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Explicitly Solvable Time-dependent Generalized Harmonic Oscillator

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Abstract: The generalized harmonic oscillator with time-dependent coefficients is considered. With two simplifying assumptions on the time-dependent coefficients, the explicit examples of the conserved quantities constituting the constituting the *SO(2,1)*-generators are given. The uncertainties ∇q and ∇p are briefly discussed

Keywords: Generators; Harmonic Oscillator; Time-dependent Coefficients

1 Introduction

The harmonic oscillator is one of the most important objects in many branches of physics [1, 2]. In its simple versions, both classical and quantal harmonic oscillators are exactly solvable. In this paper, we consider the quantal generalized harmonic oscillator defined by the Hamiltonian

$$H = \frac{1}{2}[Xq^2 + Y(qp + pq) + Zp^2], \qquad (1.1)$$

where X, Y and Z are time(t)-dependent coefficients satisfying $XZ - Y^2 > 0$ and p and q denote canonical variables at t in the Heisenberg picture.

Up to now, this system was investigated by many authors. It was shown that, although the Hamiltonian H is not invariant because of the time-dependence of X, Y, Z, there exists an invariant I in this system[3, 4, 5]. It is constructed as

$$I = \frac{1}{2}\left\{\frac{q^2}{x^2} + \left[x(p + \frac{Y}{Z}q) - \frac{q}{Z}\frac{dx}{dt}\right]^2\right\}, \qquad (1.2)$$

Where x is a solution of the equation?

$$(\frac{1}{Z}\frac{d}{dt})^2 x + [\frac{XZ - Y^2}{Z^2} - \frac{1}{Z}\frac{d}{dt}(\frac{Y}{Z})]x - \frac{1}{x^3} = 0 \cdot \qquad (1.3)$$

Furthermore, it was pointed out [6] that there exist three invariants $J_i\,(i = 1,2,3)$ satisfying the $SO(2,1)$ algebra

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2, \qquad (1.4)$$

And the constraint

$$(J_3)^2 - (J_1)^2 - (J_2)^2 = -\frac{3}{16}. \qquad (1.5)$$

Are obtained. They are constructed on the basis of the solutions of

$$(\frac{1}{Z}\frac{d}{dt})^3 \varsigma + 4[\frac{XZ - Y^2}{Z^2} - \frac{1}{Z}\frac{d}{dt}(\frac{Y}{Z})](\frac{1}{Z}\frac{d}{dt})\varsigma + 2\{\frac{1}{Z}\frac{d}{dt}[\frac{XZ - Y^2}{Z^2} - \frac{1}{Z}\frac{d}{dt}(\frac{Y}{Z})]\}\varsigma = 0. \qquad (1.6)$$

We find that the general discussion can be developed but it is rather obscure. In this paper, we seek the case in which everything becomes explicit and clear. We then obtain the explicit examples for $J_i\,(i = 1,2,3)$ and I .We also discuss the uncertainties ∇q and ∇p of these examples of the generalized harmonic oscillator.

In Sec.2, we clarify our basic assumptions. In Sec.3, we obtain explicit examples of $SO(2,1)$ generators. In Sec.4, we discuss ∇q and ∇p. The final section is devoted to summary.

2 Two Simplifying Assumptions

Our first assumption is

$$\omega = \sqrt{\frac{XZ - Y^2}{Z^2} - \frac{1}{Z}\frac{d}{dt}\left(\frac{Y}{Z}\right)} = \text{const.} \qquad (2.1)$$

With this assumption, the classical Newtonian equation corresponding to H is given by

$$\frac{d^2q}{dt^2} - \left(\frac{1}{Z}\frac{dZ}{dt}\right)\frac{dq}{dt} + \omega^2 Z^2 q = 0. \qquad (2.2)$$

Defining Q by $Q = Z^{-1/2}q$, we have

$$\frac{d^2Q}{dt^2} + \left[\omega^2 Z^2 + \frac{1}{2Z}\frac{d^2Z}{dt^2} - \frac{3}{4Z^2}\left(\frac{dZ}{dt}\right)^2\right]Q = 0. \quad (2.3)$$

This equation becomes most tractable if we have

$$\Omega \equiv \sqrt{\omega^2 Z^2 + \frac{1}{2Z}\frac{d^2Z}{dt^2} - \frac{3}{4Z^2}\left(\frac{dZ}{dt}\right)^2} = \text{const.} \quad (2.4)$$

To obtain examples in which we can develop explicit calculations, we hereafter assume (2.4) in addition to (2.1).It turns out that these assumptions are compatible if and only if Z satisfies

$$\frac{d^2Z}{dt^2} - \frac{3}{2Z}\left(\frac{dZ}{dt}\right)^2 + 2\omega^2 Z^3 - 2Z\Omega^2 = 0 \qquad (2.5)$$

If we define κ by $\kappa = \frac{1}{\sqrt{\omega Z}}$, the above equation is converted to

$$\frac{d^2\kappa}{dt^2} + \Omega^2\kappa - \frac{1}{\kappa^3} = 0. \qquad (2.6)$$

It is interesting that (1.3) to determine x and (2.6) to determine κ are of the same form. Adjusting the origin of t , we obtain

$$Z = \frac{\Omega}{\omega}\frac{1}{\sinh\gamma\cos(2\Omega t) + \cosh\gamma}, \ (\gamma \ge 0): \text{const.} \qquad (2.7)$$

To obtain the solution x of (1.3), it is convenient to define the variable τ defined by

$$\tau = \frac{\Omega}{\omega}\int_0^t \frac{dt'}{\sinh\gamma\cos(2\Omega t') + \cosh\gamma} = \frac{1}{\omega}\arctan[e^{-\gamma}\tan(\Omega t)] \qquad (2.8)$$

or

$$\tan(\tau\omega) = e^{-\gamma}\tan(\Omega t) \qquad (2.9)$$

We note that Eqs. (2.1) and (2.4) are realized when X , Y and Z satisfy

$$X = \left[\omega^2 + \frac{df}{d\tau} + f^2\right]Z, \ Y = f(\tau)Z, \qquad (2.10)$$

where f is an arbitrary function of τ and Z is given by (2.7).

3 Time-independent generators of $SO(2,1)$

Now we turn to the construction of the invariants associated with the generalized harmonic oscillators. The assumption (2.1) leads to simple solutions of (1.6) such as $\varsigma = 1, \sin(2\omega\tau), \cos(2\omega\tau)$ .According to the procedure of [6], we find that $J_i \ (i = 1,2,3)$ defined by

$$J_i = 2\eta_i T_1 + (\xi_i - \varsigma_i)T_2 + (\xi_i + \varsigma_i)T_3, \quad (i = 1,2,3) \qquad (3.1)$$

$$\begin{cases} \varsigma_1 = \sin(2\omega\tau) \\ \varsigma_2 = \cos(2\omega\tau) \\ \varsigma_3 = 1 \end{cases} \qquad (3.2)$$

$$\begin{cases} \eta_1 = \frac{Y}{Z}\sin(2\omega\tau) - \omega\cos(2\omega\tau) \\ \eta_2 = \frac{Y}{Z}\cos(2\omega\tau) + \omega\sin(2\omega\tau) \\ \eta_3 = \frac{Y}{Z} \end{cases} \qquad (3.3)$$

$$\begin{cases} \xi_1 = \left(\frac{Y^2}{Z^2} - \omega^2\right)\sin(2\omega\tau) - \frac{2Y}{Z}\omega\cos(2\omega\tau) \\ \xi_2 = \left(\frac{Y^2}{Z^2} - \omega^2\right)\cos(2\omega\tau) + \frac{2Y}{Z}\omega\sin(2\omega\tau) \\ \xi_3 = \frac{Y^2}{Z^2} + \omega^2 \end{cases} \qquad (3.4)$$

are time-independent. Here T_1, T_2 and T_3 are defined by

$$T_1 = \frac{qp + pq}{4h}, T_2 = \frac{q^2 - p^2}{4h}, T_3 = \frac{q^2 + p^2}{4h} \qquad (3.5)$$

and satisfy

$$[T_1, T_2] = -iT_3, \quad [T_2, T_3] = iT_1, \quad [T_3, T_1] = iT_2 \ \square, \qquad (3.6)$$

and the constraint

$$(T_3)^2 - (T_1)^2 - (T_2)^2 = -\frac{3}{16}. \qquad (3.7)$$

On the other hand, in terms of τ and ω, (1.3) becomes

$$\frac{d^2x}{d\tau^2}+\omega^2 x-\frac{1}{x^3}=0. \tag{3.8}$$

With the help of (2.2), we obtain the general solution x explicitly

$$x=\frac{1}{\sqrt{2\omega}}\sqrt{\sqrt{\alpha^2-4}\cos(2\omega\tau+\varepsilon)+\alpha}, \tag{3.9}$$

$(\alpha\geq 2)$, ε: const.

We then obtain various conserved quantities for various choices of α and ε. For the cases $\{\alpha=2\}$, $\{\alpha>2,\varepsilon=0\}$, $\{\alpha>2,\varepsilon=\frac{\pi}{2}\}$, x becomes as follows:

$$\begin{aligned} x&=\frac{1}{\sqrt{\omega}}: \quad \{\alpha=2\},\\ x&=\frac{1}{\sqrt{2\omega}}\sqrt{\sqrt{\alpha^2-4}\cos(2\omega\tau)+\alpha}: \quad \{\alpha>2,\varepsilon=0\},\\ x&=\frac{1}{\sqrt{2\omega}}\sqrt{-\sqrt{\alpha^2-4}\sin(2\omega\tau)+\alpha}: \quad \{\alpha>2,\varepsilon=\frac{\pi}{2}\}.\end{aligned} \tag{3.10}$$

Then it is somewhat tedious but straightforward to obtain

$$\begin{aligned} I&=2\hbar J_3: \quad \{\alpha=2\},\\ I&=\hbar(\alpha J_3+\sqrt{\alpha^2-4}J_2: \quad \{\alpha>2,\varepsilon=0\},\\ I&=\hbar(\alpha J_3-\sqrt{\alpha^2-4}J_2:: \quad \{\alpha>2,\varepsilon=\frac{\pi}{2}\}.\end{aligned} \tag{3.11}$$

According to [6], the first formula $I=2\hbar J_3$ can be adopted in general. Thanks to the simplifying assumption (2.2), we here understand how J_1 and J_2 are related to I. Although we have found the latter two formulas in the special case $\omega=$ const., it might be possible to find relations among I, J_1, J_2 in more general cases.

4. Uncertainties ∇q and ∇p

We finally comment on the uncertainties in the simplest case. The quantal eigenvalue problem $H\Psi=E\Psi$ is written down as

$$-\frac{Z\hbar^2}{2}\frac{d^2\psi}{dq^2}-i\hbar Yq\frac{d\psi}{dq}+(\frac{Xq^2}{2}-i\hbar\frac{Y}{2})\Psi=E\Psi. \tag{4.1}$$

Its normalized solution with minimal E is given by [5, 7]

$$\psi_n(q)=\sqrt{\alpha}\chi_n(aq)\exp[-\frac{i}{2}(bq)^2], \tag{4.2}$$

$$a=\frac{(XZ-Y^2)^{\frac{1}{4}}}{(Zh)^{1/2}}, b=\sqrt{\frac{Y}{Zh}} \tag{4.3}$$

$$\frac{d^2\chi_n(\xi)}{d\xi^2}+(2n+1-\xi^2)\chi_n(\xi)=0. \tag{4.4}$$

The solution in the $n=0$ case is given by

$$\psi_0(q)=\sqrt{\frac{a}{\sqrt{\pi}}}\exp[-\frac{1}{2}(a^2+ib^2)q^2]. \tag{4.5}$$

$$E_0=\frac{\hbar}{2}\sqrt{XZ-Y^2}. \tag{4.6}$$

The uncertainties $(\nabla q)^2=\int_{-\infty}^{+\infty}q^2|\psi_0(q)|^2\,dq$ and $(\nabla p)^2=\int_{-\infty}^{+\infty}\psi_0^*(q)(\frac{\hbar}{i}\frac{d}{dq})^2\psi_0(q)dq$ in this state are calculated to be

$$(\nabla q)^2=\frac{1}{2a^2}=\frac{\hbar}{2}\frac{Z}{\sqrt{XZ-Y^2}}, \tag{4.7}$$

$$(\nabla p)^2=\frac{\hbar^2}{2}\frac{a^4+b^4}{a^2}=\frac{\hbar}{2}\frac{X}{\sqrt{XZ-Y^2}}. \tag{4.8}$$

Under the assumption (2.2), they become

$$(\nabla q)^2=\frac{\hbar}{2}\frac{1}{\sqrt{\omega^2+\frac{d}{d\tau}(\frac{Y}{Z})}}, \tag{4.9}$$

$$(\nabla p)^2=\frac{\hbar}{2}\frac{\omega^2+(\frac{Y}{Z})^2+\frac{d}{d\tau}(\frac{Y}{Z})}{\sqrt{\omega^2+\frac{d}{d\tau}(\frac{Y}{Z})}}. \tag{4.10}$$

We find that ∇q and ∇p are determined solely by $f(\tau)$.

Although we have $\nabla q\nabla p>\frac{\hbar}{2}$ in general, it is interesting that, as in a squeezed state, $(\nabla q)^2$ can be smaller than $\frac{\hbar}{2\omega}$ for positive $\frac{df(\tau)}{d\tau}$. On the other hand, E_0 involves Z as well as $f(\tau)$:

$$\frac{df}{d\tau}+\omega^2=\lambda^2 f^2, \tag{4.11}$$

where λ is a positive constant. Then $f(\tau)$ is fixed as

$$f=-\frac{\omega}{\lambda}\frac{e^{\lambda\omega\tau}-Ae^{-\lambda\omega\tau}}{e^{\lambda\omega\tau}+Ae^{-\lambda\omega\tau}}, \tag{4.12}$$

Where A is an arbitrary constant. This case corresponds to the Hamiltonian

$$H=\frac{Z}{2}[(p+fq)^2+\lambda^2 f^2 q^2]. \tag{4.13}$$

Note that Z of (2.7) is simply described in terms of t, while f of (4.12) by τ with τ and t being related by (2.9).

Now $E_0, (\nabla q)^2, (\nabla p)^2$ and $\nabla q \nabla p$ become

$$E_0 = \frac{\hbar\Omega}{2}\frac{1}{\sinh\gamma\cos(2\Omega t)+\cosh\gamma}\left|\frac{e^{2\lambda\omega\tau}-A}{e^{2\lambda\omega\tau}+A}\right|, \qquad (4.14)$$

$$(\nabla q)^2 = \frac{\hbar}{2\omega}\left|\frac{e^{2\lambda\omega\tau}+A}{e^{2\lambda\omega\tau}-A}\right|, \qquad (4.15)$$

$$(\nabla q)^2 = \frac{\hbar\omega}{2}(1+\frac{1}{\lambda^2})\left|\frac{e^{2\lambda\omega\tau}-A}{e^{2\lambda\omega\tau}+A}\right|, \qquad (4.16)$$

$$\nabla q \nabla p = \frac{\hbar}{2}\sqrt{1+\frac{1}{\lambda^2}} \qquad (4.17)$$

5 Summary

In order to obtain examples in which everything is explicit, we have explored some cases of the generalized harmonic oscillator with time-dependent coefficients. By assuming (2.1) and (2.4), we have obtained the explicit examples of the time-dependent set of the $SO(2,1)$-generators associated with the system. By assuming (4.11) in addition to (2.1) and (2.4), we have obtained examples where the time-dependence of E_0 and the uncertainties ∇q and ∇p is clear.

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