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Family Potential Wells and its Applications to NLS with Harmonic Potentia

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Abstract: This paper discusses a class of nonlinear Schrödinger equation with combined power-type nonlinearities and harmonic potential. By constructing a variational problem and the so-called invariant manifolds of the evolution flow, we derive a sharp condition for blow up and global existence of the solutions by applying the potential well theory and the concavity method.

Keywords: Potential Wells; Harmonic Potentia.

1. Introduction

In this paper we study the nonlinear Schrödinger equation with combined power-type nonlinearities

$$\begin{cases} i\varphi_t + \Delta\varphi - V(x)\varphi + \sum\limits_{k=1}^{l} a_k|\varphi|^{p_k-1}\varphi \\ -\sum\limits_{j=1}^{s} b_j|\varphi|^{q_j-1}\varphi = 0,\ t>0, \\ \varphi(0,x) = \varphi_0(x),\ \ x\in R^N. \end{cases} \qquad (1)$$

Here $\varphi = \varphi(x,t)$: $R^N \times [0,T) \to C$ is a complex valued function, and $0 < T + \infty$ is the maximal existence time, N is the space dimension, $i = \sqrt{-1}$, Δ is the Laplace operator in R^N and the nonlinear power exponents p_k, q_j with the coefficients a_k, b_j satisfy

(H1)$a_k > 0,\ 1kl,\ b_j > 0,\ 1js$;
(H2)$1 < q_s < q_{s-1} < \cdots < q_1 = q < p = p_l < p_{l-1} < \cdots < p_1 < \infty$ for $N = 1, 2$;
$1 < q_s < q_{s-1} < \cdots < q_1 = q < p = p_l < p_{l-1} < \cdots < p_1 < \frac{N+2}{N-2}$ for $N3$;

and $V(x)$ satisfies

(H3) $\inf\limits_{x\in R^N} V(x)0$ is a real-valued function from R^N to R,
(H4) $\lim\limits_{|x|\to\infty} V(x) = \infty$,
(H5)$V(x) \in L^1(R^N)$.

Problem (1) arises in various physical contexts in the description of a nonlinear wave such as propagation of a laser beam, water waves at the free surface of an ideal fluid and plasma waves. And it has been investigated by many authors. R. T. Glassey [1] studied the Cauchy problem of $iu_t + \Delta u - \lambda_1|u|^{p_1}u - \lambda_2|u|^{p_2}u = 0$. He arrived at some results on local and global well-posedness, asymptotic behavior and finite time blow up of the solution in different energy spaces.

The Cauchy problem of $iu_t+\Delta u-|x|^2u+a_1|u|^{p-1}u = 0$ was discussed by Fukuizumi Reika [2], G. Chen and J. Zhang [3]. They showed some sharp criteria for global existence and blowing up of the solution. Further, the same problem was studied by Yunyun Wei and Guanggan Chen [4]. They established the existence of the solutions of the Cauchy problem and proved that the standing wave is nonlinearly unstable. In addition, J. Shu and J. Zhang [5] investigated the Cauchy problem of $iu_t + \Delta u + |u|^p u - |u|^q u = 0$ and obtained the blow up and global existence of the solution.

D. Fujiwara [6] considered the Cauchy problem of $i\varphi_t + \Delta\varphi - V(x)\varphi + |\varphi|^{p-1}\varphi = 0$, He proved that the smoothness of the Schrödinger kernel for potentials of quadratic growth. There is still much literature concerned with the existence and blow up results for the analogical equations, we refer the reader to Y. Tsutsumi and J. Zhang [7], J. Zhang [8] and the references therein.

Motivated by the above works, we are interested in problem (1). By using potential well theory introduced by

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Payne and Sattinger [10], as well as the concavity method introduced by Levine [11]. Liu [12] introduced a family of potential wells which include the known potential well as a special case. Recently, this method was extended by Xu [13] to study the Cauchy problem of nonlinear Klein-Gordon equation with dissipative term. for other related results, we refer to the reader to [14],[15],[16],[17],[18],[19],[20]. we define some invariant manifolds and derive sharp conditions for the global existence and blow up of the solution of problem (1) and generalize the results in [9].

Throughout this paper, we use $||\cdot||_{H^1}$ to denote the norm of $H^1(R^N)$ and $||\cdot||_{L^p}$ of $L^p(R^N)$. For simplicity, hereafter, we will denote $\int_{R^N}\cdot$ by $\int\cdot$ and use c to denote various positive constants.

This paper is organized as follows. In Section 2, we give some concerned preliminaries, define some functionals and prove some invariant sets. In Section 3, we give a sharp condition for the global existence and blow up of problem (1). In Section 4, we establish family of potential wells. In the last two sections, we discuss some invariant sets, global existence and finite time blow up of solutions by family of potential wells method.

2. Variational problem and invariant manifolds

For problem (1), we define the energy space in the course of nature by

$$H=\left\{\psi\in H^1(R^N):\int V(x)|\psi|^2<\infty\right\}. \quad (2)$$

Here and hereafter, H becomes a Hilbert space, continuously embedded in $H^1(R^N)$, endowed with the inner product as follows

$$<\psi,\phi>:=\int\nabla\psi\nabla\overline{\phi}+\psi\overline{\phi}+V(x)\psi\overline{\phi}, \quad (3)$$

whose associated norm we denote by $||\cdot||_H$.

We also define the following functionals

$$E(\varphi)=\frac{1}{2}\int|\nabla\varphi|^2+V(x)|\varphi|^2-2\sum_{k=1}^{l}\frac{a_k}{p_k+1}|\varphi|^{p_k+1}+2\sum_{j=1}^{s}\frac{b_j}{q_j+1}|\varphi|^{q_j+1}, \quad (4)$$

$$P(\varphi)=\frac{1}{2}\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2-2\sum_{k=1}^{l}\frac{a_k}{p_k+1}|\varphi|^{p_k+1}+2\sum_{j=1}^{s}\frac{b_j}{q_j+1}|\varphi|^{q_j+1}, \quad (5)$$

$$I(\varphi)=\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2-\sum_{k=1}^{l}a_k|\varphi|^{p_k+1}+\sum_{j=1}^{s}b_j|\varphi|^{q_j+1}. \quad (6)$$

Now we can get the local well-posedness for problem (1) in energy space H. (see [21] and [22]).

Lemma 1. *Let $\varphi_0\in H$. Then there exists a unique solution φ of problem (1) in $C([0,T];H)$ for some $T\in(0,\infty]$ (maximal existence time), and either $T=\infty$ (global existence) or else $T<\infty$ and*

$$\lim_{t\to T}||\varphi||_H=\infty \ \ (\text{finite time blow up}).$$

First we have the following lemmas by similar arguments in [1], [7].

Lemma 2. *Let $\varphi_0\in H$ and φ be a solution of problem (1) in $C([0,T];H)$. Then one has*

$$\int|\varphi|^2=\int|\varphi_0|^2, \quad (7)$$

$$E(\varphi)\equiv E(\varphi_0), \quad (8)$$

$$P(\varphi)\equiv P(\varphi_0). \quad (9)$$

Lemma 3. *Let $\varphi_0\in H$ and φ be a solution of problem (1) in $C([0,T];H)$. Set $J(t)=\int V(x)|\varphi|^2$. Then one has*

$$J''(t)=8\int|\nabla\varphi|^2-V(x)|\varphi|^2-\sum_{k=1}^{l}\frac{N(p_k-1)}{2(p_k+1)}a_k|\varphi|^{p_k+1}\varphi+\sum_{j=1}^{s}\frac{N(q_j-1)}{2(q_j+1)}b_j|\varphi|^{q_j+1}\varphi. \quad (10)$$

By similar argument as in [7], we have the following lemma.

Lemma 4. *Let $\varphi_0\in H$ and φ be a solution of problem (1) in $C([0,T];H)$. If $J''(t)<0$, then the solution $\varphi(x,t)$ of problem (1) blows up in finite time.*

We define a manifold as follows

$$M:=\{\psi\in H\backslash\{0\}:I(\psi)=0\}$$

and consider a constrained variational problem,

$$d=\inf_{\psi\in M}P(\psi). \quad (11)$$

Lemma 5. $d>0$.

proof From $I(\varphi)=0$, we have

$$P(\varphi)=\frac{1}{2}\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2-2\sum_{k=1}^{l}\frac{a_k}{p_k+1}|\varphi|^{p_k+1}+2\sum_{j=1}^{s}\frac{b_j}{q_j+1}|\varphi|^{q_j+1}$$
$$>\frac{1}{2}\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2-\frac{2}{p+1}\sum_{k=1}^{l}a_k|\varphi|^{p_k+1}+\frac{2}{p+1}\sum_{j=1}^{s}b_j|\varphi|^{q_j+1}$$
$$=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2$$
$$>c0, \tag{12}$$

i.e. $P(\varphi)>0$. Thus from (11), we get $d0$.

In the following we use the Sobolev embedding inequality

$$\int|\varphi|^{p_k+1}\le c_k\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2\right)^{\frac{p_k+1}{2}},\quad 1\le k\le l, \tag{13}$$

$$\int|\varphi|^{q_j+1}\le c_j\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2dx\right)^{\frac{q_j+1}{2}},\quad 1\le j\le s. \tag{14}$$

Here and hereafter c_k, c_j denote various positive constants. From $I(\varphi)=0$ it follows that

$$\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2$$
$$=\int\sum_{k=1}^{l}a_k|\varphi|^{p_k+1}-\sum_{j=1}^{s}b_j|\varphi|^{q_j+1}$$
$$<\int\sum_{k=1}^{l}a_k|\varphi|^{p_k+1}$$
$$\sum_{k=1}^{l}a_kc_k\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2\right)^{\frac{p_k+1}{2}}$$
$$c\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2\right)^{\frac{p_0+1}{2}}.$$

Here we use $\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2\right)^{\frac{p_0+1}{2}}$ to represent the largest one of the values $\left(\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2\right)^{\frac{p_k+1}{2}}$ for $1kl$ and c is a various positive constant. Since $p_k>1$ for $1kl$, we have $p_0>1$ and

$$\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2>c. \tag{15}$$

Hence

$$P(\varphi)>\left(\frac{1}{2}-\frac{1}{p+1}\right)\int|\nabla\varphi|^2+|\varphi|^2+V(x)|\varphi|^2$$
$$>c0,$$

which implies that $d>0$.

Theorem 1.*Define*

$$K=\{\psi\in H\backslash\{0\},P(\psi)<d,I(\psi)<0\}. \tag{16}$$

K is an invariant manifold of (1), that is, if $\varphi_0\in K$, then the solution $\varphi(x,t)$ of problem (1) also satisfies $\varphi(x,t)\in K$ for any $t\in[0,T)$.

proof Let $\varphi_0\in K$. By Lemma 1, there exists a unique $\varphi(x,t)\in C([0,T);H)$ with $T\infty$ such that $\varphi(x,t)$ is a solution of problem (1). As Lemma 2 demonstrates,

$$P(\varphi)=P(\varphi_0),\ \ t\in[0,T).$$

Thus $P(\varphi_0)<d$ implies that $P(\varphi)<d$ for any $t\in[0,T)$.

Now we show $I(\varphi)<0$ for $t\in[0,T)$. Otherwise, from the continuity of $I(\varphi(t))$ in t, there is a $t_1\in[0,T)$ such that $I(\varphi(x,t_1))=0$. By (5), (6) and

$$P(\varphi(x,t_1))>\left(\frac{1}{2}-\frac{1}{p+1}\right)\int|\nabla\varphi(x,t_1)|^2+|\varphi(x,t_1)|^2+V(x)|\varphi(x,t_1)|^2c>0,$$

we have $\varphi(x,t_1)\neq0$. Otherwise $P(\varphi(x,t_1))=0$, which contradicts with $P(\varphi(x,t_1))c>0$. From (11), it follows that $P(\varphi(x,t_1))d$. This contradicted with $P(\varphi(x,t))<d$ for any $t\in[0,T)$. Therefore $I(\varphi(x,t))<0$ for all $t\in[0,T)$. Now we prove that $\varphi(x,t)\in K$ for any $t\in[0,T)$. This completes the proof of Theorem 1.

By the same argument as Theorem 1, we can get the following results.

Theorem 2.*Define*

$$R=\{\psi\in H\backslash\{0\},P(\psi)<d,I(\psi)>0\}. \tag{17}$$

Then R is an invariant manifold of (1).

3. Sharp conditions for global existence

Theorem 3.*If $\varphi_0\in R\cup\{0\}$, then the solution $\varphi(x,t)$ of problem (1) globally exists on $t\in[0,\infty)$.*

proof First, we let $\varphi_0 \in R$. Thus Theorem 2 implies that the solution $\varphi(x,t)$ of problem (1) satisfies that $\varphi(x,t) \in R$ for $t \in [0,T)$. For fixed $t \in [0,T)$, we denote $\varphi(x,t) = \varphi$. Thus we have $P(\varphi) < d$, $I(\varphi) > 0$. It follows that from (5) and (6),

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right) \int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 \\
<&\frac{1}{2} \int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 - \frac{2}{p+1} \\
&\left(\sum_{k=1}^{l} a_k|\varphi|^{p_k+1} - \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}\right) \\
<&\frac{1}{2} \int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 - 2\sum_{k=1}^{l} \\
&\frac{a_k}{p_k+1}|\varphi|^{p_k+1} + 2\sum_{j=1}^{s} \frac{b_j}{q_j+1}|\varphi|^{q_j+1} \\
<&d,
\end{aligned}$$

which indicates

$$\int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 < \frac{2(p+1)}{p-1}d. \qquad (18)$$

Therefore in view of Lemma 1, (18) implies that φ globally exists on $t \in [0,\infty)$.

Let $\varphi_0 = 0$. From (7), we have $\varphi = 0$, which shows that φ is a trivial solution of problem (1). Theorem 3 is completed.

Theorem 4. *When $1 < q_s < q_{s-1} < \cdots < q_1 = q < p = p_l < p_{l-1} < \cdots < p_1$ and $N > \frac{2(p+1)}{p-1}$, If $\varphi_0 \in K$, then the solution $\varphi(x,t)$ of problem (1) blows up in finite time.*

proof Since $\varphi_0 \in K$, from Theorem 1, we have $\varphi(x,t) \in K$, i.e., $I(\varphi) < 0$. Therefore

$$\begin{aligned}
&\int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 \\
<&\int \sum_{k=1}^{l} a_k|\varphi|^{p_k+1} - \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{8}J''(t) =& \int |\nabla\varphi|^2 - V(x)|\varphi|^2 - \sum_{k=1}^{l} \frac{N(p_k-1)}{2(p_k+1)} \\
&a_k|\varphi|^{p_k+1}\varphi + \sum_{j=1}^{s} \frac{N(q_j-1)}{2(q_j+1)} b_j|\varphi|^{q_j+1}\varphi \\
<& \int |\nabla\varphi|^2 - V(x)|\varphi|^2 - \frac{N(p-1)}{2(p+1)} \\
&\int \sum_{k=1}^{l} a_k|\varphi|^{p_k+1}\varphi - \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}\varphi \\
<& \int |\nabla\varphi|^2 - V(x)|\varphi|^2 - \frac{N(p-1)}{2(p+1)} \\
&\int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 \\
<& -\left(1 + \frac{N(p-1)}{2(p+1)}\right) \int V(x)|\varphi|^2 \\
=& -c_* J(t) < 0,
\end{aligned}$$

where $c_* = 1 + \frac{N(p-1)}{2(p+1)} > 0$. Now we show that there exists a $T_1 \in (0,\infty)$ such that $J(t) > 0$ for $t \in [0,T_1)$ and $J(T_1) = 0$. Otherwise, $\forall t \in [0,\infty)$, $J(t) > 0$.
Set

$$g(t) = \frac{J'(t)}{J(t)}.$$

It is easy to show that

$$g'(t) = \frac{J''(t)}{J(t)} - \left(\frac{J'(t)}{J(t)}\right)^2 < -8c_* - g^2(t). \qquad (19)$$

Next let us turn to show $g(t) \neq 0$ for any $t \in [0,\infty)$. Arguing by contradiction again, suppose that there is a t_0 such that $g(t_0) = 0$, i.e., $J'(t_0) = 0$. By $J''(t) < 0$, we have $J'(t) < 0$ for $t \in (t_0,\infty)$. Hence we have $g(t) < 0$ for $t \in (t_0,\infty)$. For any fixed $t_1 > t_0$, dividing (19) by $g^2(t)$, we have

$$\frac{g'(t)}{g^2(t)} < -\frac{8c_*}{g^2(t)} - 1 < -1.$$

Further we derive

$$\int_{t_1}^{t} \frac{g'(\tau)}{g^2(\tau)} d\tau < \int_{t_1}^{t} -1 d\tau,$$

namely,

$$\frac{1}{g(t)} > \frac{1}{g(t_1)} + (t - t_1),$$

which indicates there exists a $t_2 > t_1$ such that

$$g(t) > 0 \ \text{ for any } \ t \in (t_2,\infty). \qquad (20)$$

This contradicts with $g(t) < 0$ for $t \in (t_0, \infty)$. Hence we have $g(t) \neq 0$ for any $t \in [0, \infty)$. By (20), for $t \in (0, \infty)$, we have

$$\frac{1}{g(t)} > \frac{1}{g(0)} + t.$$

Hence, $J'(t) > 0$ for $t \in \left(\left|\frac{1}{g(0)}\right|, \infty\right)$. Therefore $J(t)$ is increasing in $\left(\left|\frac{1}{g(0)}\right|, \infty\right)$ and

$$J''(t) < -8c_* J(t) < -8c_* J(0) < 0.$$

Further we have

$$\int_0^t J''(\tau)d\tau < -8c_* J(0)t,$$

i.e.,

$$J'(t) - J'(0) < -8c_* J(0)t,$$

that is,

$$J'(t) < J'(0) - 8c_* J(0)t.$$

Again, we have

$$\int_0^t J'(\tau)d\tau < J'(0)t - 4c_* J(0)t^2,$$

namely

$$J(t) - J(0) < J'(0)t - 4c_* J(0)t^2.$$

Therefore we have

$$J(t) < J(0) + J'(0)t - 4c_* J(0)t^2,$$

which contradicts with $J(t) > 0$ for $t \in [0, \infty)$. Hence we know that there exists a $T_1 \in (0, \infty)$ such that $J(t) > 0$ for $t \in [0, T_1)$ and $J(T_1) = 0$. By the inequality [7]

$$||\varphi||^2 \frac{2}{N} ||\nabla\varphi|| \cdot ||\sqrt{V(x)}\varphi||.$$

We get

$$\lim_{t \to T_1} ||\nabla\varphi|| = \infty,$$

which indicates

$$\lim_{t \to T_1} ||\varphi||_H = \infty,$$

i.e. the solution of problem (1) blows up in finite time.

*Remark.*It is clear that

$$\{\psi \in H, P(\psi) < d\} = K \cup R \cup \{0\}.$$

Thus Theorem 3 shows that Theorem 4 is sharp.

*Remark.*Apparently the results in this paper generalizes the results in [9].

4. Family of potential wells and the properties

In this section we introduce a family of potential wells and show some properties of them. Then in the following sections, these conclusions shown in this section will be used to prove the global existence and non-global existence. First we give some lemmas and by using them we introduce two families $\{W_\delta\}$ and $\{V_\delta\}$. For Cauchy problem (1) with $\|\varphi_0\| \neq 0$ we define

$$\begin{aligned}\|\varphi\|_H^2 =& \|\varphi\|_{H^1}^2 + \||V(x)|\varphi\|_{L^2}^2 \\ =& \|\nabla\varphi\|^2 + \|\varphi\|^2 + \||V(x)|\varphi\|^2,\end{aligned}$$

$$\widetilde{H} = \{\varphi \in H \mid \|\varphi\| = \|\varphi_0\|\},$$

$$\begin{aligned}J(\varphi) =& \frac{1}{2} \int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 - 2\sum_{k=1}^{l} \\ & \frac{a_k}{p_k+1}|\varphi|^{p_k+1} + 2\sum_{j=1}^{s} \frac{b_j}{q_j+1}|\varphi|^{q_j+1},\end{aligned}$$

$$\begin{aligned}I_\delta(\varphi) =& \delta \int |\nabla\varphi|^2 + |\varphi|^2 + V(x)|\varphi|^2 - \\ & \sum_{k=1}^{l} a_k|\varphi|^{p_k+1} + \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}.\end{aligned}$$

Proposition 1.*[23], [24], [25], [26] Assume that $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$, and $\varphi_0 \in H^1(R^n)$. Then the Cauchy problem (1.1) admits a unique solution $\varphi(t) \in C([0, T); H(R^n))$ for some $T \in [0, \infty)$ (maximal existence time), and $\varphi(t)$ satisfies (7), (8), (9).*

Proposition 2.*[23] Let $\varphi(t)$ be a solution of problem (1) with $\varphi_0 \in H$, T be the existence time of $\varphi(t)$,*

$$F(t) = \int |V(x)|^2|\varphi|^2.$$

Then $\varphi(t) \in H$ for $0 \leq t < T$

$$\begin{aligned}F''(t) =& 8\int |\nabla\varphi|^2 - |V(x)|^2|\varphi|^2 - \frac{n(p-1)}{2(p+1)}|\varphi|^{p+1}, \\ & 0 \leq t < T\end{aligned}$$

and

$$\|\varphi\|^2 \leq \frac{2}{n}\|\nabla\varphi\| F^{\frac{1}{2}}(t),\ 0 \leq t < T.$$

Proposition 3.*[27] Let $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$ and Q be the ground state solution of the following nonlinear elliptic equation:*

$$-\Delta u + u = |u|^{p-1}u \ \text{ in } R^n.$$

Then the best constant $c_* > 0$ of the Gagliardo-Nirenberg's inequality,

$$\|f\|_{L^{p+1}}^{p+1} \le c_* \|f\|_{L^2}^{p+1-\frac{n(p-1)}{2}} \|\nabla f\|_{L^2}^{\frac{n(p-1)}{2}}, \qquad (21)$$

is given by

$$c_* = \frac{2(p+1)}{n(p-1)} \left(\frac{4-(p-1)(n-2)}{n(p-1)} \right)^{\frac{n(p-1)-4}{4}} \times \|Q\|_{L^2}^{-(p-1)}. \qquad (22)$$

From (21) we can obtain the following lemma.

Lemma 6.*Let p satisfy (A), $\varphi \in \widetilde{H}$. Then $\sum_{j=1}^{s} b_j \|\varphi\|_{q_j+1}^{q_j+1} - \sum_{k=1}^{l} a_k \|\varphi\|_{p_k+1}^{p_k+1} \neq 0$ and $\|\nabla\varphi\| \neq 0$, where*
(A) $1+\frac{4}{n} < p < \frac{n+2}{n-2}$ for $n \ge 3$ and $1+\frac{4}{n} < p < \infty$ for $n = 1, 2$.

Next we discuss the relations between $\|\nabla\varphi\|$ and the sign of $I_\delta(\varphi)$, which are crucial for obtaining the main results in this paper.

Lemma 7.*Let p satisfy (A). Assume that $\varphi \in \widetilde{H}$ and*

$$r(\delta) = \left(\frac{\delta}{C_* M_0} \right)^{\frac{1}{q}}, \; M_0 = \|\varphi_0\|^{p+1-\frac{n(p-1)}{2}},$$

$$q = \frac{n(p-1)}{2} - 2.$$

(i)If $\|\nabla\varphi\| < r(\delta)$, then $I_\delta(\varphi) > 0$.
(ii)If $I_\delta(\varphi) < 0$, then $\|\nabla\varphi\| > r(\delta)$.
(ii)If $I_\delta(\varphi) = 0$, then $\|\nabla\varphi\| \ge r(\delta)$.

proof

(i)Since $\varphi \in \widetilde{H}$ implies $\|\nabla\varphi\| \neq 0$, from $\|\nabla\varphi\| < r(\delta)$ we get

$$\begin{aligned} &\sum_{j=1}^{s} b_j \|\varphi\|_{q_j+1}^{q_j+1} - \sum_{k=1}^{l} a_k \|\varphi\|_{p_k+1}^{p_k+1} \\ <&\sum_{j=1}^{s} b_j C_{*j}^{q_j+1} \|\nabla\varphi\|^{q_j+1} \\ \le&\delta\|\nabla\varphi\|^2, \end{aligned}$$

which gives $I_\delta(\varphi) > 0$.
(ii)From $I_\delta(\varphi) < 0$ we get

$$\begin{aligned} \delta\|\nabla\varphi\|^2 <&\sum_{j=1}^{s} b_j \|\varphi\|_{q_j+1}^{q_j+1} - \sum_{k=1}^{l} a_k \|\varphi\|_{p_k+1}^{p_k+1} \\ \le&C_* M_0 \|\nabla\varphi\|^q \|\nabla\varphi\|^2, \end{aligned}$$

which gives $\|\nabla\varphi\| > r(\delta)$.

(iii)From $I_\delta(\varphi) = 0$ we get

$$\begin{aligned} \delta\|\nabla\varphi\|^2 =&\sum_{j=1}^{s} b_j \|\varphi\|_{q_j+1}^{q_j+1} - \sum_{k=1}^{l} a_k \|\varphi\|_{p_k+1}^{p_k+1} \\ \le&C_* M_0 \|\nabla\varphi\|^q \|\nabla\varphi\|^2, \end{aligned}$$

which together with $\|\nabla\varphi\| \neq 0$ gives $\|\nabla\varphi\| \ge r(\delta)$.

As is well known that in space $H^1(R^n)$, Poincaré inequality does not hold, so that one can not use the important fact that $\|\nabla u\|$ is equivalent to $\|u\|_{H^1}$. In order to overcome this difficulty, we introduce the space $\widetilde{H}(R^n)$, so that by (7) and (21) the norms $\|\nabla\varphi\|$ and $\|\varphi\|_{H^1}$ are equivalent in some sense again.

Definition 1.*For problem (1) with $\|\varphi_0\| \neq 0$ we define*

$$d(\delta) = \inf_{\varphi \in \mathcal{N}_\delta} J(\varphi), \; \mathcal{N}_\delta = \{\varphi \in \widetilde{H} | I_\delta(\varphi) = 0\}, \delta > 0.$$

In the following Lemma 8 we estimate the value of $d(\delta)$ and give its expression by $d(1)$, which palys an important role in the proof of the main results of this paper.

Lemma 8.*Let p satisfy (A). Then*
(i)$d(\delta) \ge a(\delta) r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$;
(ii)

$$d(\delta) = \delta^{\frac{4}{n(p-1)-4}} \frac{p+1-2\delta}{p-1} d(1), \quad 0 < \delta < \frac{p+1}{2}. \qquad (23)$$

proof

(i)For any $\varphi \in \mathcal{N}_\delta$, $0 < \delta < \frac{p+1}{2}$ we have $\|\nabla\varphi\| \ge r(\delta)$ and

$$\begin{aligned} J(\varphi) =&\frac{1}{2}\|\nabla\varphi\|^2 + |\varphi|^2 + V(x)|\varphi|^2 - 2\sum_{k=1}^{l} \\ &\frac{a_k}{p_k+1}|\varphi|^{p_k+1} + 2\sum_{j=1}^{s} \frac{b_j}{q_j+1}|\varphi|^{q_j+1} \\ <&\left(\frac{1}{2} - \frac{\delta}{p+1} \right) \|\nabla\varphi\|^2 + \frac{1}{p+1} I_\delta(\varphi) \\ =&a(\delta)\|\nabla\varphi\|^2 \ge a(\delta) r^2(\delta), \end{aligned}$$

which gives $d(\delta) \ge a(\delta) r^2(\delta)$ for $0 < \delta < \frac{p+1}{2}$.
(ii) (a)From the definition of $d(1)$ it follows that for any $\varepsilon > 0$ there exists a $\varphi \in \mathcal{N}_1$ such that

$$d(1) \le J(\varphi) < d(1) + \varepsilon.$$

For $\delta > 0$, define $\lambda = \lambda(\delta)$. Then

$$\delta\|\nabla\varphi\|^2 = \lambda^{\frac{n(p-1)-4}{2}} \left(-\sum_{k=1}^{l} a_k |\varphi|^{p_k+1} \right.$$

$$+\sum_{j=1}^{s} b_j|\varphi|^{q_j+1}\Bigg).$$

Hence for each $\delta > 0$ there exists a unique

$$\lambda(\delta) = \left(\frac{\delta a(\varphi)}{b(\varphi)}\right)^{\frac{2}{n(p-1)-4}},$$

where

$$a(\varphi) = \|\nabla\varphi\|^2,$$

$$b(\varphi) = \left(-\sum_{k=1}^{l} a_k|\varphi|^{p_k+1} + \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}\right).$$

Since $\varphi \in \mathcal{N}_1$ implies $a(\varphi) = b(\varphi)$ we get

$$\lambda(\delta) = \delta^{\frac{2}{n(p-1)-4}}.$$

Note that $\|\varphi^\lambda\| = \|\varphi\| = \|\varphi_0\|, \forall\lambda > 0$ we have $\varphi\lambda(\delta) \in \mathcal{N}_\delta$. From the definition of $d(\delta)$ we get

$$\begin{aligned} &d(\delta)\\ \leq& J(\varphi^\lambda)\\ =&\frac{1}{2}\lambda^2 a(\varphi) - \frac{1}{p+1}\lambda^{\frac{n(p-1)}{2}} b(\varphi)\\ =&\frac{1}{2}\delta^{\frac{4}{n(p-1)-4}} a(\varphi) - \frac{1}{p+1}\delta^{\frac{n(p-1)}{n(p-1)-4}} b(\varphi)\\ =&\delta^{\frac{4}{n(p-1)-4}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right) a(\varphi). \end{aligned} \tag{24}$$

From (24) and

$$J(\varphi) = \frac{1}{2}a(\varphi) - \frac{1}{p+1}b(\varphi) = \frac{p-1}{2(p+1)}a(\varphi)$$

it follows that

$$\begin{aligned} &d(\delta)\\ \leq&\delta^{\frac{4}{n(p-1)-4}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}J(\varphi)\\ <&\delta^{\frac{4}{n(p-1)-4}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}\left(d(1)+\varepsilon\right), \end{aligned}$$

$\&\ 0 < \delta < \dfrac{p+1}{2}$.

From the arbitrariness of ε we obtain

$$\begin{aligned} d(\delta) \leq& \delta^{\frac{4}{n(p-1)-4}}\frac{p+1-2\delta}{p-1}d(1),\\ &0 < \delta < \frac{p+1}{2}. \end{aligned} \tag{25}$$

(b)Let $\delta > 0$. From the definition of $d(\delta)$ it follows that for any $\varepsilon > 0$ there exists a $\varphi \in \mathcal{N}_\delta$ such that

$$d(\delta) \leq J(\varphi) < d(\delta) + \varepsilon.$$

Define $\lambda = \lambda(\delta)$. Then

$$\|\nabla\varphi\|^2 = \lambda^{\frac{n(p-1)-4}{2}}\left(-\sum_{k=1}^{l} a_k|\varphi|^{p_k+1} + \sum_{j=1}^{s} b_j|\varphi|^{q_j+1}\right)$$

and

$$\lambda = \left(\frac{a(\varphi)}{b(\varphi)}\right)^{\frac{2}{n(p-1)-4}}.$$

Since $\varphi \in \mathcal{N}_\delta$ implies that $\delta a(\varphi) = b(\varphi)$ we get

$$\lambda(\delta) = \left(\frac{1}{\delta}\right)^{\frac{2}{n(p-1)-4}}.$$

From $\varphi^\lambda \in \mathcal{N}_1$ and the definition of $d(1)$ we have

$$\begin{aligned} d(1) \leq& J(\varphi^\lambda)\\ =& \frac{1}{2}\lambda^2 a(\varphi) - \frac{1}{p+1}\lambda^{\frac{n(p-1)}{2}} b(\varphi)\\ =& \frac{1}{2}\left(\frac{1}{\delta}\right)^{\frac{4}{n(p-1)-4}} a(\varphi) - \frac{1}{p+1}\left(\frac{1}{\delta}\right)^{\frac{n(p-1)}{n(p-1)-4}} b(\varphi)\\ =& \left(\frac{1}{\delta}\right)^{\frac{4}{n(p-1)-4}}\left(\frac{1}{2}a(\varphi) - \frac{1}{p+1}\frac{1}{\delta}b(\varphi)\right)\\ =& \left(\frac{1}{\delta}\right)^{\frac{4}{n(p-1)-4}}\frac{p-1}{2(p+1)}a(\varphi), \end{aligned} \tag{26}$$

it follows that

$$\begin{aligned} d(\delta) \leq& \left(\frac{1}{\delta}\right)^{\frac{4}{n(p-1)-4}}\frac{p-1}{2(p+1)}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)^{-1} J(\varphi)\\ <& \left(\frac{1}{\delta}\right)^{\frac{4}{n(p-1)-4}}\frac{p-1}{2(p+1)}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)^{-1}\left(d(\delta)+\varepsilon\right),\\ &0 < \delta < \frac{p+1}{2} \end{aligned}$$

and

$$d(\delta)+\varepsilon > \delta^{\frac{4}{n(p-1)-4}}\left(\frac{1}{2}-\frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}d(1),$$
$$0<\delta<\frac{p+1}{2}. \tag{27}$$

From (27) and the arbitrariness of ε we get

$$d(\delta)\geq \delta^{\frac{4}{n(p-1)-4}}\frac{p+1-2\delta}{p-1}d(1),\ 0<\delta<\frac{p+1}{2}. \tag{28}$$

From(25) and (28) we obtain (ii) in this lemma.

Corollary 1. *Let p satisfy (A). Then*

(i) $\lim_{\delta\to 0} d(\delta)=0,\ \lim_{\delta\to\frac{p+1}{2}} d(\delta)=0$;
(ii) $d(\delta)$ *is continuous on* $0<\delta<\frac{p+1}{2}$;
(iii) $d(\delta)$ *is increasing on* $0<\delta\leq a$, *decreasing on* $a\leq\delta<\frac{p+1}{2}$ *and takes the maximum* $d(a)$ *at* $\delta=a=\frac{2(p+1)}{n(p-1)}$.

proof Conclusions (i) and (ii) follow from (ii) in Lemma 8 immediately.

Conclusion (iii) follows from (ii) in Lemma 8 and

$$d'(\delta)=A(a-\delta)\delta^{\alpha},\ A=\frac{2n}{n(p-1)-4}d(1),$$

$$\alpha=\frac{\delta-n(p-1)}{n(p-1)-4}.$$

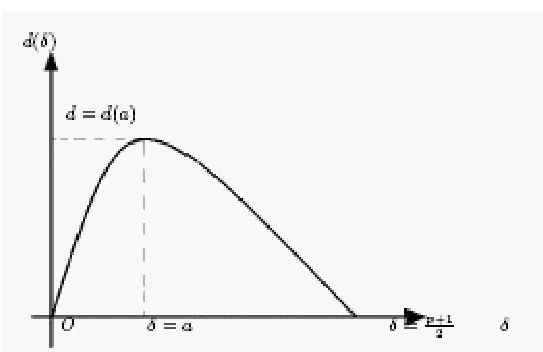

Figure 1

Definition 2. *For problem (1) with $\|\varphi_0\|\neq 0$ we define*

$$I(\varphi)=I_a(\varphi),\ d=d(a),\ a=\frac{2(p+1)}{n(p-1)},$$

$$W=\{\varphi\in\widetilde{H}|I(\varphi)>0,J(\varphi)<d\},$$

$$V=\{\varphi\in\widetilde{H}|I(\varphi)<0,J(\varphi)<d\},$$

$$W_\delta=\{\varphi\in\widetilde{H}|I_\delta(\varphi)>0,J(\varphi)<d(\delta)\},\ 0<\delta<\frac{p+1}{2},$$

$$V_\delta=\{\varphi\in\widetilde{H}|I_\delta(\varphi)<0,J(\varphi)<d(\delta)\},\ 0<\delta<\frac{p+1}{2}.$$

5. Invariant sets and vacuum isolating of solutions

In this section we discuss the invariant sets and vacuum isolating of solutions for problem (1). We consider the case $0<E(\varphi_0)<d$.

Theorem 5. *Let p satisfy (A), $\varphi_0\in H$. Assume that $0<e<d$, $\delta_1<\delta_2$ are two roots of equation $d(\delta)=e$. Then*

(i)All solutions of problem (1) with $E(\varphi_0)=e$ belong to W_δ for $\delta\in[\delta_1,\delta_2]$, provided $I(\varphi_0)>0$.
(ii)All solutions of problem (1) with $E(\varphi_0)=e$ belong to V_δ for $\delta\in[\delta_1,\delta_2]$, provided $I(\varphi_0)<0$.

proof

(i)Let $\varphi(t)\in C\left([0,T);\tilde{H}\right)$ be any solution of problem (1) with $E(\varphi_0)=e$ and $I(\varphi_0)>0$, T be the existence time of $\varphi(t)$. Firstly we prove $\varphi_0\in W_\delta$ for $\delta\in[\delta_1,\delta_2]$. From

$$\frac{1}{2}\||V(x)|\varphi_0\|^2+J(\varphi_0)=E(\varphi_0)=e\leq d(\delta),\ \delta\in[\delta_1,\delta_2] \tag{29}$$

we get $J(\varphi_0)<d(\delta)$ for $\delta\in[\delta_1,\delta_2]$. On the other hand, $I(\varphi_0)>0$ implies $\|\varphi_0\|\neq 0$. Hence from (29) we can get $I_\delta(\varphi_0)>0$ for $\delta\in[\delta_1,\delta_2]$. Otherwise there exists a $\bar\delta\in[\delta_1,\delta_2]$ such that $I_{\bar\delta}(\varphi_0)=0$ which together with $\|\varphi_0\|\neq 0$ gives $J(\varphi_0)\geq d(\bar\delta)$. This contradicts (29). Next we prove that $\varphi(t)\in W_\delta$ for $\delta\in[\delta_1,\delta_2]$, $t\in(0,T)$. Arguing by contradiction, we suppose that there exists a $t_0\in(0,T)$ such that $\varphi(t_0)\in\partial W_\delta$ for some $\delta\in[\delta_1,\delta_2]$, i.e. $I_\delta(\varphi(t_0))=0$ or $J(\varphi(t_0))=d(\delta)$. From (7) we get

$$\frac{1}{2}\||V(x)|\varphi\|^2+J(\varphi)=E(\varphi_0)\leq d(\delta),$$
$$\delta\in[\delta_1,\delta_2],\ t\in(0,T). \tag{30}$$

Hence $J(\varphi(t_0))=d(\delta)$ is impossible. If $I_\delta(\varphi(t_0))=0$, then by $\|\varphi(t_0)\|=\|\varphi_0\|\neq 0$ we get $J(\varphi(t_0))\geq d(\delta)$ which contradicts (30).

(ii)Let $\varphi(t)\in C\left([0,T);\tilde{H}\right)$ be any solution of problem (1) with $E(\varphi_0)=e$, $I(\varphi_0)<0$, T be the existence time of $\varphi(t)$. From $I(\varphi_0)<0$ and (29) we can get $\varphi_0\in V_\delta$ for $\delta\in[\delta_1,\delta_2]$. The remainder of this proof is similar to that in part (i).

From (30) it follows that if $0 < E(\varphi_0) = e < d$, $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = e$, then for any $\delta \in [\delta_1, \delta_2]$, $\varphi \in \mathcal{N}_\delta$ is impossible. Therefore for the set of all solutions of problem (1) with $0 < E(\varphi_0) = e < d$ there exists a vacuum region

$$U_e = \mathcal{N}_{\delta_1\delta_2} = \bigcup_{\delta_1 \le \delta \le \delta_2} \mathcal{N}_\delta = \{\varphi \in \widetilde{H} | I_\delta(\varphi) = 0, \delta_1 \le \delta \le \delta_2\}$$

such that $\varphi(t) \notin U_e$ for any solution $\varphi(t)$ of problem (1) with $0 < E(\varphi_0) = e < d$.

6. Global existence and finite time blow up of solutions

In this section we prove the global existence, finite time blow up of solutions and give some sharp conditions for global existence, finite time blow up of solutions for problem (1) which are completely different from those given in [29] - [31], [32], [33], [34].

Theorem 6. *Let p satisfy (A), $\varphi_0 \in H$. Assume that $\|\varphi_0\| = 0$ or $E(\varphi_0) \le d$, $I(\varphi_0) > 0$. Then problem (1) admits a unique global solution $\varphi(t) \in C\left([0,\infty); \widetilde{H}\right)$ such that*

(i) $\|\varphi(t)\| = \|\varphi_0\| = 0$ for $0 \le t < \infty$ if $\|\varphi_0\| = 0$.
Or
(ii) $\varphi(t) \in W$ for $0 \le t < \infty$ if $E(\varphi_0) \le d$, $I(\varphi_0) > 0$.

proof Firstly from Proposition 1, it follows that Cauchy problem (1) admits a unique local solution $\varphi(t) \in C([0,T); H)$ satisfying (7), (8), where T is the maximal existence time of $\varphi(t)$. Next we prove $T = +\infty$.

(i) If $\|\varphi_0\| = 0$, then by (7) we have $\|\varphi(t)\| = \|\varphi_0\| = 0$, $0 \le t < T$, which gives $\||V(x)|\varphi(t)\| = 0$ and $\|\nabla\varphi(t)\| = 0$, i.e. $\|\varphi(t)\|_H = 0$ for $0 \le t < T$. Hence by Proposition 1 we get $T = +\infty$.

(ii) If $E(\varphi_0) \le d$, $I(\varphi_0) > 0$, then

$$E(\varphi_0) = \frac{1}{2}\||V(x)|\varphi_0\|^2 + \frac{n(p-1)-4}{2n(p-1)} \|\nabla\varphi_0\|^2 + \frac{1}{p+1} I(\varphi_0) > 0.$$

Hence from Theorem 5 we have $\varphi(t) \in W$ for $0 \le t < T$. Hence from

$$\frac{1}{2}\||V(x)|\varphi\|^2 + \frac{n(p-1)-4}{2n(p-1)}\|\nabla\varphi\|^2 + \frac{1}{p+1}I(\varphi) = \frac{1}{2}\||V(x)|\varphi\|^2 + J(\varphi) = E(\varphi_0),\ 0 \le t < T,$$

we get

$$\||V(x)|\varphi\|^2 + \|\nabla\varphi\|^2 \le \frac{2n(p-1)}{n(p-1)-4}E(\varphi_0),\quad 0 \le t < T$$

and

$$\||V(x)|\varphi\|^2 + \|\nabla\varphi\|^2 + \|\varphi\|^2 \le \frac{2n(p-1)}{n(p-1)-4}E(\varphi_0) + \|\varphi_0\|^2,\ 0 \le t < T,$$

which gives $T = +\infty$.

Corollary 2. *If in Theorem 6 the assumption "$E(\varphi_0) \le d$, $I(\varphi_0) > 0$" is replaced by "$0 < E(\varphi_0) < d$, $I_{\delta_2}(\varphi_0) > 0$", where $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = E(\varphi_0)$, then problem (1) admits a unique global solution $\varphi(t) \in C\left([0,\infty); \widetilde{H}\right)$ and $\varphi(t) \in W_\delta$ for $\delta \in [\delta_1, \delta_2]$, $0 \le t < \infty$.*

Corollary 3. *Let p satisfy (A), $\varphi_0 \in H$, $a < \delta_0 < \frac{p+1}{2}$. Assume that $E(\varphi_0) \le d(\delta_0)$ and $I_{\delta_0}(\varphi_0) > 0$. Then problem (1) admits a unique global solution $\varphi(t) \in C\left([0,\infty); \widetilde{H}\right)$ and $\varphi(t) \in W_{\delta_0}$ for $0 \le t < \infty$.*

Theorem 7. *Let p satisfy (A), $\varphi_0 \in H$. Assume that $E(\varphi_0) < d$ and $I(\varphi_0) < 0$. Then the solution of problem (1.1) blows up in finite time.*

proof First Proposition 1 gives the existence of unique local solution $\varphi \in C\left([0,T); \tilde{H}\right)$, where T is the existence time of φ. Let us prove $T < \infty$. Arguing by contradiction, suppose $T = \infty$. Let

$$F(t) = \int |V(x)|^2|\varphi|^2.$$

Then by Proposition 2 we have

$$F''(t) = 8\int |\nabla\varphi|^2 - |V(x)|^2|\varphi|^2 - \frac{n(p-1)}{2(p+1)}|\varphi|^{p+1} \le 8\int |\nabla\varphi|^2 - \frac{n(p-1)}{2(p+1)}|\varphi|^{p+1} = \frac{8}{a}I_a(\varphi). \tag{31}$$

In order to finish this proof we consider the following two cases:

(i) $0 < E(\varphi_0) < d$.
In this case from Theorem 5 we have $\varphi \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \le t < \infty$, where $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = E(\varphi_0)$. Clearly we have $\delta_2 > a > 1$. Hence we have $I_\delta(\varphi) < 0$ and $\|\nabla\varphi\| > r(\delta)$ for $a < \delta < \delta_2$, $0 \le t < \infty$. And $I_{\delta_2}(\varphi) \le 0$, $\|\nabla\varphi\| \ge r(\delta_2)$ for $0 \le t < \infty$. Thus from (31) we get

$$F''(t) \le \frac{8}{a}I_a(\varphi) = \frac{8}{a}\left((a-\delta_2)\|\nabla\varphi\|^2 + I_{\delta_2}(\varphi)\right) \le \frac{8}{a}(a-\delta_2)\|\nabla\varphi\|^2 \le \frac{8}{a}(a-\delta_2)r^2(\delta_2) = -C(\delta_2) < 0,$$

$$F'(t) \le --C(\delta_2)t + F'(0),\ 0 \le t < \infty.$$

Hence there exists a $t_0 \geq 0$ such that $F'(t) < F'(t_0) < 0$ for $t_0 < t < \infty$ and

$$F(t) \leq F'(t_0)(t - t_0) + F(t_0),\ t_0 \leq t < \infty. \quad (32)$$

Since $I(\varphi_0) < 0$ implies $F(0) > 0$ from (32) it follows that there exists a $T_1 > 0$ such that $F(t) > 0$ for $0 \leq t < T_1$ and

$$\lim_{t \to T_1} F(t) = 0,$$

which together with

$$\|\varphi_0\|^2 = \|\varphi\|^2 \leq \frac{2}{n}\|\nabla\varphi\|F^{\frac{1}{2}}(t),$$

gives

$$\lim_{t \to T_1} \sup \|\nabla\varphi\| = +\infty,$$

which contradicts with $T = +\infty$.

(ii)$E(\varphi_0) \leq 0$.

Let $\varphi(t)$ be any solution of problem (1) with $E(\varphi_0) < 0$ or $E(\varphi_0) = 0$, $\|\varphi_0\| \neq 0$, T be the existence time of $\varphi(t)$. Since $E(\varphi_0) < 0$ implies $\|\varphi_0\| \neq 0$. Hence for two cases we always have $\|\varphi(t)\| = \|\varphi_0\| \neq 0$ and $\|\nabla\varphi(t)\| \neq 0$ for $0 \leq t < T$. Thus from

$$\begin{aligned} &\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla\varphi\|^2 + \frac{1}{p+1}I_\delta(\varphi) \\ =&J(\varphi) \\ =&E(\varphi_0) - \frac{1}{2}\||v(x)|\varphi\|^2, \\ &0 < \delta < \frac{p+1}{2},\ 0 \leq t < T, \end{aligned}$$

we can get $I_\delta(\varphi) < 0$ and $J(\varphi) < 0 < d(\delta)$ for $\delta \in \left(0, \frac{p+1}{2}\right)$, $t \in (0, T)$, which gives $\varphi(t) \in V_\delta$ for $\delta \in \left(0, \frac{p+1}{2}\right)$, $t \in [0, T)$, we have $\varphi \in V_\delta$ for $0 < \delta < \frac{p+1}{2}$, $0 \leq t < \infty$. If in the proof of part (i) δ_2 is replaced by $\frac{p+1}{2}$, then we also obtain $T < \infty$.

Finally from Proposition 1 we get

$$\lim_{t \to T} \|\varphi\|_H = +\infty.$$

Theorem 7 is proved.

Corollary 4. *Let p satisfy (A), $\varphi_0 \in H$ and $a < \delta < \frac{p+1}{2}$. Assume that $E(\varphi_0) \leq d(\delta)$ and $I_\delta(\varphi_0) < 0$. Then the solution of problem (1.1) blows up in finite time.*

Corollary 5. *Let p satisfy (A), $\varphi_0 \in H$. Assume that $E(\varphi_0) < 0$ or $E(\varphi_0) = 0$, $\varphi_0 \neq 0$. Then the solution of problem (1.1) blows up in finite time.*

7. Conclusions

We mainly discuss a class of nonlinear Schrödinger equation with combined power-type nonlinearities and harmonic potential, and derive a sharp condition for blow up and global existence of the solution. Compared with previous work, the nonlinear Schrödinger equation of this paper is more general. Especially the corresponding results of this paper try to explain the effects of the different nonlinear source terms to the global well-posedness of the problem.

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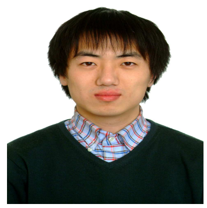

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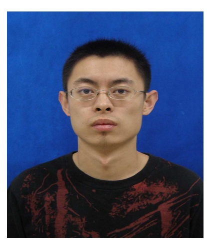

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