

Explicit Examples of Motions of Inextensible Curves in Spherical Space \mathbb{S}^3

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Abstract: In this paper we study the motion of curves in 3-dimensional spherical space \mathbb{S}^3 . We derive the evolution equations of the orthonormal frame and evolution equations for curvatures. Moreover, we give some explicit examples of motions of inextensible curves in \mathbb{S}^3 and we determine the curves from their intrinsic equations (curvature and torsion). Then we determine the surfaces that are generated by the motion of these curves. To visualize these surfaces in \mathbb{S}^3 , we use the stereographic projection in \mathbb{R}^4 .

Keywords: inextensible flows, motion of curves, evolution equations.

1 Introduction

The relationships between integrable systems and geometric motions of curves in spaces has been studied for a long time. Many integrable equations have been shown to describe the evolution invariants associated with certain movements of curves in particular geometric settings. The dynamics of shapes in physics, chemistry and biology are modeled in terms of motion of surfaces and interfaces, and some dynamics of shapes are reduced to motion of plane curves. These models are specified by velocity fields, which are local or nonlocal functionals of the intrinsic quantities of curves. In physics, it is very interesting to describe motions of patterns such as interfaces, wave fronts and defects [23]. Applications include deformations of thin vortex filaments in inviscid fluids [8, 11, 24], kinematics of interfaces in crystal growth [1, 16] and viscous fingering in a HeleShaw cell [25]. The evolution of curves and surfaces has significant applications in computer vision and image processing [26].

In [11], Hasimoto discovered the remarkable fact that the binormal motion of a nonstretching vortex filament with speed equal to its curvature produces the cubic nonlinear Schrödinger equation via the so-called Hasimoto transformation. Considering other binormal motions, Lamb [15] obtained the sine-Gordon and mKdV equations by using the Hasimoto transformation. Lakshmanan [14] interpreted the dynamics of a nonlinear

string of fixed length in \mathbb{R}^3 through consideration of the motion of an arbitrary rigid body along it. Langer and Perline [17] showed that the dynamics of a non-stretching vortex filament in \mathbb{R}^3 gives the NLS hierarchy.

Recently, Schief and Rogers [27] obtained an extended Harry Dym equation and the classical sine-Gordon equation from binormal motions of curves of constant curvature or torsion.

Goldstein and Petrich [9] showed that the celebrated mKdV equation naturally arises from inextensible motion of curves in Euclidean geometry. Nakayama, Segur and Wadati [20] set up a correspondence between the mKdV hierarchy and inextensible motions of plane curves in Euclidean geometry. Later, it was also shown that the KdV hierarchy [2, 4], Burgers hierarchy [5, 6] and Sawada-Kotera hierarchy [3], naturally arise from inextensible motions of plane curves in centro-affine geometry, similarity geometry and affine geometry, respectively. They can be regarded as a centro-affine version, similarity version and affine version of the mKdV hierarchy, respectively. Motions of curves on S^2 and S^3 were considered by Doliwa and Santini [7]. Langer et al [18] obtained the system of mKdV equations from inextensible motions of space curves in Euclidean geometries. Nakayama [21, 22] showed that the defocusing nonlinear Schrödinger equation and a coupled system of KdV equations and their hyperbolic type arise from motions of curves in hyperboloids in 4-dimensional Minkowski space $\mathbb{R}^{3,1}$.

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Recently, the study of motion of inextensible curves has arisen in a number of diverse engineering applications: Kwon et al [12, 13] studied the inextensible flows of curves and developable surfaces in \mathbb{R}^3 . Moreover, Latifi et al [19] studied inextensible flows of curves in Minkowski 3-space.

Gopal and Lakshmanan [10] considered the dynamics of moving curves in three-dimensional Minkowski space $\mathbb{R}^{2,1}$.

This paper is outlined as follows: In section 2, we give an introduction to curves in spherical space \mathbb{S}^3 . In sections 3 and 4, we study the motion of curves in \mathbb{S}^3 by computing the evolution equations for the Serret-Frenet frame and the evolution equations of curvature and torsion of these curves. In section 5, we study the motion of inextensible curves in \mathbb{S}^3 .

Finally, in section 6, we give some explicit examples of the motion of inextensible curves in \mathbb{S}^3 and we get the curvature and torsion of these curves. Then we determine and draw the surfaces that are generated by the motion of these curves.

2 Preliminaries

Definition 1. Spherical 3-space \mathbb{S}^3 is the unique simply-connected 3-dimensional complete Riemannian manifold with constant sectional curvature +1. \mathbb{S}^3 is defined by

$$\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{j=1}^4 x_j^2 = 1\}.$$

Definition 2. Consider the 3-dimensional spherical space \mathbb{S}^3 in \mathbb{R}^4 . Let $\tilde{\gamma}: I \rightarrow \mathbb{S}^3$ be a regular parametrized curve in \mathbb{R}^4 . The arc-length of the curve $\tilde{\gamma}$ with arbitrary parameter $\tilde{u} \in I$ is defined as

$$\tilde{s}(\tilde{u}) = \int_0^{\tilde{u}} \|\dot{\tilde{\gamma}}(\tilde{\sigma})\| d\tilde{\sigma}.$$

Since $\tilde{\gamma}$ is regular, then we define $\tilde{g} > 0$ by $\frac{d\tilde{s}}{d\tilde{u}} = \|\dot{\tilde{\gamma}}\| = \sqrt{\tilde{g}}$.

Definition 3. If $\|\dot{\tilde{\gamma}}\| = 1$ for all $\tilde{u} \in I$, then $\tilde{\gamma} = \tilde{\gamma}(\tilde{s})$ is said to be an arc-length parametrized or unit speed parametrized curve.

Assume that the curve $\tilde{\gamma}$ is parametrized by arc-length and assume that $\langle \tilde{\gamma}''(\tilde{s}), \tilde{\gamma}''(\tilde{s}) \rangle \neq 1$, where $' = \frac{d}{d\tilde{s}}$.

Let $\{\tilde{\gamma}, \tilde{T}, \tilde{N}, \tilde{B}\}$ be the Serret Frenet frame of the curve $\tilde{\gamma}$, where $\tilde{\gamma}(\tilde{s})$ is the position vector of the curve $\tilde{\gamma}$ and \tilde{T}, \tilde{N} and \tilde{B} are respectively, unit tangent, unit principal normal and unit binormal vector field to the curve $\tilde{\gamma}(\tilde{s})$.

Definition 4. The Frenet frame in \mathbb{S}^3 is defined by

- $\langle \tilde{\gamma}, \tilde{\gamma} \rangle = 1$, since the curve is in \mathbb{S}^3 .
- $\langle \tilde{T}, \tilde{T} \rangle = \langle \tilde{N}, \tilde{N} \rangle = \langle \tilde{B}, \tilde{B} \rangle = 1$.
- \tilde{B} is chosen so that $\{\tilde{\gamma}, \tilde{T}, \tilde{N}, \tilde{B}\}$ is an oriented orthonormal basis of \mathbb{R}^4 , so $\tilde{B} = \tilde{\gamma} \times \tilde{T} \times \tilde{N}$.

Lemma 1. The inner product and the vector product are given by:

- $\langle \tilde{\gamma}, \tilde{T} \rangle = \langle \tilde{\gamma}, \tilde{N} \rangle = \langle \tilde{\gamma}, \tilde{B} \rangle = \langle \tilde{T}, \tilde{N} \rangle = \langle \tilde{T}, \tilde{B} \rangle = \langle \tilde{N}, \tilde{B} \rangle = 0$.
- $\tilde{T} \times \tilde{B} \times \tilde{N} = \tilde{\gamma}$, $\tilde{\gamma} \times \tilde{N} \times \tilde{B} = \tilde{T}$, $\tilde{\gamma} \times \tilde{B} \times \tilde{T} = \tilde{N}$.

Definition 5. The curvature $\tilde{k}(\tilde{s})$ of the curve $\tilde{\gamma}(\tilde{s})$ is defined by:

$$\tilde{k}(\tilde{s}) = \sqrt{|\langle \tilde{T}' + \tilde{\gamma}, \tilde{T}' + \tilde{\gamma} \rangle|} = \|\tilde{T}' + \tilde{\gamma}\|.$$

Definition 6. The unit normal vector to the curve $\tilde{\gamma}(\tilde{s})$ is defined by:

$$\tilde{N} = \frac{\tilde{T}' + \tilde{\gamma}}{\tilde{k}}.$$

Lemma 2. The curvature $\tilde{k}(\tilde{s})$ of the curve $\tilde{\gamma}(\tilde{s})$ satisfies the following

$$\tilde{k}(\tilde{s}) = \langle \tilde{T}' + \tilde{\gamma}, \tilde{N} \rangle. \quad (1)$$

Definition 7. The torsion $\tilde{\tau}(\tilde{s})$ of the curve $\tilde{\gamma}(\tilde{s})$ is defined by:

$$\tilde{\tau}(\tilde{s}) = \langle \tilde{N}', \tilde{B} \rangle. \quad (2)$$

Lemma 3. The torsion $\tilde{\tau}(\tilde{s})$ of the curve $\tilde{\gamma}(\tilde{s})$ satisfies the following

$$\tilde{\tau}(\tilde{s}) = \frac{-1}{\tilde{k}^2} \det(\tilde{\gamma}, \tilde{\gamma}', \tilde{\gamma}'', \tilde{\gamma}'''). \quad (3)$$

Lemma 4. The Frenet Frame for the curve in \mathbb{S}^3 satisfies the following:

$$\tilde{F}_{\tilde{s}} = \tilde{M} \cdot \tilde{F}, \quad (4)$$

$$\text{where } \tilde{F} = \begin{pmatrix} \tilde{\gamma} \\ \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{pmatrix} \text{ and } \tilde{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \tilde{k} & 0 \\ 0 & -\tilde{k} & 0 & \tilde{\tau} \\ 0 & 0 & -\tilde{\tau} & 0 \end{pmatrix}.$$

Proof. The vectors $\{\tilde{\gamma}', \tilde{T}', \tilde{N}', \tilde{B}'\}$ can be uniquely decomposed as follows:

$$\begin{aligned} \tilde{\gamma}' &= \tilde{T}, \\ \tilde{T}' &= \tilde{a}_{11}\tilde{T} + \tilde{a}_{12}\tilde{N} + \tilde{a}_{13}\tilde{B} + \tilde{a}_{14}\tilde{\gamma}, \\ \tilde{N}' &= \tilde{a}_{21}\tilde{T} + \tilde{a}_{22}\tilde{N} + \tilde{a}_{23}\tilde{B} + \tilde{a}_{24}\tilde{\gamma}, \\ \tilde{B}' &= \tilde{a}_{31}\tilde{T} + \tilde{a}_{32}\tilde{N} + \tilde{a}_{33}\tilde{B} + \tilde{a}_{34}\tilde{\gamma}. \end{aligned} \quad (5)$$

We will compute the coefficients \tilde{a}_{ij} , where $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

Since $\langle \tilde{T}, \tilde{T} \rangle = 1$, then $\langle \tilde{T}', \tilde{T} \rangle = 0$. By the inner product of the second equation of (5) with \tilde{T} , we have

$$\langle \tilde{T}', \tilde{T} \rangle = \tilde{a}_{11},$$

then $\tilde{a}_{11} = 0$. Similarly, $\tilde{a}_{22} = \tilde{a}_{33} = 0$.

Using the first property of Lemma (1), then we have

$$\tilde{a}_{14} = -1, \quad \tilde{a}_{24} = 0, \quad \tilde{a}_{34} = 0, \quad \tilde{a}_{21} = -\tilde{a}_{12},$$

$$\tilde{a}_{31} = -\tilde{a}_{13}, \quad \tilde{a}_{32} = -\tilde{a}_{23}.$$

Since $\tilde{B}' \perp \tilde{B}$, and also $\tilde{B}' \perp \tilde{T}$, hence $\tilde{a}_{13} = 0$. Using (1), then we have: $\tilde{a}_{12} = \tilde{k}$. Using (2), so $\tilde{a}_{23} = \tilde{\tau}$. Hence, the lemma holds.

Lemma 5. Consider the curve $\tilde{\gamma}(\tilde{s}(\tilde{u}))$ with arbitrary parameter $u \in I$. Then the Serret-Frenet frame satisfies

$$\tilde{F}_{\tilde{u}} = \sqrt{\tilde{g}} \tilde{M} \cdot \tilde{F}, \tag{6}$$

where \tilde{M} and \tilde{F} are given as in (4).

3 Motions of curves in \mathbb{S}^3

Let $\tilde{\gamma}_0 : I \rightarrow \mathbb{S}^3$ be a regular curve in \mathbb{S}^3 . Consider the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t) : I \times [0, \infty) \rightarrow \mathbb{S}^3$, with initial curve $\tilde{\gamma}_0 = \tilde{\gamma}(\tilde{s}, 0)$. The position vector of a point on the curve at time t and an arc-length parameter \tilde{s} is denoted by $\tilde{\gamma}(\tilde{s}, t)$. The time parameter t is the parameter for the deformation $\tilde{\gamma}(\tilde{s}, t)$ of the curve.

The arc-length of the curve $\tilde{\gamma}(\tilde{s}, t)$ is defined as

$$\tilde{s}(\tilde{u}, t) = \int_0^{\tilde{u}} \sqrt{\tilde{g}(\tilde{\sigma}, t)} d\tilde{\sigma},$$

where $\sqrt{\tilde{g}} = \|\dot{\tilde{\gamma}}(\tilde{\sigma}, t)\|$. Thus the element of arc-length is $d\tilde{s} = \sqrt{\tilde{g}(\tilde{u}, t)} d\tilde{u}$, and

$$\frac{\partial}{\partial \tilde{s}} = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{u}}, \quad \frac{\partial \tilde{s}}{\partial \tilde{u}} = \sqrt{\tilde{g}}.$$

Consider the curve flow (the time evolution of the curve) specified by the velocity field

$$\frac{\partial \tilde{\gamma}}{\partial t} = \tilde{W} \tilde{T} + \tilde{U} \tilde{N} + \tilde{V} \tilde{B}, \tag{7}$$

where $\{\tilde{\gamma}, \tilde{T}, \tilde{N}, \tilde{B}\}$ is the orthonormal Frenet frame to the curve $\tilde{\gamma}(\tilde{s}, t)$, and \tilde{W}, \tilde{U} and \tilde{V} are the velocity vectors in the direction of \tilde{T}, \tilde{N} and \tilde{B} respectively, and they are functions of the curvature \tilde{k} and torsion $\tilde{\tau}$ of the curve, and they are also functions of the derivatives of \tilde{k} and $\tilde{\tau}$ (including higher order derivatives).

Remark

- The derivatives with respect to \tilde{u} and t commute,

$$\frac{\partial}{\partial \tilde{u}} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial \tilde{u}}.$$

- The derivatives with respect to \tilde{s}, t in general do not commute,

$$\frac{\partial}{\partial \tilde{s}} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial \tilde{s}} + \frac{\tilde{g}_t}{2\tilde{g}} \frac{\partial}{\partial \tilde{s}}. \tag{8}$$

4 Main results

Theorem 1. The time evolution of the Serret-Frenet frame can be written in matrix form as follows:

$$\tilde{F}_t = \tilde{Q} \cdot \tilde{F}, \tag{9}$$

where

$$\tilde{F} = \begin{pmatrix} \tilde{\gamma} \\ \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & \tilde{W} & \tilde{U} & \tilde{V} \\ -\tilde{W} & 0 & \tilde{f}_1 & \tilde{f}_2 \\ -\tilde{U} & -\tilde{f}_1 & 0 & \tilde{\xi} \\ -\tilde{V} & -\tilde{f}_2 & -\tilde{\xi} & 0 \end{pmatrix},$$

and $\tilde{f}_1 = \tilde{k} \tilde{W} + \tilde{U}_{\tilde{s}} - \tilde{\tau} \tilde{V}$, $\tilde{f}_2 = \tilde{V}_{\tilde{s}} + \tilde{\tau} \tilde{U}$,

$$\tilde{\xi} = \frac{1}{\tilde{k}} \left(\tilde{V} + \frac{\partial \tilde{f}_2}{\partial \tilde{s}} + \tilde{f}_1 \tilde{\tau} \right).$$

The time evolution of the curvature and torsion of the curve $\tilde{\gamma}(\tilde{s}, t)$ can be given by

$$\begin{pmatrix} \tilde{k} \\ \tilde{\tau} \end{pmatrix}_t = \begin{pmatrix} -\frac{\tilde{g}_t}{2\tilde{g}} & -\tilde{f}_2 \\ \tilde{f}_2 & -\frac{\tilde{g}_t}{2\tilde{g}} \end{pmatrix} \begin{pmatrix} \tilde{k} \\ \tilde{\tau} \end{pmatrix} + \begin{pmatrix} \tilde{U} + \tilde{f}_1 \tilde{s} \\ \tilde{\xi}_{\tilde{s}} \end{pmatrix}. \tag{10}$$

Proof. Take the \tilde{u} derivative of (7), then

$$\begin{aligned} \tilde{\gamma}_{\tilde{u}} &= \sqrt{\tilde{g}} \left((-\tilde{W}) \tilde{\gamma} + (\tilde{W}_{\tilde{s}} - \tilde{k} \tilde{U}) \tilde{T} + (\tilde{\tau} \tilde{U} + \tilde{V}_{\tilde{s}}) \tilde{B} \right. \\ &\quad \left. + (\tilde{k} \tilde{W} + \tilde{U}_{\tilde{s}} - \tilde{\tau} \tilde{V}) \tilde{N} \right). \end{aligned}$$

Put $\tilde{k} \tilde{W} + \tilde{U}_{\tilde{s}} - \tilde{\tau} \tilde{V} = \tilde{f}_1$ and $\tilde{V}_{\tilde{s}} + \tilde{\tau} \tilde{U} = \tilde{f}_2$, so

$$\tilde{\gamma}_{\tilde{u}} = \sqrt{\tilde{g}} \left(-\tilde{W} \tilde{\gamma} + (\tilde{W}_{\tilde{s}} - \tilde{k} \tilde{U}) \tilde{T} + \tilde{f}_1 \tilde{N} + \tilde{f}_2 \tilde{B} \right). \tag{11}$$

Since $\tilde{\gamma}_{\tilde{u}} = \sqrt{\tilde{g}} \tilde{\gamma}_{\tilde{s}} = \sqrt{\tilde{g}} \tilde{T}$, then

$$\tilde{\gamma}_{\tilde{u}} = \sqrt{\tilde{g}} \left(\tilde{T}_t + \frac{\tilde{g}_t}{2\tilde{g}} \tilde{T} \right). \tag{12}$$

Since the derivatives with respect to \tilde{u} and t commute, then

$$\tilde{\gamma}_{\tilde{u}} = \tilde{\gamma}_{\tilde{u}}. \tag{13}$$

Substitute from (11) and (12) into (13), then

$$\begin{aligned} \frac{\partial \tilde{g}}{\partial t} &= 2\tilde{g}(\tilde{W}_{\tilde{s}} - \tilde{k} \tilde{U}), \\ \frac{\partial \tilde{T}}{\partial t} &= -\tilde{W} \tilde{\gamma} + \tilde{f}_1 \tilde{N} + \tilde{f}_2 \tilde{B}. \end{aligned} \tag{14}$$

To compute the time evolution equations for the curvature \tilde{k} and the unit normal vector \tilde{N} , we take the \tilde{u} derivative of the second equation of (14), then

$$\begin{aligned} \tilde{T}_{\tilde{u}} &= \sqrt{\tilde{g}} \left((-\tilde{W}_{\tilde{s}}) \tilde{\gamma} - (\tilde{W} + \tilde{k} \tilde{f}_1) \tilde{T} + \left(\frac{\partial \tilde{f}_1}{\partial \tilde{s}} - \tilde{f}_2 \tilde{\tau} \right) \tilde{N} \right. \\ &\quad \left. + \left(\frac{\partial \tilde{f}_2}{\partial \tilde{s}} + \tilde{f}_1 \tilde{\tau} \right) \tilde{B} \right). \end{aligned} \tag{15}$$

Since

$$\tilde{T}_{\tilde{u}} = \sqrt{\tilde{g}} \tilde{T}_{\tilde{s}} = \sqrt{\tilde{g}} (-\tilde{\gamma} + \tilde{k} \tilde{N}).$$

Taking the t derivative of this equation, then we have

$$\tilde{T}_{\tilde{u}} = \sqrt{\tilde{g}} \left(\tilde{k} \tilde{N}_t - \frac{\tilde{g}_t}{2\tilde{g}} \tilde{\gamma} - \tilde{W} \tilde{T} + \left(\tilde{k}_t + \tilde{k} \frac{\tilde{g}_t}{2\tilde{g}} - \tilde{U} \right) \tilde{N} - \tilde{V} \tilde{B} \right). \tag{16}$$

Since

$$\tilde{T}_{i\tilde{u}} = \tilde{T}_{i\tilde{u}}. \quad (17)$$

Substitute from (15) and (16) in (17), and put

$$\frac{1}{k} \left(\tilde{V} + \frac{\partial \tilde{f}_2}{\partial \tilde{s}} + \tilde{f}_1 \tilde{\tau} \right) = \tilde{\xi},$$

then

$$\begin{aligned} \frac{\partial \tilde{k}}{\partial t} &= \tilde{U} + \frac{\partial \tilde{f}_1}{\partial \tilde{s}} - \tilde{\tau} \tilde{f}_2 - \frac{\tilde{g}_t}{2\tilde{g}} \tilde{k}, \\ \frac{\partial \tilde{N}}{\partial t} &= -\tilde{U} \tilde{\gamma} - \tilde{f}_1 \tilde{T} + \tilde{\xi} \tilde{B}. \end{aligned} \quad (18)$$

The time evolution equation for the unit binormal vector \tilde{B} to the curve $\tilde{\gamma}(\tilde{s}, t)$ is given as follows:

Since $\tilde{B} = \tilde{\gamma} \times \tilde{T} \times \tilde{N}$, so

$$\tilde{B}_t = \tilde{\gamma}_t \times \tilde{T} \times \tilde{N} + \tilde{\gamma} \times \tilde{T}_t \times \tilde{N} + \tilde{\gamma} \times \tilde{T} \times \tilde{N}_t. \quad (19)$$

Substitute from (7) and the second equation of both (14) and (18) into (19), then

$$\frac{\partial \tilde{B}}{\partial t} = -\tilde{V} \tilde{\gamma} - \tilde{f}_2 \tilde{T} - \tilde{\xi} \tilde{N}. \quad (20)$$

Take the \tilde{u} derivative of (20), then

$$\begin{aligned} \tilde{B}_{i\tilde{u}} &= -\sqrt{\tilde{g}} \left(-(\tilde{\tau} \tilde{U}) \tilde{\gamma} + (\tilde{V} + \frac{\partial \tilde{f}_2}{\partial \tilde{s}} - \tilde{k} \tilde{\xi}) \tilde{T} \right. \\ &\quad \left. + (\tilde{k} \tilde{f}_2 + \tilde{\xi} \tilde{s}) \tilde{N} + (\tilde{\xi} \tilde{\tau}) \tilde{B} \right). \end{aligned} \quad (21)$$

Since

$$\tilde{B}_{i\tilde{u}} = \sqrt{\tilde{g}} \tilde{B}_{\tilde{s}} = -\sqrt{\tilde{g}} \tilde{\tau} \tilde{N}.$$

Taking the t derivative of this equation, then we have

$$\tilde{B}_{i\tilde{u}} = -\sqrt{\tilde{g}} \left(\left(\frac{\tilde{g}_t}{2\tilde{g}} \tilde{\tau} + \tilde{\tau}_t \right) \tilde{N} + \tilde{\tau} \tilde{N}_t \right). \quad (22)$$

Since $\tilde{B}_{i\tilde{u}} = \tilde{B}_{i\tilde{u}}$, using (21) and (22), then the time evolution equation for the torsion $\tilde{\tau}$ of the curve $\tilde{\gamma}(\tilde{s}, t)$ is

$$\frac{\partial \tilde{\tau}}{\partial t} = \tilde{k} \tilde{f}_2 + \tilde{\xi} \tilde{s} - \frac{\tilde{g}_t}{2\tilde{g}} \tilde{\tau}. \quad (23)$$

Theorem 2. Consider the Serret-Frenet matrix \tilde{F} that satisfies (6) and (9). Then we have the integrability condition:

$$\tilde{M}_t - \tilde{Q}_{\tilde{s}} + [\tilde{M}, \tilde{Q}] = \frac{-\tilde{g}_t}{2\tilde{g}} \tilde{M}, \quad (24)$$

where $[\tilde{M}, \tilde{Q}] = \tilde{M} \cdot \tilde{Q} - \tilde{Q} \cdot \tilde{M}$ is the Lie bracket.

Proof. Since

$$\tilde{F}_{i\tilde{u}} = \tilde{F}_{i\tilde{u}}. \quad (25)$$

Differentiating (6) with respect to t , then

$$\tilde{F}_{i\tilde{u}} = \sqrt{\tilde{g}} \left(\tilde{M}_t + \tilde{M} \cdot \tilde{Q} + \frac{\tilde{g}_t}{2\tilde{g}} \tilde{M} \right) \cdot \tilde{F}. \quad (26)$$

Differentiating (9) with respect to u , then

$$\tilde{F}_{i\tilde{u}} = \sqrt{\tilde{g}} \left(\tilde{Q}_{\tilde{s}} + \tilde{Q} \cdot \tilde{M} \right) \cdot \tilde{F}. \quad (27)$$

Substitute from (26) and (27) into (25), then the theorem holds.

Lemma 6. If the integrability condition (24) is satisfied, then we have the PDE system (10).

Proof. Since

$$\tilde{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \tilde{k} & 0 \\ 0 & -\tilde{k} & 0 & \tilde{\tau} \\ 0 & 0 & -\tilde{\tau} & 0 \end{pmatrix}, \tilde{Q} = \begin{pmatrix} 0 & \tilde{W} & \tilde{U} & \tilde{V} \\ -\tilde{W} & 0 & \tilde{f}_1 & \tilde{f}_2 \\ -\tilde{U} & -\tilde{f}_1 & 0 & \tilde{\xi} \\ -\tilde{V} & -\tilde{f}_2 & -\tilde{\xi} & 0 \end{pmatrix}. \quad (28)$$

Then

$$\tilde{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{k}_t & 0 \\ 0 & -\tilde{k}_t & 0 & \tilde{\tau}_t \\ 0 & 0 & -\tilde{\tau}_t & 0 \end{pmatrix}, \tilde{Q}_{\tilde{s}} = \begin{pmatrix} 0 & \tilde{W}_{\tilde{s}} & \tilde{U}_{\tilde{s}} & \tilde{V}_{\tilde{s}} \\ -\tilde{W}_{\tilde{s}} & 0 & \tilde{f}_{1\tilde{s}} & \tilde{f}_{2\tilde{s}} \\ -\tilde{U}_{\tilde{s}} & -\tilde{f}_{1\tilde{s}} & 0 & \tilde{\xi}_{\tilde{s}} \\ -\tilde{V}_{\tilde{s}} & -\tilde{f}_{2\tilde{s}} & -\tilde{\xi}_{\tilde{s}} & 0 \end{pmatrix}. \quad (29)$$

The Lie bracket $[\tilde{M}, \tilde{Q}]$ is given by:

$$[\tilde{M}, \tilde{Q}] = \begin{pmatrix} 0 & \tilde{b}_{12} & \tilde{b}_{13} & \tilde{b}_{14} \\ -\tilde{b}_{12} & 0 & \tilde{b}_{23} & \tilde{b}_{24} \\ -\tilde{b}_{13} & -\tilde{b}_{23} & 0 & \tilde{b}_{34} \\ -\tilde{b}_{14} & -\tilde{b}_{24} & -\tilde{b}_{34} & 0 \end{pmatrix}, \quad (30)$$

where

$$\tilde{b}_{12} = \tilde{k} \tilde{U}, \quad \tilde{b}_{13} = \tilde{f}_1 - \tilde{k} \tilde{w} + \tilde{\tau} \tilde{V}, \quad \tilde{b}_{14} = \tilde{f}_2 - \tilde{\tau} \tilde{U},$$

$$\tilde{b}_{23} = -\tilde{U} + \tilde{\tau} \tilde{f}_2, \quad \tilde{b}_{24} = \tilde{k} \tilde{\xi} - \tilde{\tau} \tilde{f}_1 - \tilde{V}, \quad \tilde{b}_{34} = -\tilde{k} \tilde{f}_2.$$

Substitute from (28), (29) and (30) into (24), then we get the PDE system (10).

5 Motions of inextensible curves in \mathbb{S}^3

Definition 8. The curve $\tilde{\gamma}(\tilde{s}, t)$ is said to be inextensible if $\frac{\partial}{\partial t} \|\tilde{\gamma}(\tilde{s}(\tilde{u}, t))\| = 0$, i.e. $\tilde{g}_t = 0$.

Lemma 7. If the curve $\tilde{\gamma}(\tilde{s}, t)$ is inextensible, then the arclength of the curve $\tilde{\gamma}(\tilde{s}, t)$ is preserved.

Lemma 8. If the curve $\tilde{\gamma}(\tilde{s}, t)$ is inextensible, then the derivatives with respect to \tilde{s} and t commute.

Lemma 9. The curve $\tilde{\gamma}(\tilde{s}, t)$ is inextensible if and only if $\tilde{W}_{\tilde{s}} = \tilde{k} \tilde{U}$.

Lemma 10. If the curve $\tilde{\gamma}(\tilde{s}, t)$ is inextensible, then the evolution equations for the curvature and torsion (10), take the following formula

$$\begin{pmatrix} \tilde{k} \\ \tilde{\tau} \end{pmatrix}_t = \begin{pmatrix} 0 & -\tilde{f}_2 \\ \tilde{f}_2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{k} \\ \tilde{\tau} \end{pmatrix} + \begin{pmatrix} \tilde{U} + \tilde{f}_{1\tilde{s}} \\ \tilde{\xi}_{\tilde{s}} \end{pmatrix}. \quad (31)$$

6 Examples of motions of inextensible curves in \mathbb{S}^3

We consider the motion of inextensible curves in \mathbb{S}^3 . Then the system (31) of PDE can be written explicitly in the following form:

$$\begin{aligned} \tilde{k}_t &= (1 + \tilde{k}^2 - \tilde{\tau}^2)\tilde{U} + \tilde{U}_{\tilde{s}\tilde{s}} + \tilde{k}_{\tilde{s}}\tilde{W} - \tilde{\tau}_{\tilde{s}}\tilde{V} - 2\tilde{\tau}\tilde{V}_{\tilde{s}}, \\ \tilde{\tau}_t &= \tilde{k}(\tilde{V}_{\tilde{s}} + \tilde{\tau}\tilde{U}) + \frac{\partial}{\partial \tilde{s}}\left(\frac{1}{\tilde{k}}(1 - \tilde{\tau}^2)\tilde{V} + \frac{\tilde{\tau}}{\tilde{k}}(\tilde{k}\tilde{W} + 2\tilde{U}_{\tilde{s}}) \right. \\ &\quad \left. + \frac{1}{\tilde{k}}\tilde{V}_{\tilde{s}\tilde{s}} + \frac{\tilde{\tau}_{\tilde{s}}}{\tilde{k}}\tilde{U}\right). \end{aligned} \tag{32}$$

Example 1.If

$$\tilde{W} = constant = b \neq 0, \quad \tilde{U} = 0 \quad and \quad \tilde{V} = constant = a. \tag{33}$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= b\tilde{k}_{\tilde{s}} - a\tilde{\tau}_{\tilde{s}}, \\ \tilde{\tau}_t &= \frac{\partial}{\partial \tilde{s}}\left(\frac{1}{\tilde{k}}(1 - \tilde{\tau}^2)a + b\tilde{\tau}\right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= \frac{ac_1}{bc_1 - c_2}\left(-1 + c_5 + c_4 \tanh(c_1\tilde{s} + c_2t + c_3)\right), \\ \tilde{\tau}(\tilde{s}, t) &= c_5 + c_4 \tanh(c_1\tilde{s} + c_2t + c_3), \end{aligned} \tag{34}$$

where c_1, c_2, c_3, c_4 and c_5 are constants. Substitute from (33) and (34) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves. To visualize this surface in \mathbb{S}^3 , we use the stereographic projection in \mathbb{R}^4 (Fig. 1).

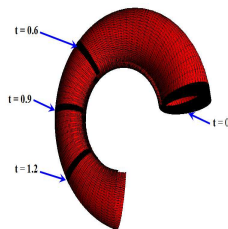


Fig. 1: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 2]$, $t \in [0, 1.5]$, $a = 3, b = 0.1, c_1 = c_2 = 0.1, c_3 = 0, c_4 = 1$ and $c_5 = 0$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.6, 0.9, 1.2$.

Example 2.If

$$\tilde{W} = \tilde{U} = 0 \quad and \quad \tilde{V} = const = a. \tag{35}$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= -a\tilde{\tau}_{\tilde{s}}, \\ \tilde{\tau}_t &= \frac{\partial}{\partial \tilde{s}}\left(\frac{1}{\tilde{k}}(1 - \tilde{\tau}^2)a\right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= -\frac{ac_1}{c_2}\left(-1 + c_5 + c_4 \tanh(c_1\tilde{s} + c_2t + c_3)\right), \\ \tilde{\tau}(\tilde{s}, t) &= c_5 + c_4 \tanh(c_1\tilde{s} + c_2t + c_3), \end{aligned} \tag{36}$$

where c_1, c_2, c_3, c_4 and c_5 are constants. Substitute from (35) and (36) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves (Fig. 2).

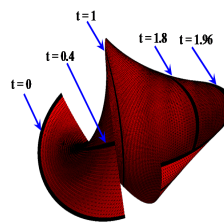


Fig. 2: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 2.5]$, $t \in [0, 2]$, $a = 1.8, c_1 = 1, c_2 = c_3 = 0.1, c_4 = 0.001$ and $c_5 = 1$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.4, 1, 1.8, 1.96$.

Example 3.If

$$\tilde{W} = \tilde{U} = 0 \quad and \quad \tilde{V} = \tilde{k}(\tilde{s}, t). \tag{37}$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= -2\tilde{\tau}\tilde{k}_{\tilde{s}} - \tilde{k}\tilde{\tau}_{\tilde{s}}, \\ \tilde{\tau}_t &= \tilde{k}\tilde{k}_{\tilde{s}} + \frac{\partial}{\partial \tilde{s}}\left((1 - \tilde{\tau}^2) + \frac{\tilde{k}_{\tilde{s}\tilde{s}}}{\tilde{k}}\right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= 2c_1 \operatorname{sech}(c_1\tilde{s} + c_2t + c_3), \\ \tilde{\tau}(\tilde{s}, t) &= \frac{-c_2}{2c_1}, \end{aligned} \tag{38}$$

where $c_1, c_2,$ and c_3 are constants. Substitute from (37) and (38) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves (Fig. 3).

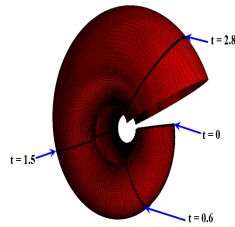


Fig. 3: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 2]$, $t \in [0, 3]$, $c_1 = 1$, $c_2 = 0.2$ and $c_3 = 0$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.6, 1.5, 2.8$.

Example 4. If

$$\tilde{W} = \frac{1}{2}\tilde{k}^2, \quad \tilde{U} = \tilde{k}_{\tilde{s}} \quad \text{and} \quad \tilde{V} = 0. \quad (39)$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= \left(1 + \left(\frac{3}{2}\right)\tilde{k}^2 - \tilde{\tau}^2\right)\tilde{k}_{\tilde{s}} + \tilde{k}_{\tilde{s}\tilde{s}}, \\ \tilde{\tau}_t &= \tilde{\tau}\tilde{k}_{\tilde{s}} + \frac{\partial}{\partial \tilde{s}} \left(\frac{\tilde{\tau}}{\tilde{k}} \left(\frac{1}{2}\tilde{k}^3 + 2\tilde{k}_{\tilde{s}\tilde{s}} \right) + \frac{\tilde{\tau}_{\tilde{s}}}{\tilde{k}}\tilde{k}_{\tilde{s}} \right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= -\sqrt{3}\operatorname{sech}\left(\frac{1}{\sqrt{2}}\tilde{s} + \frac{3}{2\sqrt{2}}t - c_1\right), \\ \tilde{\tau}(\tilde{s}, t) &= -\sqrt{\frac{3}{2}}\operatorname{sech}\left(\frac{1}{\sqrt{2}}\tilde{s} + \frac{3}{2\sqrt{2}}t - c_1\right), \end{aligned} \quad (40)$$

where c_1 is constant. Substitute from (39) and (40) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves (Fig. 4).

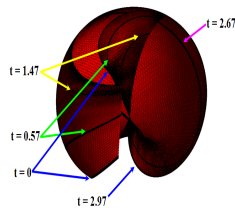


Fig. 4: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 6]$, $t \in [0, 3]$ and $c_1 = 3$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.57, 1.47, 2.67, 2.97$.

Example 5. If

$$\tilde{W} = \text{const} = a, \quad \tilde{U} = 0 \quad \text{and} \quad \tilde{V} = \tilde{k} + a. \quad (41)$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= (a - 2\tilde{\tau})\tilde{k}_{\tilde{s}} - (a + \tilde{k})\tilde{\tau}_{\tilde{s}}, \\ \tilde{\tau}_t &= \tilde{k}\tilde{k}_{\tilde{s}} + \frac{\partial}{\partial \tilde{s}} \left(\frac{1}{\tilde{k}}(1 - \tilde{\tau}^2)(\tilde{k} + a) + a\tilde{\tau} + \frac{1}{\tilde{k}}\tilde{k}_{\tilde{s}\tilde{s}} \right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= -2c_1\operatorname{sech}(c_1\tilde{s} + c_2t + c_3), \\ \tilde{\tau}(\tilde{s}, t) &= -1, \\ c_1 &= \frac{c_2}{a+2}, \end{aligned} \quad (42)$$

where c_1 , c_2 and c_3 are constants. Substitute from (41) and (42) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves (Fig. 5).

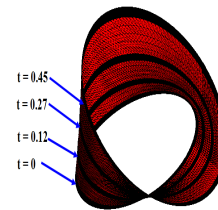


Fig. 5: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 7]$, $t \in [0, 0.5]$, $a = 1.5$, $c_2 = 0.01$ and $c_3 = 1$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.12, 0.27, 0.45$.

Example 6. If

$$\tilde{W} = \text{const} = a, \quad \tilde{U} = 0 \quad \text{and} \quad \tilde{V} = \frac{\tilde{k}}{a}. \quad (43)$$

Then (32) takes the form:

$$\begin{aligned} \tilde{k}_t &= \frac{1}{a}(a^2 - 2\tilde{\tau})\tilde{k}_{\tilde{s}} - \frac{1}{a}\tilde{k}\tilde{\tau}_{\tilde{s}}, \\ \tilde{\tau}_t &= \frac{1}{a}\tilde{k}\tilde{k}_{\tilde{s}} + \frac{\partial}{\partial \tilde{s}} \left(\frac{1}{a}(1 - \tilde{\tau}^2) + a\tilde{\tau} + \frac{1}{a\tilde{k}}\tilde{k}_{\tilde{s}\tilde{s}} \right). \end{aligned}$$

One solution of this system is

$$\begin{aligned} \tilde{k}(\tilde{s}, t) &= -2c_1\operatorname{sech}(c_1\tilde{s} + c_2t + c_3), \\ \tilde{\tau}(\tilde{s}, t) &= \frac{a}{2c_1}(ac_1 - c_2), \end{aligned} \quad (44)$$

where c_1 , c_2 and c_3 are constants. Substitute from (43) and (44) into the systems (4) and (9), and solve them numerically. Then we can get the family of curves $\tilde{C}_t = \tilde{\gamma}(\tilde{s}, t)$, so we can determine the surface that is generated by this family of curves (Fig. 6).

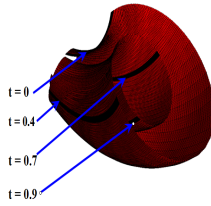


Fig. 6: The surface that is generated by motion of the family of curves \tilde{C}_t for $\tilde{s} \in [0, 1]$, $t \in [0, 1]$ and $a = 0.1$, $c_1 = 0.4$, $c_2 = 1$ and $c_3 = 1$. The bold black curves in the surface represent the family of curves \tilde{C}_t for $t = 0, 0.4, 0.7, 0.9$.

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