta]\psi\right)$.

Proposition 1. *Let ψ be a fuzzy subset of S. If $\alpha = [\in]$ and $\beta = [\in] \vee [q]$ in Definition 7. Then $(B5)$ and $(B6)$, respectively, are equivalent to the following conditions:*

$(B7)$ $(\forall x, y \in S)(\psi(xy) \leq \max\{\psi(x), \psi(y), 0.5\})$,

$(B8)$ $(\forall x, y, a \in S)(\psi((xa)y) \leq \max\{\psi(a), 0.5\})$.

Proof: It is very easy to prove. □

Remark. A fuzzy subset ψ of S is an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S if and only if it satisfy conditions $(B7)$ and $(B8)$ of the Proposition 1.

We have the following characterization of $([\in], [\in] \vee [q])$-fuzzy interior ideals of an LA-semigroup S.

Lemma 5. *Let S be an LA-semigroup and $\emptyset \neq I \subseteq S$. Then I is an interior ideal of S if and only if the characteristic function χ_{I^c} of I is an $([\in], [\in] \vee [q])$ -fuzzy interior ideal of S.*

The converse of Theorem 1 is not true in general, as shown in the following example.

Example: Let $S = \{a, b, c, d, e\}$ be an LA-semigroup with the following table:

Example 1.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

The interior ideals of S are $\{a\}$, $\{a, c, d, e\}$, Φ and S. Define a fuzzy subset $\psi : S \to [0, 1]$ by $\psi(a) = 0.1, \psi(c) = 0.2, \psi(d) = 0.4, \psi(e) = 0.6, \psi(b) = 0.8$. Then

$$L(A;t) = \left\{ \begin{array}{ll} S & \text{if } t \in [0.8, 1) \\ \{a, c, d, e\} & \text{if } t \in [0.7, 1) \\ \{a\} & \text{if } t \in [0.1, 0.12) \\ \Phi & \text{if } t \in [0.01, 0.02) \end{array} \right\}$$

Obviously ψ is an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S. By Lemma 5. But we have the following ,

(1) ψ is not an $([\in], [\in])$-fuzzy interior ideal of S. Since $\frac{0.5}{d}[\in]\psi$ but $\frac{\max\{0.5,0.5\}}{(dd)} = \frac{0.5}{e}\overline{[\in]}\psi$.

(2) ψ is not an $([\in], [q])$-fuzzy interior ideal of S. Since $\frac{0.7}{d}[\in]\psi$ but $\frac{\max\{0.7,0.7\}}{(dd)} = \frac{0.7}{e}\overline{[q]}\psi$.

(3) ψ is not an $([q], [\in])$-fuzzy interior ideal of S. Since $\frac{0.2}{c}[q]\psi$ and $\frac{0.2}{c}[q]\psi$ but $\frac{\max\{0.2,0.2\}}{(cc)} = \frac{0.2}{e}\overline{[\in]}\psi$.

(4) ψ is not an $([q], [\in] \vee [q])$-fuzzy interior-ideal of S. Since $\frac{0.2}{c}[q]\psi$ and $\frac{0.2}{c}[q]\psi$ but $\frac{\max\{0.2,0.2\}}{(cc)} = \frac{0.2}{e}\overline{[\in]}\psi$.

(5) ψ is not an $([\in] \vee [q], [\in] \wedge [q])$ -fuzzy interior ideal of S. Since $\frac{0.5}{c}[\in] \vee [q]\psi$ but $\frac{\max\{0.5,0.5\}}{(cc)} = \frac{0.5}{e}\left(\overline{[\in], [q]}\right)\psi$.

(6) ψ is not an $([\in] \vee [q], [q])$-fuzzy interior ideal of S. Since $\frac{0.5}{c}[\in] \vee [q]\psi$ and $\frac{0.9}{c}[\in] \vee [q]\psi$ but $\frac{\max\{0.5,0.9\}}{(cc)} = \frac{0.9}{e}\overline{[q]}\psi$.

(7) ψ is not an $([\in] \vee [q], [\in])$-fuzzy interior ideal of S. Since $\frac{0.4}{c}[\in] \vee [q]\psi$, but $\frac{\max\{0.4,0.4\}}{(cc)} = \frac{0.4}{e}\overline{[\in]}\psi$.

(8) ψ is not an $([\in] \wedge [q], [\in])$-fuzzy interior ideal of S. Since $\frac{0.3}{c}[\in] \wedge [q]\psi$, but $\frac{\max\{0.3,0.3\}}{(cc)} = \frac{0.3}{e}\overline{[\in]}\psi$.

(9) ψ is not an $([q], [q])$ -fuzzy interior ideal of S. Since $\frac{0.5}{c}[q]\psi$ and $\frac{0.5}{c}[q]\psi$ but $\frac{\max\{0.5,0.5\}}{(cc)} = \frac{0.5}{e}\overline{[q]}\psi$.

(10) ψ is not an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S. Since $\frac{0.5}{c}[q]\psi$ but $\frac{\max\{0.5,0.5\}}{(cc)} = \frac{0.5}{e}\left(\overline{[\in], [q]}\right)\psi$.

(11) ψ is not an $([\in] \vee [q], [\in] \vee [q])$-fuzzy interior ideal of S. Since $\frac{0.5}{c}[\in]\psi$ and $\frac{0.7}{e}[\in]$ but $\frac{\max\{0.5,0.7\}}{(ce)} = \frac{0.7}{d}\left(\overline{[\in], [q]}\right)\psi$. □

Remark. Every anti fuzzy interior ideal of S is an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S. However, the converse is not true, in general.

Example: Consider the LA-semigroup given in Example 4 and define a fuzzy subset $\psi : S \to [0, 1]$ by $\psi(a) = 0.1, \psi(c) = 0.2, \psi(d) = 0.4, \psi(e) = 0.6, \psi(b) = 0.8$. Clearly ψ is an $([\in], [\in] \vee [q])$ -fuzzy interior ideal of S. But ψ is not an $([\alpha], [\beta])$-fuzzy interior ideal of S as shown in Example 4. □

Theorem 2. *Every $([\in], [\in])$-fuzzy interior ideal of S is an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S.*

Proof: It is straightforward. □

Theorem 3. *Every $([\in] \vee [q], [\in] \vee [q])$ -fuzzy interior ideal of S is $([\in], [\in] \vee [q])$-fuzzy interior ideal of S.*

Proof: Let ψ be an $([\in] \vee [q], [\in] \vee [q])$-fuzzy interior ideal of S. Let $x, y \in S$ and $t_1, t_2 \in [0, 1)$ be such that $\frac{t_1}{x}$, $\frac{t_2}{y}[\in]\psi$. Then $\frac{t_1}{x}$, $\frac{t_2}{y}[\in] \quad \vee [q]\psi$ which implies that $\frac{\max\{t_1,t_2\}}{xy}[\in] \vee [q]\psi$. Let $x, y, a \in S$ and $t \in [0, 1)$ be such that $\frac{t}{a}[\in]\psi$. Then $\frac{t}{a}[\in] \vee [q]\psi$ and we have $\frac{t}{((xa)y)}[\in] \vee [q]\psi$. □

Theorem 4. *Let ψ be a non-zero $([\alpha], [\beta])$-fuzzy interior ideal of S. Then the set $\psi_1 = \{x \in S | \psi(x) < 1\}$ is an interior ideal of S.*

Proof: Let $x, y \in \psi_1$. Then $\psi(x) < 1$ and $\psi(y) < 1$. Let $\psi(xy) = 1$.If $\alpha \in \{[\in], [\in] \vee [q]\}$, then $\frac{\psi(x)}{x}[\alpha]\psi$, $\frac{\psi(y)}{y}[\alpha]\psi$. But $\psi(xy) = 1 > \max\{\psi(x), \psi(y)\}$ and $\psi(xy) + \max\{\psi(x), \psi(y)\} > 1$. So $\frac{\max\{\psi(x),\psi(y)\}}{xy}\overline{[\beta]}\psi$, for every $\beta \in \{[\in], [q], [\in] \vee [q], [\in] \wedge [q]\}$ which is contradiction. Hence $\psi(xy) < 1$. So $xy \in \psi_1$. Let $a \in \psi_1$ and $x, y \in S$. Then $\psi(a) < 1$. Assume that $\psi((xa)y) = 0$. If $\alpha \in \{[\in], [\in] \vee [q]\}$ then, $\frac{\psi(a)}{a}[\alpha]\psi$ but $\frac{\psi(a)}{((xa)y)}\overline{[\beta]}\psi$ for every $\beta \in \{[\in], [q], [\in] \vee [q], [\in] \wedge [q]\}$,a contradiction. Note that $\frac{1}{a}[q]\psi$ but $\frac{\max\{1,1\}}{((xa)y)} = \frac{1}{((xa)y)}\overline{[\beta]}\psi$ for every $\beta \in \{[\in], [q], [\in] \vee [q], [\in] \wedge [q]\}$,a contradiction. Hence $\psi((xa)y) < 1$,that is $(xa)y \in \psi_1$. Consequently ψ_1 is an interior ideal of S. □

Theorem 5. *Let I be an interior ideal and ψ a fuzzy subset of S such that*

(1) $(\forall x \in S \backslash I)(\psi(x) = 1)$.
(2) $(\forall x \in I)(\psi(x) \leq 0.5)$.
Then
(a) ψ is a $([q],[\in] \vee [q])$ -fuzzy interior ideal of S.
(b) ψ is an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S.

Proof: (a) Let $x,y \in S$ and $t_1,t_2 \in [0,1)$ be such that $\frac{t_1}{x}[q]\psi$ and $\frac{t_2}{y}[q]\psi$. Then $x,y \in I$ and we have $xy \in I$. If $\max\{t_1,t_2\} \geq 0.5$, then $\psi(xy) \leq 0.5 \leq \max\{t_1,t_2\}$ and hence $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$. If $\max\{t_1,t_2\} < 0.5$, then $\psi(xy) + \max\{t_1,t_2\} < 0.5 + 0.5 = 1$ and so $\frac{\max\{t_1,t_2\}}{xy}[q]\psi$. Therefore $\frac{\max\{t_1,t_2\}}{xy}[\in] \vee [q]\psi$. Let $x,y,a \in S$ and $t \in [0,1)$ be such that $\frac{t}{a}[q]\psi$. Then $a \in I$ and we have $(xa)y \in (SI)S \subseteq I$. If $t \geq 0.5$, then $\psi((xa)y) \leq 0.5 \leq t$ and hence $\frac{t}{((xa)y)}[\in]\psi$. If $t < 0.5$, then $\psi((xa)y) + t < 0.5 + 0.5 = 1$ and so $\frac{t}{((xa)y)}[q]\psi$. Therefore $\frac{t}{((xa)y)}[\in] \vee [q]\psi$. Therefore ψ is a $([q],[\in] \vee [q])$ -fuzzy interior ideal of S.

(b) Let $x,y \in S$ and $t_1,t_2 \in [0,1)$ be such that $\frac{t_1}{x}[\in]\psi$ and $\frac{t_2}{y}[\in]\psi$. Then $x,y \in I$ and we have $xy \in I$. If $\max\{t_1,t_2\} \geq 0.5$, then $\psi(xy) \leq 0.5 \leq \max\{t_1,t_2\}$ and hence $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$. If $\max\{t_1,t_2\} < 0.5$, then $\psi(xy) + \max\{t_1,t_2\} < 0.5 + 0.5 = 1$ and so $\frac{\max\{t_1,t_2\}}{xy}[q]\psi$. Therefore $\frac{\max\{t_1,t_2\}}{xy}[\in] \vee [q]\psi$. Now let $x,y,a \in S$ and $t \in [0,1)$ be such that $\frac{t}{a}[\in]\psi$ .Then $a \in I$ and we have $((xa)y) \in I$. If $t \geq 0.5$, then $\psi((xa)y) \leq 0.5 \leq t$ and hence $\frac{t}{((xa)y)}[\in]\psi$. If $t < 0.5$, then $\psi((xa)y) + t < 0.5 + 0.5 = 1$, and so $\frac{t}{((xa)y)}[q]\psi$. Therefore $\frac{t}{((xa)y)}[\in] \vee [q]\psi$ and so ψ is an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S. □

Example: We see from Example 4 an $([\in],[\in] \vee [q])$-fuzzy interior ideal is not an $([q],[\in] \vee [q])$-fuzzy interior ideal. □

In the following theorem we give a condition for an $([q],[\in] \vee [q])$-fuzzy interior ideal to be an $([\in],[\in])$-fuzzy interior ideal of S.

Theorem 6. *Let ψ be an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S such that $\psi(x) \geq 0.5$ for all $x \in S$. Then ψ is an $([\in],[\in])$-fuzzy interior ideal of S.*

Proof: Let $x,y \in S$ and $t_1,t_2 \in [0,1)$ be such that $\frac{t_1}{x},\frac{t_2}{y}[\in]\psi$. Then $\psi(x) \leq t_1$ and $\psi(y) \leq t_2$ and so $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{t_1,t_2,0.5\} = \max\{t_1,t_2\}$ and hence $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$. Now, let $x,y,a \in S$ and $t \in [0,1)$ be such that $\frac{t}{a}[\in]\psi$. Then $\psi(a) \leq t$ and we have $\psi((xa)y) \leq \psi(a) \leq t$. Consequently, $\frac{t}{((xa)y)}[\in]\psi$. Therefore ψ is an $([\in],[\in])$-fuzzy interior ideal of S. □

– For any fuzzy subset ψ of an LA-semigroup S and $t \in [0,1)$, we denote $Q(\psi;t) = \{x \in S | \frac{t}{x}[q]\psi\}$, $[\psi]_t = \{x \in S | \frac{t}{x}[\in] \vee [q]\psi\}$. Obviously $[\psi]_t = L(\psi;t) \cup Q(\psi;t)$. We call $[\psi]_t$ an $([\in] \vee [q])$-level interior ideal of ψ and $Q(\psi;t)$ an $[q]$-level interior ideal of ψ. We have given a characterization of $([\in],[\in] \vee [q])$-fuzzy interior ideals by using level subsets (see Proposition 1). Now we provide another characterization of $([\in],[\in] \vee [q])$-fuzzy interior ideals by using the set $[\psi]_t$.

Theorem 7. *Le ψ be a fuzzy subset of S. Then ψ is an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S if and only if $[\psi]_t$ is an interior ideal of S for all $t \in [0,1)$.*

Proof: Let ψ be an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S. Let $x,y \in [\psi]_t$ for $t \in [0,1)$.Then $\frac{t}{x}[\in] \vee [q]\psi$ and $\frac{t}{y}[\in] \vee [q]\psi$, that is, $\psi(x) \leq t$ or $\psi(x) + t < 1$,and $\psi(y) \leq t$ or $\psi(y) + t < 1$. Since ψ is an $([\in],[\in] \vee [q])$-fuzzy interior ideal of S, we have $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\}$. We discuss the following cases .

Case 1. Let $\psi(x) \leq t$ and $\psi(y) \leq t$. If $t \leq 0.5$ then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} = 0.5$ and we have $\frac{t}{xy}[q]\psi$. If $t > 0.5$ then we have $\psi(xy) \leq \max\{\psi(x),\psi(y),o.5\} = t$ then and so $\frac{t}{xy}[\in]\psi$. Hence $\frac{t}{xy}[\in] \vee [q]\psi$.

Case 2. Let $\psi(x) \leq t$ and $\psi(y) + t < 1$. If $t \leq 0.5$, then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{t,1-t,0.5\} = 1 - t$ and so $\psi(xy) < 1 - t \Longrightarrow \psi(x) + t < 1$ and so $\frac{t}{xy}[q]\psi$. If $t > 0.5$ then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{t,1-t,0.5\} = 0.5 \leq t$, so $\frac{t}{x}[\in]\psi$. Thus in both cases, we have $\frac{t}{x}[\in] \vee [q]\psi$.

Case 3. Let $\psi(x) + t < 1$and $\psi(y) \leq t$. If $t \geq 0.5$, then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{\psi(x),0.5\} = 1 - t$, so $\psi(xy) + t < 1$ and hence $\frac{t}{xy}[q]\psi$. If $t < 0.5$, then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{1-t,t,0.5\} = t$, and so $\psi(xy) \leq t \Rightarrow \frac{t}{xy}[\in]\psi$. Hence $\frac{t}{xy}[\in] \vee [q]\psi$.

Case 4 Let $\psi(x) + t < 1$and $\psi(y) + t < 1$. If $t \leq 0.5$, then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{1-t,0.5\} = 1 - t$, and so $\psi(xy) + t < 1$, hence $\frac{t}{xy}[q]\psi$. If $t > 0.5$ then $\psi(xy) \leq \max\{\psi(x),\psi(y),0.5\} \leq \max\{1-t,0.5\} = 0.5 \leq t$, hence $\frac{t}{xy}[\in]\psi$. Thus $\frac{t}{xy}[\in] \vee [q]\psi$. Therefore in any case we have $\frac{t}{xy}[\in] \vee [q]\psi$. Hence $xy \in [\psi_t]$.

Now let Let $x,y \in S$ and $t \in [0,1)$ be such that $a \in [\psi;t]$. Then $\frac{t}{a}[\in] \vee [q]\psi$. Hence $\psi(a) \leq t$ or $\psi(a) + t < 1$ and since ψ is an $([\in],[\in] \vee [q])-$fuzzy interior ideal of S, we have $\psi((xa)y) \leq \psi(a)$. Then we have the following cases:

Case 1 Let $\psi(a) < t$.If $t \leq 0.5$. Then $\psi((xa)y) \leq \psi(a) = 0.5$ and hence $\frac{t}{a}[q]\psi$. If $t > 0.5$ Then $\psi((xa)y) \leq \psi(a) \leq t$ and so $\frac{t}{a}[\in]\psi$. Hence $\frac{t}{a}[\in] \vee [q]\psi$.

Case 2 Let $\psi(a) \leq t$and $\psi(a) + t < 1$. If $t \leq 0.5$, then $\psi((xa)y) \leq \psi(a) = 1 - t$, so $\psi((xa)y) + t < 1$ and hence $\frac{t}{((xa)y)}[q]\psi$. If $t > 0.5$. Then $\psi((xa)y) \leq \psi(a) \leq t$ and hence $\frac{t}{((xa)y)}[\in]\psi$. Thus $\frac{t}{((xa)y)}[\in] \vee [q]\psi$. Hence in any case $((xa)y) \in [\psi]_t$. Conversely, let ψ be a fuzzy subset of S and let $x,y,a \in S$ be such that $\psi(xy) \geq t \geq \max\{\psi(x), \psi(y), 0.5\}$ for some $t \in [0, 0.5)$. Then $x,y \in L(\psi;t) \subseteq [\psi]_t$ it implies that $xy \in [\psi]_t$. Thus $\psi(xy) \leq t$ or $\psi\ (xy)+t < 1$, a contradiction. Hence $\psi(xy) \leq \max\{\psi(x), \psi(y), 0.5\}$ for all $x,y \in S$. Now let $\psi((xa)y) \geq \psi(a)$ for some $x,y,a \in S$. Choose t such that $\psi((xa)y) > t \geq \psi(a)$. Then $a \in L\ (\psi;t) \subseteq [\psi]_t$. It follows that $((xa)y) \in [\psi]_t$. This implies that $\psi((xa)y) \leq t$ or $\psi((xa)y)+t < 1$ which is a contradiction. Hence $\psi((xa)y) \leq \max\{\psi(a), 0.5\}$ for all $x,y,a \in S$. By Proposition 1, it follows that ψ is an $([\in], [\in] \vee [q])$-fuzzy interior ideal of S. □

–$L(\psi;t)$ and $[\psi]_t$ are interior ideals of S for all $t \in [0,1)$, but $Q[\psi;t]$ is not an interior ideal of S for all $t \in [0,1)$, in general as we show in the following Example.

Example: Consider the LA-semigroup S as given in Example 4. Define a fuzzy subset $\psi : S \to [0,1]$ by $\psi(a) = 0.1, \psi(c) = 0.2, \psi(d) = 0.4, \psi(e) = 0.6, \psi(b) = 0.8$. Then $Q(\psi;t) = \{a,c,d\}$ for all $0.4 \leq t \leq 0.5$. Since $\frac{0.5}{c}[\in]\psi$ and $\frac{0.7}{e}[\in]\psi$ but $\frac{\max\{0.5,0.7\}}{ce} = \frac{0.7}{d}\overline{[q]}\psi$. □

Proposition 2. *If $\{\psi_i\}_{i\in I}$ is a family of $([\in], [\in] \vee [q])$-fuzzy interior-ideals of S, then $\cap_{i\in I}\psi_i$ is an $([\in], [\in] \vee [q])$-fuzzy interior-ideal of S.*

Proof: Let $\{\psi_i\}_{i\in I}$ be a family of $([\in], [\in] \vee [q])$-fuzzy interior-ideals of S. Let $x,y \in S$. Then

$$\begin{aligned}(\cap_{i\in I}\psi_i)(xy) &= {}_{i\in I}\psi_i(xy) \\ &\leq {}^{\psi_i}_{i\in I}\{\psi_i(x), \psi_i(y)\} \\ &= \left({}^{\psi_i}_{i\in I}(x) \wedge {}^{\psi_i}_{i\in I}(y)\right) \\ &= (\cap_{i\in I}\psi_i)(x) \wedge \cap_{i\in I}\psi_i\end{aligned}$$

Let $x,y,a \in S$. Then

$$\begin{aligned}(\cap_{i\in I}\psi_i)((xa)y) &= {}_{i\in I}\psi_i((xa)y) \\ &\leq {}_{i\in I}(\psi_i(a)) \\ &= (\cap_{i\in I}\psi_i)(a)\end{aligned}$$

Thus $\cap_{i\in I}\psi_i$ is an $([\in], [\in] \vee [q])$ fuzzy interior ideal of S. □

Definition 8. *Let ψ be a fuzzy subset of S. Then ψ is called a strongly anti fuzzy interior ideal of S, if it satisfies the following conditions.*

$(B9)$ $(\forall x,y \in S)(\psi(xy) < \max\{\psi(x), \psi(y)\})$.
$(B10)$ $(\forall x,y,a \in S)(\psi((xa)y) < \psi(a))$.

Lemma 6. *Every anti fuzzy interior ideal of S is strongly anti fuzzy interior ideal of S.*

Theorem 8. *For any fuzzy subset ψ of S. The conditions$(B9)$ and $(B10)$ of Definition 8 are equivalent to the following.*

$(B11)$ $(\forall x,y \in S)(\forall t_1,t_2 \in [0,1))(\frac{t_1}{x}[\in]\psi, \frac{t_2}{y}[\in]\psi \to \frac{\max\{t_1,t_2\}}{xy}[\in]\psi)$.

$(B12)$ $(\forall x,y,a \in S)(\forall t \in [0,1))(\frac{t}{a}[\in]\psi \to \frac{t}{((xa)y)}[\in]\psi)$.

Proof: $(B9) \Rightarrow (B11)$ Let ψ be a fuzzy subset of S. Let $x,y \in S$ and $t_1,t_2 \in [0,1)$ be such that $\frac{t_1}{x}[\in]\psi$, $\frac{t_2}{y}[\in]\psi$.

Then $\psi(x) < t_1$ and $\psi(y) < t_2$. Using $(B9)$ we have $\psi(xy) < \max\{\psi(x), \psi(y)\} < \max\{t_1,t_2\}$, and so $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$.

$(B11) \Rightarrow (B9)$ Let $x,y \in S$. Since $\frac{\psi(x)}{x}[\in]\psi$ and $\frac{\psi(y)}{y}[\in]\psi$. Then by$(B9,)$ we have $\frac{\max\{t_1,t_2\}}{xy}[\in]\psi$ and so $\psi(xy) < max\{t_1,t_2\}$.

$(B10) \Rightarrow (B12)$. Let $x,y,a \in S$ and $t \in [0,1)$ be such that $\frac{t}{a}[\in]\psi$. Then $\psi(a) < t$. By $(B10)$ we have $\psi((xa)y) < \psi(a) < t$ and so $\frac{t}{((xa)y)}[\in]\psi$.

$(B12) \to (B10)$. Let $x,y \in S$. Since $\frac{\psi(a)}{a}[\in]\psi$, by $(B11)$, we have $\frac{\psi(a)}{((xa)y)}[\in]\psi$ and so $\psi((xa)y) < \psi(a)$. □

5 $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-Fuzzy Interior Ideals.

In this section we define the notions of $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-fuzzy interior ideals of an LA-semigroup and investigate some of their properties in terms of $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-fuzzy interior ideals.

Let ψ be a fuzzy subset of S and $\psi(x) \geq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in [0,1)$ be such that $\frac{t}{x}\underline{[\in]} \wedge \underline{[q]}\psi$. Then $\frac{t}{x}\underline{[\in]}\psi$ and $\frac{t}{x}\underline{[q]}\psi$ and so $\psi(x) < t$ and $\psi(x)+t < 1$. It follows that $1 > \psi(x)+t > \psi(x)+\psi(x) = 2\psi(x)$, and so $\psi(x) < 0.5$, which is a contradiction. This means that $\{x \in S | \frac{t}{x}\underline{[\in]} \wedge \underline{[q]}\psi\} = \emptyset$.

Definition 9. *A fuzzy subset ψ of S is called an $(\underline{[\alpha]}, \underline{[\beta]})$-fuzzy interior ideal of S, where $\alpha \neq \underline{[\in]} \wedge \underline{[q]}$, if it satisfy the following conditions.*

$(B13)$ $(\forall x,y \in S)(\forall t_1,t_2 \in [0,1))\left(\frac{t_1}{x}\underline{[\alpha]}\psi, \frac{t_2}{y}\underline{[\alpha]}\psi \to \frac{\max\{t_1,t_2\}}{xy}\underline{[\beta]}\psi\right)$.

$(B14)$ $(\forall x,y,\psi \in S)(\forall t \in [0,1))\left(\frac{t}{\psi}\underline{[\alpha]}\psi \to \frac{t}{((xa)y)}\underline{[\beta]}\psi\right)$.

Proposition 3. *Let ψ be a fuzzy subset of S. If $\alpha = \underline{[\in]}$ and $\beta = \underline{[\in]} \wedge \underline{[q]}$ in Definition 9. Then $(B13)$,and $(B14)$, respectively, of Definition 9, are equivalent to the following conditions.*

$(B15)$ $(\forall x,y \in S)(\psi(xy) \leq \max\{\psi(x), \psi(y), 0.5\})$.
$(B16)$ $(\forall x,y,a \in S)$ $(\psi((xa)y) \leq \max\{\psi(a), 0.5\})$.

*Remark.*A fuzzy subset ψ of an LA-semigroup S is an $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-fuzzy interior ideal of S if and only if it satisfies conditions $(B15)$ and $(B16)$ of the above proposition.

Using Proposition 3, we have the following characterization of $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-fuzzy interior ideals of an LA-semigroup S.

Lemma 7.*Let S be an LA-semigroup and $\emptyset \neq I \subseteq S$. Then I is an interior ideal of S if and only if the characteristic function X_I of I is an $\left(\underline{[\in]}, \underline{[\in]} \vee \underline{[q]}\right)$-fuzzy interior ideal of S.*

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