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Certain Family of Weighted Averages with Beta Random Weights

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Abstract: A new specification for weighted averages of two independent and continuous random variables X_1 and X_2 with beta random weights is provided. Furthermore, it can be concluded that the randomly weighted averages with beta random proportions which is defined by Homei [H. Homei, Randomly weighted averages with beta random proportions **82**, 15151520 (2012)] is a special case of the achievements of Soltani and Roozegar[A.R. Soltani, R. Roozegar, On distribution of randomly ordered uniform incremental weighted averages: Divided difference approach **82**, 10121020 (2012)]. A family including several specific examples is provided.

Keywords: Randomly weighted average, Stieltjes transform, generalized Stieltjes transform, beta distribution

1 Introduction

The research of finding the distribution of random mixture is nowadays a hot challenge which has been object of several investigations. Applications of mixture distributions in sociology and biology are described in [1] and [2]; Tukey [3] used mixtures in creating models of ”contaminated” distributions; and, generally, mixtures provide models that lead to clear interpretation in the teaching of notions such as total probability, Bayes’s theorem and posterior distributions. Randomly weighted averages have been naturally appeared in many different areas; for example in sampling, density estimators, Bayesian and distribution characterization, among others. In the theory of general regression and neural networks the multivariate kernel density estimations and multivariate kernel regressions are randomly weighted averages, [4]. More applications can be found in [5] and [6] and in some references of them.

A random variable X is said to have a beta distribution on $(-a,a)$ with parameters $p>0$ and $q>0$, $beta(p,q)$, if its probability density function is given by

$$f(x)=\frac{1}{(2a)^{p+q-1}B(p,q)}(x+a)^{p-1}(a-x)^{q-1}, \tag{1}$$

where $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is beta function and $\Gamma(m)=\int\limits_0^\infty s^{m-1}e^{-s}ds$ is gamma function. If we consider $Y=\frac{X+1}{2}$ and $a=1$, then Y is said to have an ordinary beta distribution on $(0,1)$. A random variable X is said to have a power semicircle distribution with shape parameter θ and range parameter σ , $PS(\theta,\sigma)$, if its density function is given by

$$f(x)=C_{\theta,\sigma}(\sigma^2-x^2)^{\theta+\frac{1}{2}},\quad x\in(-\sigma,\sigma), \tag{2}$$

where $C_{\theta,\sigma}=\frac{\Gamma(\theta+2)\Gamma(\theta+\frac{3}{2})}{\sqrt{\pi}\sigma^{2\theta+2}}$. Indeed, a $PS(\theta,\sigma)$ distribution is a special case of beta distribution when $p=q=\theta+\frac{3}{2}$ and $a=\sigma$. We recall that for $\theta=-1$, (2) is Arcsine density on $(-\sigma,\sigma)$, for $\theta=\frac{-1}{2}$, it is the uniform density on $(-\sigma,\sigma)$ and for $\theta=0$, is the semicircle or Wigner density function.

Suppose that X_1 and X_2 are continuous random variables of independent distributions with distribution functions F_1 and F_2, respectively. Recently, Homei [7] proposed a randomly weighted averages (RWA) of two independent and

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continuous random variables X_1 and X_2 with random proportions which have beta distributions. More precisely, given two independent random variables X_1 and X_2, this author considers

$$Z = WX_1 + (1-W)X_2, \tag{3}$$

where the random proportion W is $beta\ (\alpha_1, \alpha_2)$ distributed and is independent of X_1 and X_2. By using two lemmas that will be stated in Section 2 and the following formula taken from the Schwartz distribution theory, he found that

$$B(\alpha_1,\alpha_2)\frac{d^{\alpha_1+\alpha_2-1}}{dz^{\alpha_1+\alpha_2-1}}\mathscr{S}[F_Z](z) = -\frac{d^{\alpha_1-1}}{dz^{\alpha_1-1}}\mathscr{S}[F_1](z)\frac{d^{\alpha_2-1}}{dz^{\alpha_2-1}}\mathscr{S}[F_2](z), \tag{4}$$

for which $z \in \mathbb{C} \cap (\text{Supp } F_1)^c \cap (\text{Supp } F_2)^c$, where F_Z is distribution function of Z. This result was the main theorem of Homei's [7] work.

Also $\mathscr{S}[H](z)$ denotes the Stieltjes transform (ST) of a distribution H and it is defined as

$$\mathscr{S}[H](z) = \int_{\mathbb{R}} \frac{1}{z-x} H(dx), \ \ z \in \mathbb{C} \cap (\text{supp}H)^c,$$

where $\mathbb{C}$ is the set of complex numbers and supp$H$ stands for the support of H. Similarly, the generalized ST of a distribution $H(x)$ is defined as

$$\mathscr{S}[H;n](z) = \int_{\mathbb{R}} \frac{1}{(z-x)^n} H(dx), \ z \in \mathbb{C} \cap (\text{supp}H)^c, \ n > 0.$$

The mentioned formula taken from the Schwartz distribution theory is given by

$$\int_{-\infty}^{\infty} \varphi(x)\Lambda^{[n]}(dx) = \frac{(-1)^n}{n!}\int_{-\infty}^{\infty} \frac{d^n}{dx^n}\varphi(x)\Lambda(dx),$$

where Λ is a distribution function and $\Lambda^{[n]}$ is the n-th distributional derivative of Λ.

Soltani and Roozegar [6] consider a general form of RWA of independent and continuous random variables $X_1, \cdots, X_m$, namely

$$S_{n;\,k_1,\ldots,k_{m-1}} = \sum_{j=1}^{m} V_j\, X_j, \tag{5}$$

where the random weights V_j are defined by

$$V_j = U_{(k_j)} - U_{(k_{j-1})}, \ \ j = 1,2,\ldots,m, \ \ m \le n,$$

$U_{(1)},\ldots,U_{(n-1)}$ are order statistics of a random sample $U_1,\ldots,U_n$ from a uniform distribution on $[0,1]$, $U_{(0)} = 0$, $U_{(n)} = 1$ and $k_0 = 0 < k_1 < \ldots < k_{m-1} < k_m = n$ points in $\{1,\ldots,n\}$. In particular, they use certain techniques in divided differences and show that

$$\mathscr{S}[F_{S_{n;\,k_1,\ldots,k_{m-1}}};n](z) = \prod_{j=1}^{m}\mathscr{S}[F_j;r_j](z), \quad z \in \mathbb{C}\bigcap_{i=1}^{m}(\text{supp } F_i)^c, \tag{6}$$

where $r_j = k_j - k_{j-1}$, $j = 1,2,\cdots,m$, $\sum_{j=1}^{m} r_j = n$ and each X_j have distribution function F_j for $j = 1,2,\ldots,m$.

In this paper, we present a simple specification for introducing the randomly weighted averages with beta random weights to find its distribution. The present paper is organized as follows: After this introduction, in Section 2 we investigate the main result. In Section 3 by applying the main Theorem in Section 2, we characterize a new family of RWA distributions for beta distribution and more specifically, we present several examples of this family.

2 Main Result

As already mentioned in the introduction, in this section we present a simple specification for RWA with beta random weights. First we state the following two lemmas by [7].

Lemma 2.1. Assume Z is a randomly weighted average given in (3). Then the conditional distribution of Z, for given distinct values $X_1 = x_1$ and $X_2 = x_2$ at z, $-\infty < z < +\infty$, will be equal to

$$\frac{1}{B(\alpha_1,\alpha_2)} \sum_{i=0}^{1} \sum_{k=0}^{\alpha_{2-i}-1} \frac{\binom{\alpha_{2-i}-1}{k}(-1)^k (z-x_{2-i})^{\alpha_{i+1}+k} U(z-x_{2-i})}{(\alpha_{i+1}+k)[C(x_{2-i};x_1,x_2)]^{\alpha_{i+1}+k}},$$

for which $z \in [min\{x_1,x_2\}, max\{x_1,x_2\}]$, where $C(x_{2-i};x_1,x_2) = \prod_{k=1}^{2-i-1}(x_k - x_{2-i}) \prod_{k=2-i+1}^{2}(x_k - x_{2-i})$ and $U(x) = 0, x < 0, = 1, x \geqslant 0$ is the Heaviside function.

Lemma 2.2. For given distinct real x_1, x_2, z and integers $n_j \geq 1, j = 1,2$, we have the following formula:

$$\sum_{i=1}^{2} \frac{(-1)^{n_i}}{(n_i-1)!} \left[\frac{d^{n_i-1}}{dx_i^{n_i-1}} \frac{1}{(z-x_i)} \frac{1}{\prod_{j\neq i}^{2}(x_j-x_i)^{n_j}} \right] = \prod_{i=1}^{2} \frac{1}{(x_i-z)^{n_i}}.$$

The following theorem indicates how the Stieltjes transforms of Z and X_1, X_2 are related.

Theorem 2.1. Let Z be the randomly weighted averages given in (3). Assume random variables X_1 and X_2 are independent and continuous with distribution functions F_1 and F_2, respectively. Then

$$B(\alpha_1,\alpha_2)\frac{d^{\alpha_1+\alpha_2-1}}{dz^{\alpha_1+\alpha_2-1}}\mathscr{S}[F_Z](z) = -\frac{d^{\alpha_1-1}}{dz^{\alpha_1-1}}\mathscr{S}[F_1](z)\frac{d^{\alpha_2-1}}{dz^{\alpha_2-1}}\mathscr{S}[F_2](z) \quad z \in \mathbb{C}\bigcap_{i=1}^{2}(\text{supp } F_i)^c. \tag{7}$$

Proof. Let $m = 2$ in (5), then we have

$$S_{n;\,k_1} = V_1\, X_1 + V_2\, X_2, \tag{8}$$

where the random weights V_1 and V_2 are defined by

$$V_1 = U_{(k_1)} - U_{(k_0)} = U_{(k_1)} \quad \text{and} \quad V_2 = U_{(k_2)} - U_{(k_1)} = 1 - V_1.$$

Therefore (8) is equivalent to (3) with $\alpha_1 = k_1$ and $\alpha_2 = k_2 - k_1$.
In accordance with (6), we let $m = 2$, $\alpha_1 = k_1$ and $\alpha_2 = k_2 - k_1$. Therefore,

$$\mathscr{S}[F_Z;\alpha_1+\alpha_2](z) = \mathscr{S}[F_1;\alpha_1](z)\mathscr{S}[F_2;\alpha_2](z) \quad z \in \mathbb{C}\bigcap_{i=1}^{2}(\text{supp } F_i)^c, \tag{9}$$

which is (7), because

$$\mathscr{S}[H;n](z) = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\mathscr{S}[H](z).$$

This completes the proof. □

3 A characterization for beta distribution

The Stieltjes transform can be applied in many circumstances to find the distribution of RWA. Here we apply this method. The following theorem characterizes the distribution of randomly weighted averages on beta distribution.

Theorem 3.1. Let independent random variables X_1 and X_2 have $beta(m_1,n_1)$ and $beta(m_2,n_2)$ distributions, respectively. The RWA Z given by (3) has $beta(m_1+m_2, n_1+n_2)$ distribution when the random weight W has $beta(m_1+n_1, m_2+n_2)$ distribution.

Proof. Since this result is achieved for both supports $(-1,1)$ and $(0,1)$, without loss of generality we assume the support of beta distributions are $(-1,1)$. Consider $\alpha_1 = m_1+n_1$, $\alpha_2 = m_2+n_2$, $m = m_1+m_2$ and $n = n_1+n_2$. The generalized ST of beta distribution is

$$\mathscr{S}[F_i;\alpha_i](z) = \int_{-1}^{1}\frac{F_i(dx)}{(z-x)^{\alpha_i}} = \frac{1}{(z+1)^{n_i}}\frac{1}{(z-1)^{m_i}},$$

where $X_i \sim F_i$, $i = 1,2$. Thus, from (9) we have

$$\mathscr{S}[F_Z;\alpha_1+\alpha_2](z) = \frac{1}{(z+1)^{n}}\frac{1}{(z-1)^{m}},$$

implying that Z has a $beta(m,n)$ distribution, which completes the proof. □

The following two examples by [7] and [8] are immediate results of Theorem 3.1. In addition to shedding new light on some characterizations in Section 3 of [7], Roozegar and Soltani [9] characterize large family of RWA distributions. Some other examples are provided in [7] and [6].

Example 3.1. For independent random variables X_1 and X_2 with standard uniform $(0,1)$ distributions, and W with $beta(2,2)$ distribution, the RWA Z has $beta(2,2)$ distribution.

Proof Put $m_i = n_i = 1$ for $i = 1,2$ in Theorem 3.1.

Example 3.2. Let X_1 and X_2 are independent random variables with common Arcsine distribution on $(0,1)$. Also assume the random weight W has uniform $(0,1)$ distribution. The RWA Z has a power semicircle distribution with parameters $\theta = \frac{-1}{2}$ and $\sigma = 1$ that is $U(0,1)$ distribution.

Proof Put $m_i = n_i = \frac{1}{2}1$ for $i = 1,2$ in Theorem 3.1.

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