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On Rough Multi-Level Linear Programming Problem

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Abstract: This paper presents a multi-level linear programming problem with random rough coefficients in objective functions. At the first phase of the solution approach and to avoid the complexity of this problem, we begin by converting the rough nature of this problem into equivalent crisp problem. At the second phase, we use the concept of tolerance membership function at each level to solve a Tchebcheff problem till an optimal solution is obtained. Finally, an illustrative example is given to show the application of the proposed model.

Keywords: Linear programming, Fuzzy programming, Multi-level programming, Rough variable

1 Introduction

Multi-level programming techniques are developed to solve decentralized planning problems with multiple decision makers in a hierarchical organization where each unit or department seeks its own interests.

Three level programming (TLP) problem, whether from the stand point of the three planner Stackelberg behavior or from the interactive organizational behavior, is a very practical problem and encountered frequently in actual practice. Osman et al. [1] proposed a three-planner multi-objective decision-making model and solution method for solving this problem.

Emam presented a bi-level integer non-linear programming problem with linear or non-linear constraints [2] and proposed an interactive approach to solve a bi-level integer multi-objective fractional programming problem in [3]. Baky [4] introduced two new algorithms to solve multi-level multi-objective linear programming problems through the fuzzy goal programming approach. The membership functions for the defined fuzzy goals of all objective functions at all levels were developed. Then the fuzzy goal programming approach was used to obtain the satisfactory solution for all decision makers.

Mathematical programming problems (MPPs) in the crisp form aim to maximize or minimize an objective function over a certain set of feasible solutions. But in many practical situations, the decision maker may not be in a position to specify the objective and/or the feasible set precisely but rather can specify them in a rough sense [5].

Rough set theory, introduced by Pawlak in the early 1980s, is a new mathematical tool to deal with vagueness and uncertainty. This approach seems to be of fundamental importance to artificial intelligence and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, decision support systems, inductive reasoning, and pattern recognition [6].

Osman et al. [5] presented a framework to hybridize the rough set theory with the bi-level programming problem. They designed a genetic algorithm for solving the problem by constructing the fitness function of the upper level programming problems based on the definition of the rough feasible degree.

Xu et al. [7] discussed a class of multi-objective programming problems with random rough coefficients. They showed how to turn a constrained model with random rough variables into crisp equivalent models. Then they introduced an interactive algorithm to obtain the decision maker's satisfying solution.

This paper is organized as follows: Section 2 presents a problem formulation and solution concept of multi-level linear programming (MLLP) problem with random rough coefficients. In Section 3, a fuzzy decision model for the equivalent crisp problem is suggested. In Section 4, an

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illustrative example is given to show the application of the proposed model. Section 5 will be for the conclusion and some open points are stated for future research work in the field of multi-level linear optimization.

2 Problem Formulation and Solution Concept

2.1 Model Formulation

In fact, sometimes a decision needs be made based on uncertain data. In this case, a multi-level linear programming problem with random rough coefficients can be described as:

[1^{th} level]

$$\max_{x_1} f_1(x,\ \xi)\ , \tag{1}$$

[2^{th} level]

$$\max_{x_2} f_2(x,\ \xi)\ ,$$

$$\vdots$$

[m^{th} level]

$$\max_{x_m} f_m(x,\ \xi)\ ,$$

Subject to

$$x \in G.$$

where x is an n-dimensional decision vector, $\xi = (\xi_1,\ \xi_2, \ldots, \xi_n)$ is a random rough vector, $f_i(x,\ \xi)$ are objective functions, i = 1, 2,…, m, which are not well defined and the concept of maximizing $f_i(x,\ \xi)$, i = 1, 2,…, m is not obvious due to the presence of the random rough vector ξ. G is a linear convex constraint set.

Definition 1([7]). *Let* $\xi = (\xi_1,\ \xi_2, \ldots, \xi_n)$ *is a random rough vector on the rough space* $(\Lambda,\ \triangle,\ A,\ \pi)$, *and* $f_i\ : A^n \rightarrow A$ *be continuous functions,* $i = 1, 2, \ldots, m$. *Then the primitive chance of random rough event characterized by* $f_i(\xi) \leq 0$, $i = 1, 2, \ldots, m$, *is a function from [0, 1] to [0, 1] and defined as*

$$\mathrm{Ch}\{f_i(\xi) \leq 0,\ i = 1,\ 2, \ldots,\ m\}(\propto) = \sup\left\{ \beta|\mathrm{Tr}\left\{\lambda \in \Lambda|\Pr\left\{\begin{array}{c} f_i(\xi(\lambda)) \leq 0 \\ i = 1,\ 2, \ldots, m \end{array}\right\} \geq \beta\right\} \geq \propto\right\}.$$

Based on the definition of primitive chance, the random rough chance constrained multi-level programming (RRCCMLP) model will be expressed as follows:

[1^{th} level]

$$\max_{x_1} f_1\ , \tag{2}$$

[2^{th} level]

$$\max_{x_2} f_2\ ,$$

$$\vdots$$

[m^{th} level]

$$\max_{x_m} f_m\ ,$$

Subject to

$$\mathrm{Ch}\{f_i(x,\xi) \geq f_i\}(\gamma_i) \geq \delta_i, i = 1,\ 2,\ \ldots, m,$$

$$\mathrm{Ch}\{g_r(x) \leq 0\}(\eta_r) \geq \theta_r, r = 1,\ 2,\ \ldots, p,$$

$$x \in G.$$

where γ_i, δ_i, η_r and θ_r are predetermined confidence levels, $i = 1, 2, \ldots, m$, $r = 1, 2, \ldots, p$.

The multi-level linear programming problem with random rough coefficients is presented as:

[1^{th} level]

$$\max_{x_1} \xi_1^T x\ , \tag{3}$$

[2^{th} level]

$$\max_{x_2} \xi_2^T x\ ,$$

$$\vdots$$

[m^{th} level]

$$\max_{x_m} \xi_m^T x\ ,$$

Subject to

$$e_r^T x \leq b_r, r = 1,\ 2, \ldots, p,$$

$$x \geq 0,$$

$$x \in G.$$

where $\xi_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{im})^T$, $e_r = (e_{r1},\ e_{r2},\ \ldots,\ e_{rm})^T$ and b_r are random rough vectors, $(r = 1, 2, \ldots, p)$.

Then

$$\mathrm{Ch}\left\{e_r^T x \leq b_r\right\}(\eta_r) \geq \theta_r \Leftrightarrow \mathrm{Tr}\left\{\lambda \mid \Pr\left\{e_r(\lambda)^T x \leq b_r(\lambda)\right\} \geq \theta_r\right\} \geq \eta_r\}, \quad r = 1, 2, \ldots, p.$$

For given confidence levels η_r, θ_r, using the primitive chance measure the chance constraints will be as follows:

$$\mathrm{Tr}\left\{\lambda \mid \Pr\left\{e_r(\lambda)^T x \leq b_r(\lambda)\right\} \geq \theta_r\right\} \geq \eta_r, \quad r = 1, 2, \ldots, p. \tag{4}$$

Thus a point $x(\geq 0)$ is called feasible for problem (3) if and only if the trust measures of the rough events $\{\lambda \mid \Pr\{e_r(\lambda)^T x \leq b_r(\lambda)\} \geq \theta_r\}$ are at least $\eta_r, r = 1, 2, \ldots, p$.

Since

$$\begin{aligned} &\mathrm{Ch}\left\{\xi_i^T x \geq f_i\right\}(\gamma) \geq \delta_i \\ &\Leftrightarrow \mathrm{Tr}\left\{\lambda \mid \Pr\left\{\xi_i(\lambda)^T x \geq f_i\right\} \geq \delta_i\right\} \geq \gamma_i. \end{aligned} \quad (5)$$

The RRCCMLP model for problem (3) which is called tr-pr constrained multi-level programming model can be formulated as follows:

[1th level]

$$\max_{x_1} f_1, \quad (6)$$

[2th level]

$$\max_{x_2} f_2,$$

$$\vdots$$

[m^{th} level]

$$\max_{x_m} f_m,$$

Subject to

$$\mathrm{Tr}\{\lambda|\Pr\{\xi_i(\lambda)^T x \geq f_i\} \geq \delta_i\} \geq \gamma_i,\ i = 1,\ 2,\ \ldots, m,$$

$$\mathrm{Tr}\{\lambda|\Pr\{e_r(\lambda)^T x \leq b_r(\lambda)\} \geq \theta_r\} \geq \eta_r,\ r = 1,\ 2,\ \ldots, p,$$

$$x \geq 0,$$

$$x \in G.$$

where δ_i, γ_i, θ_r, η_r are predetermined confidence levels, $i= 1,\ 2,\ \ldots, m, r= 1,\ 2,\ \ldots, p$. $\mathrm{Tr}\{\cdot\}$ indicates the trust measure of the event in $\{\cdot\}$, and $\Pr\{\cdot\}$ indicates the probability of the event in $\{\cdot\}$.

2.2 Crisp Equivalent Model

In order to solve a tr-pr constrained multi-level programming model, a conversion into its crisp equivalent model is required. However, this procedure may be difficult in some cases.

Theorem 1([7]). *Assume that the random rough variable ξ_{ij} is characterized by $\xi_{ij}(\lambda) \sim \mathcal{N}(\xi_{ij}(\lambda), V_i^c)$, where $(\xi_{ij}(\lambda))_{n\times 1} = (\xi_{i1}(\lambda),\ \xi_{i2}(\lambda),\ \ldots,\ \xi_{in}(\lambda))$ is a rough variable and V_i^c is a positive definite covariance matrix, it follows that $\xi_i(\lambda)^T x = ([a,\ b],\ [c,\ d])$ (where $c \leq a \leq b \leq d$) is a rough variable and characterized by the following trust measure function:*

$$\mathrm{Tr}\left\{\xi_i(\lambda)^T x \geq t\right\} = \begin{cases} 0 & \text{if } d \leq t, \\ \frac{d-t}{2(d-c)} & \text{if } b \leq t \leq d, \\ \frac{1}{2}\left(\frac{d-t}{d-c} + \frac{b-t}{b-a)}\right) & \text{if } a \leq t \leq b, \\ \frac{1}{2}\left(\frac{d-t}{d-c} + 1\right) & \text{if } c \leq t \leq a, \\ 1 & \text{if } t \leq c. \end{cases}$$

Then, $\mathrm{Tr}\{\lambda|\Pr\left\{\xi_i(\lambda)^T x \geq f_i\right\} \geq \delta_i\} \geq \gamma_i$ *if and only if*

$$\begin{cases} b + R \leq f_i \leq d - 2\gamma_i(d-c) + R & \text{if b} \leq \text{M} \leq \text{d}, \\ a + R \leq f_i & \\ \quad \leq \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} + R & \text{if } a \leq M \leq b, \\ c + R \leq f_i & \\ \quad \leq d - (d-c)(2\gamma_i - 1) + R & \text{if } c \leq M \leq a, \\ f_i \leq c + R & \text{if } M \leq c. \end{cases}$$

where $h_i = f_i - \Phi^{-1}(1-\delta_i)\sqrt{x^T V_i^c x}$, Φ is the standardized normal distribution and δ_i, $\gamma_i \in [0,\ 1]$ are predetermined confidence levels.

For the Proof of Theorem 1, the reader is referred to [7].

The crisp equivalent model of the MLLP problem with random rough coefficients with trust more than or equal $\mathrm{Tr}\{\xi\}$ will be as follows:

[1th level]

$$\max_{x_1} h_1(x), \quad (7)$$

[2th level]

$$\max_{x_2} h_2(x),$$

$$\vdots$$

[mth level]

$$\max_{x_m} h_m(x),$$

Subject to

$$x \in G.$$

Definition 2. *Assume that the random rough variable ξ_{ij} is characterized by $\xi_{ij}(\lambda) \sim \mathcal{N}(\xi_{ij}(\lambda), V_i^c)$, where $(\xi_{ij}(\lambda))_{n\times 1} = (\xi_{i1}(\lambda),\ \xi_{i2}(\lambda), \ldots, \xi_{in}(\lambda))$ is a rough variable and $\mathrm{Tr}\{\xi_i(\lambda)^T x \geq t\} = (\omega_{i1}, \ldots, \omega_{in})$. Then ω is the minimum of $(\omega_{i1}, \ldots, \omega_{in})$.*

Definition 3. *If x^* is a feasible solution of a three level programming problem with random rough coefficients in the objective functions; no other feasible solution $x \in G$ exists, such that $f(x^*) \leq f(x)$, then x^* is the optimal solution of the problem with trust value more than or equal ω.*

3 Fuzzy Decision Models for The Equivalent Crisp Problem

To solve an equivalent crisp problem of the multi-level linear programming problem with rough parameters in its objective functions based on fuzzy decision model [1], it is needed to obtain the satisfactory solution that is acceptable to the first level decision maker (FLDM), and provide the second level decision maker (SLDM) with the FLDM decision variables and goals with some leeway to look for the satisfactory solution. After that, the SLDM should provide the third level decision maker (TLDM) with the decision variables and goals with some leeway to look for the satisfactory solution, and to reach the solution that is nearest to the satisfactory solution of the FLDM.

3.1 First Level Decision Maker Problem

First, the FLDM solves the following problem:

$$\max \quad h_1(x), \tag{8}$$

Subject to

$$x \in G,$$

where

$$x = (x_1, x_2, x_3).$$

The individual best solution (h_1^*) and individual worst solution (h_1^-) will be found for the objective function $h_1(x)$, where:

$$h_1^* = \max h_1(x), \quad h_1^- = \min h_1(x). \tag{9}$$

Then goals and tolerances will be determined for individual solutions and the differences between the best solution and the worst solution, respectively. This can be formulated as the following membership function of fuzzy set theory:

$$\mu_{h_1}[h_1(x)] = \begin{cases} 1 & \text{if } h_1(x) > h_1^*, \\ \frac{h_1(x) - h_1^-}{h_1^* - h_1^-} & \text{if } h_1^- \leq h_1(x) \leq h_1^*, \\ 0 & \text{if } h_1^- \geq h_1(x). \end{cases} \tag{10}$$

Then the solution of the FLDM problem can be reached by solving the following Tchebycheff problem [1]:

$$\max \quad \lambda, \tag{11}$$

Subject to

$$x \in G,$$

$$\mu_{h_1}[h_1(x)] \geq \lambda,$$

$$\lambda \in [0,\ 1].$$

3.2 Second Level Decision Maker Problem

Second, the SLDM solves the following problem:

$$\max \quad h_2(x), \tag{12}$$

Subject to

$$x \in G,$$

where

$$x = (x_1, x_2, x_3).$$

The individual best solution (h_2^*) and individual worst solution (h_2^-) will be found for the objective function $h_2(x)$, where:

$$h_2^* = \max h_2(x), \quad h_2^- = \min h_2(x). \tag{13}$$

The membership function will be constructed as follows:

$$\mu_{h_2}[h_2(x)] = \begin{cases} 1 & \text{if } h_2(x) > h_2^*, \\ \frac{h_2(x) - h_2^-}{h_2^* - h_2^-} & \text{if } h_2^- \leq h_2(x) \leq h_2^*, \\ 0 & \text{if } h_2^- \geq h_2(x). \end{cases} \tag{14}$$

Then the solution of the SLDM problem can be reached by solving the following Tchebycheff problem:

$$\max \quad \beta, \tag{15}$$

Subject to

$$\text{x} \in \text{G},$$

$$\mu_{h_2}[h_2(x)] \geq \beta,$$

$$\beta \in [0,\ 1].$$

3.3 Third Level Decision Maker Problem

Third, the TLDM solves the following problem:

$$\max \quad h_3(x), \tag{16}$$

Subject to

$$x \in G,$$

where

$$x = (x_1, x_2, x_3).$$

The individual best solution (h_3^*) and individual worst solution (h_3^-) will be found for the objective function $h_3(x)$, where:

$$h_3^* = \max h_3(x), \quad h_3^- = \min h_3(x). \tag{17}$$

The membership function will be constructed as follows:

$$\mu_{h_3}[h_3(x)] = \begin{cases} 1 & \text{if } h_3(x) > h_3^*, \\ \frac{h_3(x) - h_3^-}{h_3^* - h_3^-} & \text{if } h_3^- \le h_3(x) \le h_3^*, \\ 0 & \text{if } h_3^- \ge h_3(x). \end{cases} \tag{18}$$

Then the solution of the TLDM problem can be reached by solving the following Tchebycheff problem:

$$\max \quad \gamma, \tag{19}$$

Subject to

$$x \in G,$$
$$\mu_{h_3}[h_3(x)] \ge \gamma,$$
$$\gamma \in [0,\ 1].$$

3.4 Three Level Programming Problem

The FLDM, SLDM, and TLDM solutions are now discovered. Nevertheless, they are not usually similar, due to the identity of the objective function of each level. It is not reasonable for the FLDM and SLDM to provide the TLDM with the optimal decisions x_1^F, x_2^S as control factors. They should offer some tolerance, so that TLDM can have an extent feasible region to seek his/her optimal solution, and minimize the time of searching as well.

That way, the maximum tolerances t_1 and t_2 will be provided, so that the decision variables x_1 and x_2 range will be around x_1^F and x_2^S respectively and the following membership function describes x_1^F as

$$\mu_{x_1}(x_1) = \begin{cases} \frac{x_1 - (x_1^F - t_1)}{t_1} & x_1^F - t_1 \le x_1 \le x_1^F, \\ \frac{(x_1^F + t_1) - x_1}{t_1} & x_1^F \le x_1 \le x_1^F + t_1, \end{cases} \tag{20}$$

where x_1^F is the best solution; $(x_1^F - t_1)$ and $(x_1^F + t_1)$ are the worst satisfactory solutions. In addition, this satisfaction rises linearly with the interval $[x_1^F - t_1, x_1]$ and diminishes linearly with the interval $[x_1, x_1^F + t_1]$, and thus other solutions are unacceptable.

The membership function that describes x_2 can be formulated as

$$\mu_{x_2}(x_2) = \begin{cases} \frac{x_2 - (x_2^S - t_2)}{t_2} & x_2^S - t_2 \le x_2 \le x_2^S, \\ \frac{(x_2^S + t_2) - x_2}{t_2} & x_2^S \le x_2 \le x_2^S + t_2, \end{cases} \tag{21}$$

where x_2^S is the best solution; $(x_2^S - t_2)$ and $(x_2^S + t_2)$ are the worst acceptable solutions. To guide the TLDM towards the solution through the correct path:

First, the FLDM goals consider $h_1 \ge h_1^F$ is certainly acceptable and $h_1 < h_1' = h_1(x_1^S, x_2^S, x_3^S)$ is unacceptable, and that the preference with $[h_1',\ h_1^F]$ is linearly increasing. This because the SLDM got the optimum at (x_1^S, x_2^S, x_3^S), that offers the FLDM the objective function values h_1', makes any $h_1 < h_1'$ practically undesirable.

The membership functions of the FLDM can be formulated as

$$\mu'_{h_1}[h_1(x)] = \begin{cases} 1 & \text{if } h_1(x) > h_1^F, \\ \frac{h_1(x) - h_1'}{h_1^F(x) - h_1'} & \text{if } h_1' \le h_1(x) \le h_1^F, \\ 0 & \text{if } h_1(x) \le h_1'. \end{cases} \tag{22}$$

Second, the SLDM goals consider $h_2 \ge h_2^S$ is certainly acceptable and $h_2 < h_2' = h_2(x_1^T, x_2^T, x_3^T)$ is unacceptable, and that the preference with $[h_2',\ h_1^S]$ is linearly increasing. This because the TLDM got the optimum at (x_1^T, x_2^T, x_3^T), that offers the SLDM the objective function values h_2', makes any $h_2 < h_2'$ practically undesirable.

The membership functions of the SLDM can be formulated as

$$\mu^{\backslash}_{h_2}[h_2(x)] = \begin{cases} 1 & \text{if } h_2(x) > h_2^S, \\ \frac{h_2(x) - h_2^{\backslash}}{h_2^S(x) - h_2^{\backslash}} & \text{if } h_2^{\backslash} \le h_2(x) \le h_2^S, \\ 0 & \text{if } h_2(x) \le h_2^{\backslash}. \end{cases} \tag{23}$$

Third, the TLDM needs to construct a membership function for his/her objective function in order to evaluate the fulfillment of each possible solution which will be as follows:

$$\mu^{\setminus}_{h_3}[h_3(x)] = \begin{cases} 1 & \text{if } h_3(x) > h_3^{\mathrm{T}}, \\ \frac{h_3(x) - h_3^{\setminus}}{h_3^{\mathrm{T}}(x) - h_3^{\setminus}} & \text{if } h_3^{\setminus} \le h_3(x) \le h_3^{\mathrm{T}}, \\ 0 & \text{if } h_3(x) \le h_3^{\setminus}. \end{cases} \quad (24)$$

where $h_3^{\setminus} = h_3(x_1^{\mathrm{S}},\ x_2^{\mathrm{S}},\ x_3^{\mathrm{S}})$.

Finally, to get the satisfactory solution, that is also a Pareto optimal solution with overall satisfaction for all DMs, the following Tchebycheff problem [1] will be solved:

$$\max \quad \delta, \quad (25)$$

Subject to

$$\frac{[(x_1^{\mathrm{F}} + t_1) - x_1]}{t_1} \ge \delta I,$$

$$\frac{[x_1 - (x_1^{\mathrm{F}} - t_1)]}{t_1} \ge \delta I,$$

$$\frac{[(x_2^{\mathrm{S}} + t_2) - x_2]}{t_2} \ge \delta I,$$

$$\frac{[x_2 - (x_2^{\mathrm{S}} - t_2)]}{t_2} \ge \delta I,$$

$$\mu'_{h_1}[h_1(x)] \ge \delta I,$$

$$\mu^{\setminus}_{h_2}[h_2(x)] \ge \delta I,$$

$$\mu^{\setminus}_{h_3}[h_3(x)] \ge \delta I,$$

$$t_1 > 0, t_2 > 0,$$

$$\delta \in [0,\ 1],$$

$$x \in G.$$

where δ is the overall satisfaction and I the unit column vector.

A satisfactory solution is found if the FLDM is satisfied with this solution. Otherwise, he/she needs to provide the SLDM with new membership function for the control variables and objectives, and accordingly the SLDM needs to provide the TLDM with new membership function for the control variables and objectives. This process will continue until a satisfactory solution is found.

4 Numerical Example

A three level linear programming problem with random rough coefficients can be written as:

[First level]

$$\max_{x_1} \ \mathrm{Tr}\{\lambda | \Pr\{k_1\xi_1x_1 + k_2\xi_2x_2 + k_3\xi_3x_3 \ge f_1\} \ge \delta_1\} \ge \gamma_1,$$

where x_1, x_2 solve

[Second level]

$$\max_{x_2} \ \mathrm{Tr}\{\lambda | \Pr\{k_4\xi_4x_1 + k_5\xi_5x_2 + k_6\xi_6x_3 \ge f_2\} \ge \delta_2\} \ge \gamma_2,$$

where x_3 solves

[Third level]

$$\max_{x_3} \ \mathrm{Tr}\{\lambda | \Pr\{\xi_7x_1 + \xi_8x_2 + \xi_9x_3 \ge f_3\} \ge \delta_3\} \ge \gamma_3,$$

Subject to

$$4x_1 + 5x_2 - x_3 \le 100,$$
$$2x_1 + x_2 + x_3 \le 35,$$
$$x_1 + x_2 + x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0.$$

where $(k_1, k_2, k_3, k_4, k_5, k_6) = (1.3, 0.5, 1.0, 0.8, 1.6, 2.0)$, the predetermined levels are respectively $\delta_j = \gamma_j = 0.4$, $j = 1, 2, 3$, and

$$\begin{aligned}
&\xi_1 \sim \mathrm{N}(\rho_1,\ 1), && \text{with } \rho_1 = ([1,\ 2],\ [1,\ 4]),\\
&\xi_2 \sim \mathrm{N}(\rho_2,\ 2), && \text{with } \rho_2 = ([3,\ 4],\ [2,\ 5]),\\
&\xi_3 \sim \mathrm{N}(\rho_3,\ 1), && \text{with } \rho_3 = ([2,\ 3],\ [0,\ 3]),\\
&\xi_4 \sim \mathrm{N}(\rho_4,\ 4), && \text{with } \rho_4 = ([4,\ 5],\ [2,\ 5]),\\
&\xi_5 \sim \mathrm{N}(\rho_5,\ 3), && \text{with } \rho_5 = ([3,\ 4],\ [1,\ 4]),\\
&\xi_6 \sim \mathrm{N}(\rho_6,\ 1), && \text{with } \rho_6 = ([1,\ 2],\ [0,\ 3]),\\
&\xi_7 \sim \mathrm{N}(\rho_7,\ 1), && \text{with } \rho_7 = ([0,\ 1],\ [0,\ 3]),\\
&\xi_8 \sim \mathrm{N}(\rho_8,\ 2), && \text{with } \rho_8 = ([2,\ 3],\ [1,\ 4]),\\
&\xi_9 \sim \mathrm{N}(\rho_9,\ 1), && \text{with } \rho_9 = ([2,\ 3],\ [2,\ 5]),
\end{aligned}$$

ρ_i $(i = 1, 2, \ldots, 9)$ are rough variables. By setting $\delta_j = \gamma_j = 0.4$; then $\Phi^{-1}(1 - \delta_j) = 0.26, j = 1, 2, 3.$

And its solution will be as follows:

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9) = (1.8, 3.64, 0.86, 3.9, 2.1, 0.73, 0.86, 2.12, 2.86).$$

$$(Tr\{\xi_1\}, Tr\{\xi_2\}, Tr\{\xi_3\}, Tr\{\xi_4\}, Tr\{\xi_5\}, Tr\{\xi_6\}, Tr\{\xi_7\}, Tr\{\xi_8\}, Tr\{\xi_9\}) = (0.46, 0.41, 0.85, 0.68, 0.82, 0.88, 0.43, 0.75, 0.43).$$

The equivalent crisp problem with trust more than or equal 0.41 can be written as:

[First level]

$$\max_{x_1}\ 2.08x_1 + 1.3x_2 + 0.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2},$$

where x_1, x_2 solve

[Second level]

$$\max_{x_2}\ 2.08x_1 + 2.56x_2 + 1.2x_3 + 0.26\sqrt{4x_1^2 + 3x_2^2 + x_3^2},$$

where x_3 solves

[Third level]

$$\max_{x_3}\ 0.6x_1 + 1.6x_2 + 2.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2},$$

Subject to

$$4x_1 + 5x_2 - x_3 \le 100,$$
$$2x_1 + x_2 + x_3 \le 35,$$
$$x_1 + x_2 + x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0.$$

First, the FLDM solves the following problem:

$$\max\ 2.08x_1 + 1.3x_2 + 0.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2},$$

Subject to

$$4x_1 + 5x_2 - x_3 \le 100,$$
$$2x_1 + x_2 + x_3 \le 35,$$
$$x_1 + x_2 + x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0.$$

The best and worst solution of FLDM will be found:

$$h_1^* = 42, \quad h_1^- = 0.$$

The membership function $\mu_{h_1(.)}$ will be constructed using (10) and (11) will be solved as follows:

$$\max \lambda,$$

Subject to

$$(x_1, x_2, x_3) \in G,$$

$$2.08x_1 + 1.3x_2 + 0.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2} - 42\lambda \ge 0,$$

$$\lambda \in [0, 1].$$

Whose solution is

$$(x_1^F, x_2^F, x_3^F) = (0.1, 0.92, 0.63),$$
$$h_1^F = 2.2, \quad \lambda = 0.052.$$

Second, the SLDM solves the following problem:

$$\max\ 2.08x_1 + 2.56x_2 + 1.2x_3 + 0.26\sqrt{4x_1^2 + 3x_2^2 + x_3^2},$$

Subject to

$$4x_1 + 5x_2 - x_3 \le 100,$$
$$2x_1 + x_2 + x_3 \le 35,$$
$$x_1 + x_2 + x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0.$$

The best and worst solution of SLDM will be found:

$$h_2^* = 60.2, \quad h_2^- = 0.$$

The membership function $\mu_{h_2(.)}$ will be constructed using (14) and (15) will be solved as follows:

$$\max \beta,$$

Subject to

$$(x_1, x_2, x_3) \in G,$$

$$2.08x_1 + 2.56x_2 + 1.2x_3 + 0.26\sqrt{4x_1^2 + 3x_2^2 + x_3^2} - 60.2\beta \ge 0,$$

$$\beta \in [0, 1].$$

Whose solution is

$$(x_1^S, x_2^S, x_3^S) = (0.3, 0.1, 0.31),$$
$$h_2^S = 1.433, \quad \beta = 0.024.$$

Third, the TLDM solves the following problem:

$$\max\ 0.6x_1 + 1.6x_2 + 2.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2},$$

Subject to

$$4x_1 + 5x_2 - x_3 \le 100,$$
$$2x_1 + x_2 + x_3 \le 35,$$
$$x_1 + x_2 + x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0.$$

The best and worst solution of TLDM will be found:

$$h_3^* = 57.2, \quad h_3^- = 0.$$

The membership function $\mu_{h_3(.)}$ will be constructed using (18) and (19) will be solved as follows:

$$\max \gamma,$$

Subject to

$$(x_1, x_2, x_3) \in G,$$

$$0.6x_1 + 1.6x_2 + 2.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2} - 57.2\,\gamma \geq 0,$$

$$\gamma \in [0,\ 1].$$

Whose solution is

$$(x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_3^{\mathrm{T}}) = (0.132,\ 0.166,\ 0.1),$$
$$h_3^{\mathrm{T}} = 0.679, \quad \gamma = 0.012.$$

Assuming that both the FLDM control decision x_1^{F} and the SLDM control decision x_2^{S} are around 0.1 with tolerance 1, the TLDM solves the following problem:

$$\max \delta,$$

Subject to

$$(x_1, x_2, x_3) \in G,$$

$$x_1 + \delta \geq 1.1,$$

$$-x_1 + \delta \geq 0.9,$$

$$x_2 + \delta \geq 1.1,$$

$$-x_2 + \delta \geq 0.9,$$

$$2.08x_1 + 1.3x_2 + 0.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2} - 1.14\delta \geq 1.06,$$

$$2.08x_1 + 2.56x_2 + 1.2x_3 + 0.26\sqrt{4x_1^2 + 3x_2^2 + x_3^2} - 1.33\delta \geq 0.105,$$

$$0.6x_1 + 1.6x_2 + 2.6x_3 + 0.26\sqrt{x_1^2 + 2x_2^2 + x_3^2} + 0.52\delta \geq 1.2,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0,$$

$$\delta \in [0, 1].$$

Whose compromise solution with trust more than or equal 0.41 is X^0 = (0.385, 0.192, 1.234) and $\delta = 0.94$ (overall satisfaction for all DMs).

$$f_1^0 = (2.13), \quad f_2^0 = (3.16), \quad f_3^0 = (4.09).$$

5 Conclusion

This paper presented a multi-level linear programming problem with random rough coefficients in objective functions. At the first phase of the solution approach and to avoid the complexity of this problem, we began by converting the rough nature of this problem into equivalent crisp problem. At the second phase, we used the concept of tolerance membership function at each level to solve a Tchebcheff problem till an optimal solution is obtained.

There are however several open points for future research in the area of rough multi-level linear optimization, in our opinion, to be studied. Some of these points of interest are stated in the following:

1. An algorithm for solving multi-level integer linear multi-objective decision-making problems with rough parameters in the objective functions, in the constraints and in both using Taylor series.
2. An algorithm for solving multi-level mixed-integer linear multi-objective decision-making problems with rough parameters in the objective functions, in the constraints and in both using Taylor series.

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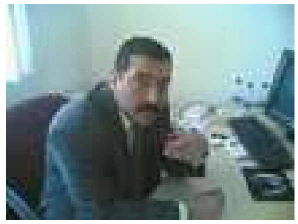

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