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On Δ-Asymptotically Statistical Equivalent Sequences

Ayhan Esi
Department of Mathematics, Science and Art Faculty, Adiyaman University
02040, Adiyaman, Turkey
Email Address: aesi23@adiyaman.edu.tr

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This paper presents new definitions which are a natural combination of the definition for asymptoticall equivalence and $\Delta-$statistical convergence of sequences. Let $\theta = (k_r)$ be a lacunary sequence. Then the sequences x and y are said to be $[w]^L_{\theta,\Delta}$ $-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L\right| \geq \varepsilon\right\}\right| = 0\,.$$

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1 Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$. A sequence $x = (x_k) \in l_\infty$ is said to be *almost convergent* if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [8] proved that

$$\hat{c} = \left\{x = (x_k) \in l_\infty : \ \lim_k t_{km}(x) \text{ exists uniformly in } m\right\},$$

where

$$t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1}.$$

The space of strongly almost convergent sequences was introduced by Maddox [9] as

$$[\hat{c}] = \left\{x = (x_k) \in l_\infty : \ \lim_k t_{km}(|x - le|) \text{ exists uniformly in } m, \text{for some } l\right\},$$

where $e = (1, 1, 1, \ldots)$.

The notion of difference sequence space was introduced by Kizmaz [7] as

$$X(\Delta) = \{x = (x_k) : \ (\Delta x_k) \in X\}$$

for $X = l_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$ for all k.

The idea of statistical convergence was introduced by Fast [4] and studied by Fridy [5] Fridy and Orhan [6], Connor [2], Salat [13], among others. A sequence $x = (x_k)$ is said to be *statistically convergent to number L* if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : \ |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

A complex number sequence $x = (x_k)$ is said to be $\Delta-$statistically convergent to the number L [3] if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : \ |\Delta x_k - L| \geq \varepsilon\}| = 0,$$

in this case, we write $S_\Delta - \lim x = L$ or $x_k \to L(S_\Delta)$ and S_Δ denotes the set of all statistically convergent sequences, where

$$\Delta^1 x_k = \Delta x_k = x_k - x_{k+1}, \quad \Delta^0 x_k = x_k,$$

for all $k \in N$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r.

In 1993, Marouf [10] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [11] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, Patterson and Savaş [12] extended these definitions by using lacunary sequences. In 2008 Altundag and Basarir [1] defined and studied new definitions which are natural combination of the definition for asymptotically equivalence and $[w]_\theta$ $-$statistically convergence.

2 Definitions and Notations

Definition 2.1 ([10]). Two nonnegative sequences x and y are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1 \text{ (denoted by } x \sim y).$$

Definition 2.2 ([1]). Two nonnegative sequences x, y are said to be st-$[w]^L$ asymptotically equivalent of multiple L provided that for $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } m \text{ (denoted by } x \overset{st-[w]^L}{\sim} y)$$

and simply st-$[w]$ asymptotically equivalent, if $L = 1$.

Definition 2.3 ([1]). Let $\theta = (k_r)$ be a lacunary sequence, the two nonnegative sequences x and y are said to be st-$[w]_\theta^L$ asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } m \text{ (denoted by } x \overset{st-[w]_\theta^L}{\sim} y)$$

and simply st-$[w]_\theta$asymptotically equivalent, if $L = 1$.

Definition 2.4 ([1]). Let $\theta = (k_r)$ be a lacunary sequence, the two nonnegative sequences x and y are said to be $[w]_\theta^L$ -asymptotically equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| = 0 \text{ (denoted by } x \overset{[w]_\theta^L}{\sim} y)$$

and simply $[w]_\theta -$ asymptotically equivalent, if $L = 1$.

Following the above definitions, we shall now introduce some new ones.

Let (Δx_k) and (Δy_k) be first order difference sequences of x and y, respectively.

Definition 2.5. The sequences x and y are said to be $w_\Delta -$asymptotically equivalent if

$$\lim_k \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} = 1 \text{ uniformly in } m \text{ (denoted by } x \overset{w_\Delta}{\sim} y).$$

Example 2.1. Let

$$x = (x_k) = (-1, -2, -3, \ldots) \text{ and } y = (y_k) = (1, 0, -1, 2, 1, 0, -1, 2, 1, 0, -1, 2, \ldots).$$

Then $\Delta x_k = \Delta y_k = 1$ for all k, so

$$\lim_k \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} = 1$$

uniformly in m, i.e., $x \overset{w_\Delta}{\sim} y$.

Definition 2.6. The sequences x and y are said to be $st - [w]_\Delta^L$ $-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L\right| \geq \varepsilon\right\}\right| = 0 \text{ uniformly in } m \text{ (denoted by } x \overset{st-[w]_\Delta^L}{\sim} y)$$

and simply $[w]_\Delta$ $-$asymptotically statistical equivalent, if $L = 1$.

Definition 2.7. Let $\theta = (k_r)$ be a lacunary sequence. Then the sequences x and y are said to be $st - [w]^L_{\theta,\Delta}$ $-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| \geq \varepsilon \right\} \right| = 0 \text{ uniformly in } m \text{ (denoted by } x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y)$$

and simply st-$[w]_{\theta,\Delta}$ $-$asymptotically equivalent, if $L = 1$.

Definition 2.8. Let $\theta = (k_r)$ be a lacunary sequence. Then the sequences x and y are said to be are $-$asymptotically equivalent to multiple L provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| = 0 \text{ uniformly in } m \text{ (denoted by } x \overset{[w]^L_{\theta,\Delta}}{\sim} y)$$

and simply $[w]_{\theta,\Delta}$ $-$asymptotically equivalent, if $L = 1$.

3 Main Results

Theorem 3.1. *Let $\theta = (k_r)$ be a lacunary sequence, then*

(a) *If $x \overset{[w]^L_{\theta,\Delta}}{\sim} y$ then $x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y$,*
(b) *If $x, y \in l_\infty(\Delta)$ and $x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y$ then $x \overset{[w]^L_{\theta,\Delta}}{\sim} y$,*
(c) $[w]^L_{\theta,\Delta} \cap l_\infty(\Delta) = st - [w]^L_{\theta,\Delta} \cap l_\infty(\Delta)$,

where $l_\infty(\Delta) = \{x = (x_k) : (\Delta x_k) \in l_\infty\}$.

Proof. (a). If $\varepsilon > 0$ and $x \overset{[w]^L_{\theta,\Delta}}{\sim} y$, then

$$\sum_{k \in I_r} \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| \geq \sum_{\substack{k \in I_r \\ \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon}} \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right|$$

$$\geq \varepsilon \left| \left\{ k \in I_r : \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| \geq \varepsilon \right\} \right| .$$

Therefore $x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y$.

(b). Suppose that $x, y \in l_\infty(\Delta)$ and $x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y$. Then we can assume that

$$\left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| \leq M \text{ for all } k \text{ and } m.$$

Given $\varepsilon > 0$

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right| \geq \varepsilon}} \left| \frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)} - L \right|$$

$$+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|<\varepsilon}}\left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|$$

$$\leq \frac{M}{h_r}\left|\left\{k\in I_r:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|+\varepsilon.$$

Therefore $x \overset{[w]^L_{\theta,\Delta}}{\sim} y$.

(c). This immediately follows from (a) and (b). □

Theorem 3.2. *Let* $\theta=(k_r)$ *be a lacunary sequence with* $\liminf q_r>1$, *then*

$$x \overset{st-[w]^L_{\Delta}}{\sim} y \ \textit{implies}\ x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y .$$

Proof. Suppose that $\liminf q_r>1$, then there exists a $\delta>0$ such that $q_r\geq 1+\delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}.$$

If $x \overset{st-[w]^L_{\Delta}}{\sim} y$, then for every $\varepsilon>0$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k\leq k_r:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|\geq\frac{1}{k_r}\left|\left\{k\in I_r:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|$$

$$\geq\frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|,$$

which completes the proof. □

Theorem 3.3. *Let* $\theta=(k_r)$ *be a lacunary sequence with* $\limsup q_r<\infty$,

$$x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y \ \textit{implies}\ x \overset{st-[w]^L_{\Delta}}{\sim} y .$$

Proof. Suppose that $\limsup q_r<\infty$, then there exists $B>0$ such that $q_r<B$ for all $r\geq 1$. Let $x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y$ and $\varepsilon>0$. There exists $R>0$ such that for every $j\geq R$

$$A_j=\frac{1}{h_j}\left|\left\{k\in I_j:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|<\varepsilon.$$

We can also find $K>0$ such that $A_j<K$ for all $j=1,2,\ldots$. Now let n be any integer with $k_{r-1}<n<k_r$, where $r>R$. Then

$$\frac{1}{n}\left|\left\{k\leq n:\left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|\leq\frac{1}{k_{r-1}}\left|\left\{k\leq k_r:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|$$

$$=\frac{1}{k_{r-1}}\left|\left\{k\in I_1:\ \left|\frac{t_{km}(\Delta x_k)}{t_{km}(\Delta y_k)}-L\right|\geq\varepsilon\right\}\right|$$

$$
\begin{aligned}
&+\frac{1}{k_{r-1}}\left|\left\{k\in I_2:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
&+\cdots+\frac{1}{k_{r-1}}\left|\left\{k\in I_r:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
=&\frac{k_1}{k_{r-1}k_1}\left|\left\{k\in I_1:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
&+\frac{k_2-k_1}{k_{r-1}\left(k_2-k_1\right)}\left|\left\{k\in I_2:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
&+\cdots+\frac{k_R-k_{R-1}}{k_{r-1}\left(k_R-k_{R-1}\right)}\left|\left\{k\in I_R:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
&+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}\left(k_r-k_{r-1}\right)}\left|\left\{k\in I_r:\ \left|\frac{t_{km}\left(\Delta x_k\right)}{t_{km}\left(\Delta y_k\right)}-L\right|\geq\varepsilon\right\}\right| \\
=&\frac{k_1}{k_{r-1}k_1}A_1+\frac{k_2-k_1}{k_{r-1}\left(k_2-k_1\right)}A_2 \\
&+\cdots+\frac{k_R-k_{R-1}}{k_{r-1}\left(k_R-k_{R-1}\right)}A_R+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}\left(k_r-k_{r-1}\right)}A_r \\
\leq&\left(\sup_{j\geq 1}A_j\right)\frac{k_R}{k_{r-1}}+\left(\sup_{j\geq R}A_j\right)\frac{k_r-k_R}{k_{r-1}} \\
\leq&K\frac{k_R}{k_{r-1}}+\varepsilon B.
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *Let θ be a lacunary sequence with $1<\liminf q_r\leq\limsup q_r<\infty$, then*

$$x \overset{st-[w]^L_{\theta,\Delta}}{\sim} y \Leftrightarrow x \overset{st-[w]^L_{\Delta}}{\sim} y\,.$$

Proof. This immediately follows from Theorem 3.2 and Theorem 3.3. □

References

[1] S. Altundag and M. Basarir, On $[w]_{L,\theta}$ –Lacunary asymptotically equivalent sequences, *Int. Journal of Math. Analysis* **2** (2008), 373–378.

[2] J. S. Connor, The statistical and strong p-Cesaro convergence of sequences, *Anaysis* **8** (1988), 47–63.

[3] M. Et and F. Nuray, Δ^m-statistical convergence, *Indian J. Pure and Appl. Math.* **32** (2001), 961–969.

[4] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.

[5] J. A. Fridy, On statistical convergence, *Analysis* **5** (1985), 301–313.

[6] J. A. Fridy and C. Orhan, Lacunary statistical convergence, *Pasific J. Math.* **160** (1993), 43–51.

[7] H. Kizmaz, On certain sequence spaces, *Canad. Math. Bull.* **24** (1981), 169–176.

[8] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Math.* **80** (1948), 167–190.
[9] I. J. Maddox, Spaces of strongly summable sequences, *Quart. J. Math.* **18** (1967), 345–355.
[10] M. Marouf, Asymptotic equivalence and summability, *Internat. J. Math. Math. Sci.* **16** (1993), 755–762.
[11] R. F. Patterson, On asymptotically statistically equivalent sequences, *Demonstratio Math.* **36** (2003), 149–153.
[12] R. F. Patterson and E. Savaş, On asymptotically lacunary statistical equivalent sequences, *Thai Journal of Mathematics* **4** (2006), 267–272.
[13] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca* **30** (1980), 139–150.

Ayhan Esi was born in Istanbul, Turkey, on March 5, 1965. Ayhan Esi got his B.Sc. from Inonu University in 1987 and M. Sc. and Ph.D. degree in pure mathematics from Elazig University, Turkey in 1990 and 1995, respectively. His research interests include Summability Theory, Sequences and Series in Analysis and Functional Analysis. In 2000, Esi was appointed to Education Faculty in Gaziantep University. In 2002, Esi was appointed as the head of Department of Mathematics in Science and Art Faculty in Adiyaman of the Inonu University. In 2006, Esi joined the Department of Mathematics of Adiyaman University. He is married and has 2 children.